1. Consider the initial value problem: find $y(t)$ such that
$$y' = y^2 t, \quad y(0) = 1.$$
Take a step size $h = 0.1$ and verify that the *forward Euler* approximation to $y(0.1)$ is $y_1 = 1.0$ and that the approximations to subsequent points are $y_2 = 1.01$, $y_3 = 1.0304$ and $y_4 = 1.0623$. Compare these estimates with the exact values $y(0.1)$, $y(0.2)$, $y(0.3)$ and $y(0.4)$ obtained by solving the ODE algebraically.

2. By taking a step size of $h = 0.5$, compute the *forward Euler* approximation of $z(5)$, where $z(t)$ satisfies the initial value problem:
$$z' = \frac{1}{50(1+t)} z, \quad z(0) = 1.$$

3. Consider the initial value problem: find $y(t)$ such that
$$y' = \pi\sqrt{1-y^2}, \quad y(0) = 0.$$
Take a step size $h = 0.1$ and show that computing the *forward Euler* approximation to $y(1)$ is problematic.

4. Consider the initial value problem: find $y(x)$ such that
$$y' = \frac{1}{2} e^{x-y}, \quad y(0) = 0.$$
Compute the *forward Euler* approximation of $y(1)$ using (i) a single step of size $h = 1$, (ii) two steps each of size $h = 0.5$, (iii) four steps each of size $h = 0.25$, and (iv) ten steps each of size $h = 0.1$. Compare your estimates with the exact value.

5. By taking a step size of $h = 0.5$, compute the *modified Euler* approximation of $z(2)$, where $z(t)$ satisfies the initial value problem:
$$z' = \frac{1}{50(1+t)} z, \quad z(0) = 1.$$

6. Consider the initial value problem: find $y(x)$ such that
$$y' = 17 - 12x - 3y, \quad y(0) = 9.$$
Using four steps of size $h = 0.25$, estimate $y(1)$ using (i) the *forward Euler* method, (ii) the *modified Euler* method. Compare your estimates with the exact value.

7. *Heun's method* for approximating the solution of an initial value problem is given by

$$K_1 = f(x_n, y_n), \quad K_2 = f(x_n + h, y_n + hK_1)$$
$$y_{n+1} = y_n + \frac{h}{2}(K_1 + K_2).$$

Carry out two steps of the method (with step size $h = 0.5$) to compute an estimate of the solution $y(x)$ of the problem

$$y' = y - x, \quad y(0) = 20.$$

8. Consider the initial value problem: find $y(t)$ such that

$$y' = 1 + \ln y, \quad y(0) = 1.$$

Carry out a step of the 4th-order *Runge–Kutta* method (with step size $h = 0.1$) and verify that $K_1 = 1$, $K_2 = 1.048790$, $K_3 = 1.051111$ and $K_4 = 1.099946$ so that the approximation to $y(0.1)$ is given by $y_1 = 1.104996$.

For the next step of the method, show that $K_1 = 1.099842$, $K_2 = 1.148410$, $K_3 = 1.150501$ and $K_4 = 1.198888$ so that $y_2 = 1.219938$. Continue for one more step and verify that $y_3 = 1.344702$.

9. Consider fluid draining out of a tank according to the equation $y' = -k\sqrt{y}$ with $k = 0.1$ and $y = 20$ at $t = 0$. Use the 4th-order *Runge–Kutta* method with step size $h = 10$ to estimate the values of $y(t)$ (the height, in metres, of fluid in the tank) for $t = 10$, 20 and 30 measured in seconds.

10. Consider the initial value problem: find $y(x)$ such that

$$y' = -2xy^2 \quad y(0) = 1.$$

Carry out two steps of each of the following methods for solving the equation. (i) forward Euler with $h = 1$ (ii) forward Euler with $h = 0.5$ (iii) modified Euler with $h = 1$ (iv) modified Euler with $h = 0.5$ (v) Runge–Kutta with $h = 1$ (vi) Runge–Kutta with $h = 0.5$. Compare your results with the exact solution.

11. Express the following high-order equations as systems of first-order equations.

(a) $\dfrac{d^2y}{dx^2} + 12y = 0; \quad y(0) = 1, \quad \dfrac{dy}{dx}(0) = 0.$

(b) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 4y = x; \quad y(0) = 0, \quad \dfrac{dy}{dx}(0) = 0.$

(c) $\dfrac{d^3y}{dx^3} - \dfrac{dy}{dx} + y = x^3, \quad y(0) = 1, \quad \dfrac{dy}{dx}(0) = 0, \quad \dfrac{d^2y}{dx^2}(0) = 0.$

(d) $\dfrac{d^2p}{dt^2} = \dfrac{1}{(p+t)}\dfrac{dp}{dt}; \quad p(0) = 1, \quad \dfrac{dp}{dt}(0) = 2.$

(e) the coupled system

$$\frac{d^2y}{dt^2} - 2\frac{dz}{dt} = y + z; \quad y(t_0) = y_0, \quad \frac{dy}{dt}(t_0) = v_A,$$
$$\frac{d^2z}{dt^2} + 2\frac{dy}{dt} = y - z; \quad z(t_0) = z_0, \quad \frac{dz}{dt}(t_0) = v_B.$$

12. Consider the coupled initial value problem

$$\frac{dx}{dt} = -3y, \quad \frac{dy}{dt} = 3x, \quad x(0) = 1, \quad y(0) = 0.$$

Approximate the solution of this system using the Runge–Kutta method with step size $h = 0.1$. Show that at the first step, the K values are $(0, 3)$, $(-0.45, 3)$, $(-0.45, 2.9325)$, $(-0.87975, 2.865)$ giving new values x_1 and y_1 of 0.955338 and 0.2955. Show that the second step gives values x_2 and y_2 of 0.825349 and 0.564604. Compare these results with the exact values given by $x = \cos 3t$, $y = \sin 3t$.

13. Consider the coupled initial value problem

$$\frac{dx}{dt} = \frac{1}{3}y, \quad x(0) = 1;$$
$$\frac{dy}{dt} = 6x^3, \quad y(0) = -3.$$

Compute an approximation to the solution vector (x, y) at the final time $t = 0.5$ using the Runge–Kutta method with step size $h = 0.1$.* Then repeat the calculation for $h = 0.05$. Compare the numerical results with the exact values given by $x = 1/(t + 1)$, $y = -3/(t + 1)^2$.

14. Consider the second-order initial value problem

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = 0; \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 0$$

with the value $\epsilon = 0.1$. (This is an example of a *Duffing oscillator*.) Compute three steps of Runge–Kutta with a step size $h = 0.1$.*

15. A sphere floating in water (in the absence of friction and with its centre y above the water level) experiences a force due to gravity of $-\frac{4}{3}\pi r^3 \rho g$ and a force due to buoyancy of $\frac{1}{3}\rho_{water}\pi\left(2r^3 + y(y^2 - 3r^2)\right) g$. It's motion is determined by the ODE

$$\frac{d^2y}{dt^2} = -g + g\frac{\rho_{water}}{4\rho}\left[2 + \frac{y^3}{r^3} - 3\frac{y}{r}\right]$$

A 0.4 metre radius sphere with density 0.7 of that of water is initially at rest with its centre 25 cm below the level of the water. You may assume that $g = 10\ ms^{-2}$.

*An efficient way of doing this is to modify the function `rhsf.m` that is associated with Computational Exercise III.

(a) Express the equation of motion as a system of two first-order equations and state the appropriate initial conditions.

(b) Take a step size of $h = 0.1$ and compute two steps of Runge–Kutta to estimate the position of the centre of the sphere when $t = 0.1$ and when $t = 0.2$.

16. Compute a centered difference approximation to the BVP

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 4y = x^3, \quad x \in (1, 2); \quad y(1) = 1,\ y(2) = 6,$$

using a grid of size $h = 0.25$. Compare your result with the exact solution values $y(1.25) = 4.771$, $y(1.5) = 5.9834$, $y(1.75) = 6.1867$.

17. Compute a centered difference approximation to the BVP

$$\frac{d^2y}{dx^2} + \frac{\pi^2}{4}y = \epsilon y^3, \quad x \in (0, 1); \quad y(0) = 1,\ y(1) = 0,$$

for the value $\epsilon = 0.1$, using a grid of size $h = 0.5$. Compare your computed result with the exact solution at the interior grid point.

18. An iron bar is kept at temperatures T_0 and T_1 at its two ends. The radiation of heat at point x is proportional its temperature $-AT(x)$ and this effect is balanced by thermal diffusion, that is, $-\kappa\frac{d^2T}{dx^2}$. Thus, when the temperature $T(x)$ has reached equilibrium, it satisfies the BVP

$$-\kappa\frac{d^2T}{dx^2} = -AT, \quad x \in (0, 1); \quad T(0) = T_0,\ T(1) = T_1.$$

Write down a centered difference approximation to $T(x)$ for values of x of 0.25, 0.5 and 0.75 and solve the resulting linear system in the case where $A = 1$, $\kappa = 1$, $T_0 = 1$ and $T_1 = 2$.

19. Imagine two metal bars, one with ends maintained at temperature $T = 600$ and the other with a temperature maintained at $T = 400$. As well as thermal diffusion and general radiation, there is heat transferred from one bar to the other. The bars thus satisfy a coupled system of ODEs

$$-0.01\,\frac{d^2T_1}{dx^2} + 0.01\,T_1 + 0.005\,(T_1 - T_2) = 0,$$
$$-0.01\,\frac{d^2T_2}{dx^2} + 0.01\,T_2 + 0.005\,(T_2 - T_1) = 0,$$

subject to $T_1(0) = T_1(1) = 600$ and $T_2(0) = T_2(1) = 400$.

Take a grid with points $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$ and $x_4 = 1$ and compute a centered difference approximation to T_1 and T_2.