Engineering Mathematics | CHEN30101 Computational Exercise X

1. The IVP associated with the heat equation

$$u_t - u_{xx} = 0 \quad \text{in } (0,1)\times(0,\tau)$$
$$u(0,t) = 0; \quad u(1,t) = 0, \qquad t > 0$$
$$u(x,0) = f(x), \qquad x \in (0,1)$$

has an exact temperature solution $u(x,t)$ that decays to zero in the limit $\tau \to \infty$. A plot of the temperature solution for a specific initial function $f(x)$ can be generated using the M-file `heat.m` shown below.

```
function [u,tau]= heat(fc,nstep,tau)
% Fourier solution of 1-D heat equation
%   input parameters
%          fc       row vector of fourier coefficients of f(x)
%          nstep    number of time steps
%          tau      final time
ndiv=256;  h=1/ndiv;  x=0:h:1; % set spatial resolution
dt=tau/nstep; tau=0; % set temporal resolution
% define and plot initial condition
ell=length(fc); j=1:ell; X=sin(j.*pi.*x'); ww=fc.*X;
uu=sum(ww'); stepsolplot(tau,x,uu), shg
% timestep loop
for step=1:nstep
tau=tau+dt;  T=exp(-j.*j.*pi.*pi.*tau); ww=fc.*X.*T;
uu=sum(ww'); stepsolplot(tau,x,uu)
end

function stepsolplot(tau,x,u)
if tau==0, % plot initial condition
plot(x,u,'-b'), axis('square'), title('initial profile'),
xlabel('x'), ylabel('temperature'), shg, pause(2)
else % plot solution
hold on, plot(x,u,'-b'),
title(['temperature: time is ',num2str(tau)])
pause(0.5), hold off
end
return
```

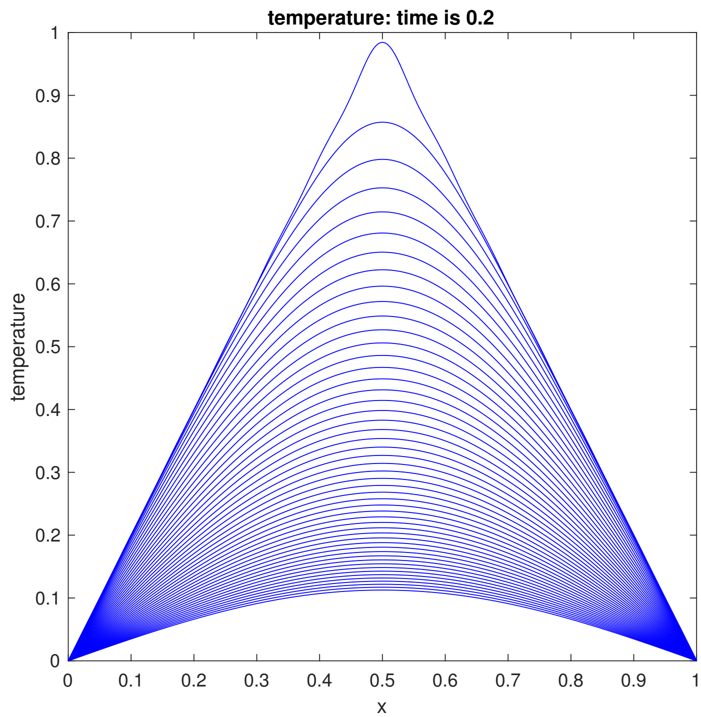
Run the code to show the solution in the case of the initial data given by

$$f(x) = \begin{cases} 2x & \text{when } 0 \le x \le 1/2 \\ 2-2x & \text{when } 1/2 \le x \le 1, \end{cases}$$

by typing the following

```
>> fc=zeros(1,25);
>> j=1:4:25; ffc=8./(j.*j*pi*pi); fc(j)=ffc
>> j=3:4:25; ffc=-8./(j.*j*pi*pi); fc(j)=ffc
>> heat(fc,50,0.2)
```

You should generate the solution plot shown below.



Next, experiment by running **heat** with a randomly generated set of Fourier coefficients (corresponding to a random initial profile) by typing

```
>>  j=1:10; fc=rand(1,10)./j;  heat(fc,50,0.2)
```

2. The IVP associated with the wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \quad \text{in } (0,1)\times(0,\tau) \\ u(0,t) &= 0; \quad u(1,t) = 0, \qquad t > 0 \\ u(x,0) &= 0 \quad u_t(x,0) = f(x), \qquad x \in (0,1) \end{aligned}$$

has an exact displacement solution $u(x,t)$ that is periodic in time.

Your task is to modify the provided function to plot solutions of the wave equation instead of the heat equation. You can do this by going through the following steps.

- Generate a copy of M-file, call it `wave.m` and change the name of the main function from `heat` to `wave`.

- Generate a vector of *scaled* Fourier coefficients by dividing the input Fourier coefficients by the factor $j\pi$ using componentwise division: that is include the line `fct=fc./(j*pi);`.
- Set the initial displacement vector `uu` to zero before plotting it (that is, immediately prior to the timestep loop).
- Inside the timestep loop change the definition of the vector T from `exp(-j.*j.*pi.*pi.*tau)` to `sin(j.*pi.*tau)` and then edit the code so as to multiply by the scaled Fourier coefficient vector `fct` when generating `ww`.
- Change the `ylabel` and the `title` in the function `stepsolplot`.

Test your function by typing the following

```
>> fc=zeros(1,25);
>> j=1:4:25; ffc=8./(j.*j*pi*pi); fc(j)=ffc
>> j=3:4:25; ffc=-8./(j.*j*pi*pi); fc(j)=ffc
>> wave(fc,39,2)
```

You should generate a solution that looks similar to the one shown below.