Manchester Numerical Analysis

Crank–Nicolson | 70 years on

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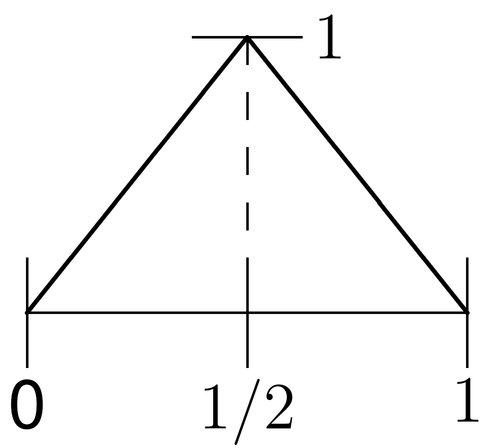
The heat equation I

$$\left.\begin{array}{rll} u_t - u_{xx} & = 0 & (x,t) \in (0,1) \times (0,\tau] \\ u(0,t) = 0, \quad u(1,t) & = 0 & t \in (0,\tau] \\ u(x,0) & = u_0(x) & x \in [0,1]. \end{array}\right\}$$

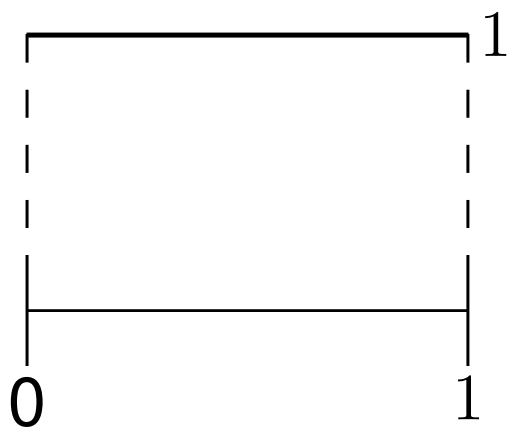
$$u_0(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \le x \le \ell \end{cases}$$



The heat equation II

$$\left.\begin{aligned} u_t - u_{xx} &= 0 && (x,t) \in (0,1) \times (0,\tau] \\ u(0,t) = 0, \quad u(1,t) &= 0 && t \in (0,\tau] \\ u(x,0) &= u_0(x) && x \in [0,1]. \end{aligned}\right\}$$

$$u_0(x) = 1 \quad 0 \le x \le 1$$



PART I

Infinite series solutions

$$\left.\begin{aligned} u_t - u_{xx} &= 0 && (x,t) \in (0,1) \times (0,\tau] \\ u(0,t) = 0, \quad u(1,t) &= 0 && t \in (0,\tau] \\ u(x,0) &= u_0(x) && x \in [0,1]. \end{aligned}\right\} (H)$$

Infinite series solutions

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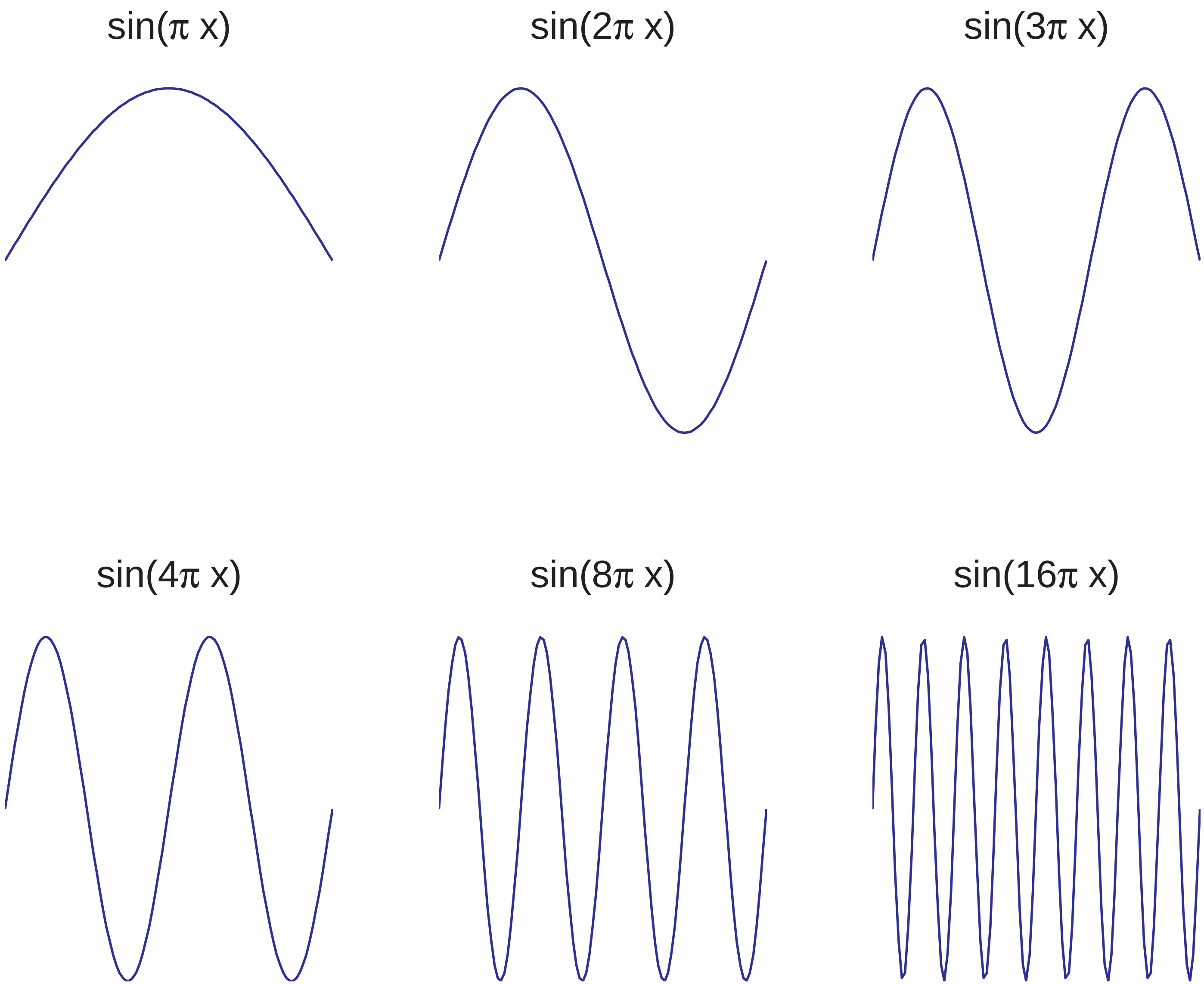
- Joseph Fourier (1768–1830) showed that if the initial condition is given by a combination of waves

$$u_0(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x),$$

then the general solution to (H) is also an infinite series of waves :

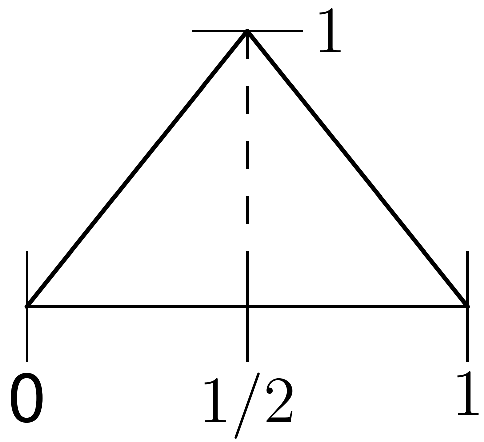
$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x).$$

Fourier Modes



Example I

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t}\sin(n\pi x)$$



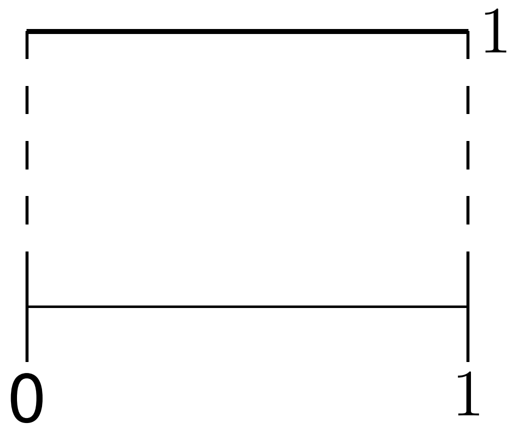
$$c_n = \frac{8}{\pi^2}\left\{\frac{\mathbf{1}}{\mathbf{1^2}}, \frac{0}{2^2}, \frac{\mathbf{-1}}{\mathbf{3^2}}, \frac{0}{4^2}, \frac{\mathbf{1}}{\mathbf{5^2}}, \frac{0}{6^2}, \frac{\mathbf{-1}}{\mathbf{7^2}}, \dots\right\} \quad \sim \frac{1}{n^2}.$$

Hence

$$u(x,t) = \frac{8}{\pi^2}\left\{\frac{1}{\mathbf{1}}e^{-\pi^2 t}\sin(\pi x) - \frac{1}{\mathbf{9}}e^{-9\pi^2 t}\sin(3\pi x) + \frac{1}{\mathbf{25}}\dots\right\}$$

Example II

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x)$$



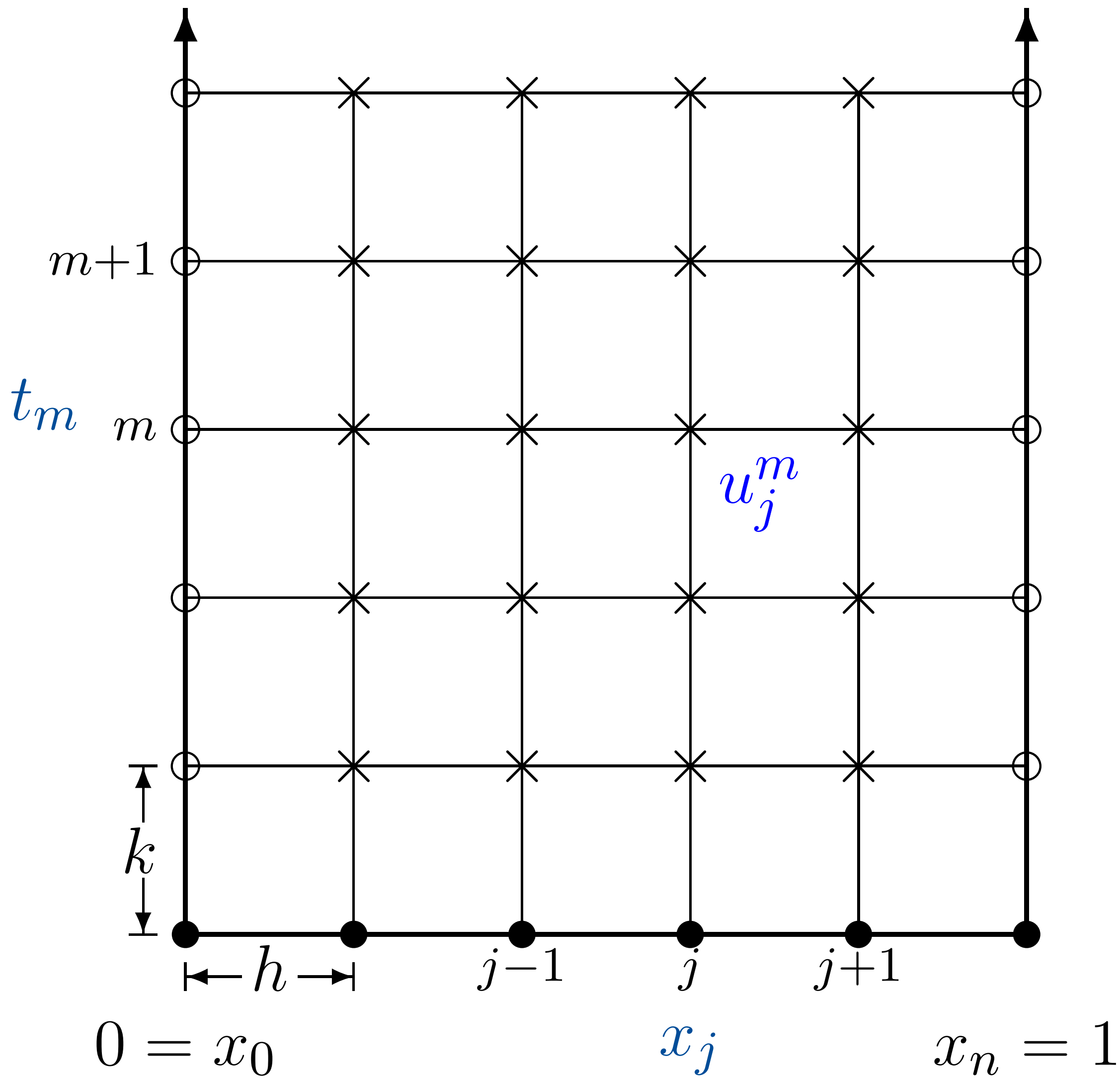
$$c_n = \frac{4}{\pi}\left\{\frac{1}{\mathbf{1}}, \frac{0}{2}, \frac{1}{\mathbf{3}}, \frac{0}{4}, \frac{1}{\mathbf{5}}, \frac{0}{6}, \ldots\right\} \quad \sim \frac{1}{n}.$$

$$u(x,t) = \frac{4}{\pi}\left\{\frac{1}{\mathbf{1}} e^{-\pi^2 t} \sin(\pi x) + \frac{1}{\mathbf{3}} e^{-9\pi^2 t} \sin(3\pi x) + \frac{1}{\mathbf{5}} \ldots\right\}$$

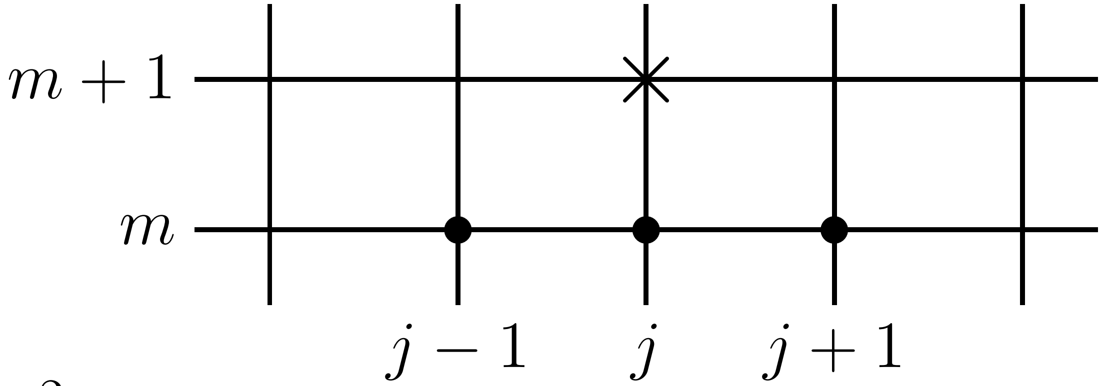
PART II

Numerical solutions

Discretisation of (x,t) space : $u_j^m \approx u(x_j, t_m)$

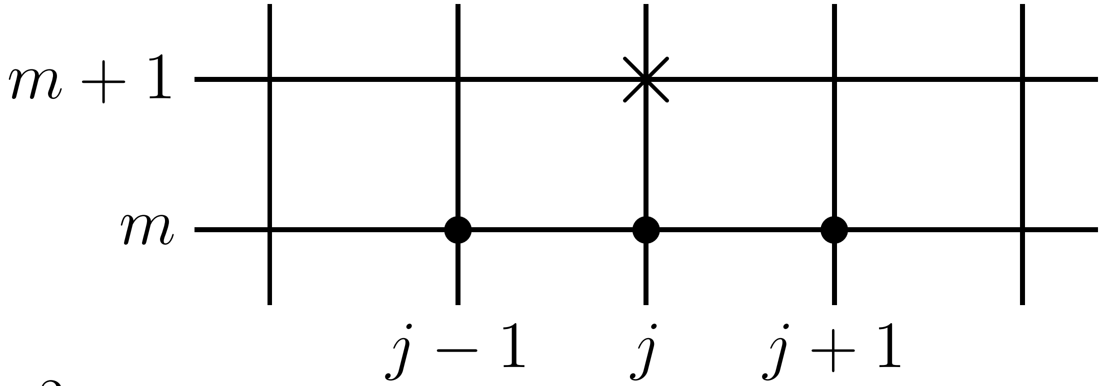


$$h = \frac{1}{n}$$



For $\mu = k/h^2$

$$u_j^{m+1} = \mu u_{j-1}^m + (1-2\mu)u_j^m + \mu u_{j+1}^m, \qquad \begin{array}{l} j = 1, 2, \ldots, n-1 \\ m = 0, 1, 2, \ldots \end{array}$$



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This is a matrix-vector multiply at each time level:

$$\begin{bmatrix} u_1^{m+1} \\ \vdots \\ u_j^{m+1} \\ \vdots \\ u_{n-1}^{m+1} \end{bmatrix} = \begin{bmatrix} 1-2\mu & \mu & 0 & & \\ & \ddots & \ddots & & \\ & \mu & 1-2\mu & \mu & \\ & & \ddots & \ddots & \\ & & 0 & \mu & 1-2\mu \end{bmatrix} \begin{bmatrix} u_1^m \\ \vdots \\ u_j^m \\ \vdots \\ u_{n-1}^m \end{bmatrix}.$$

- Is the explicit approximation consistent?

Yes ...

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Yes ...

$$\left|T_j^m\right| \leq \frac{1}{2}kM_{tt} + \frac{1}{12}h^2 M_{xxxx} \qquad \begin{array}{l} j = 1, 2, \ldots, n-1 \\ m = 1, 2, \ldots \end{array},$$

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Moreover, for a fixed ratio $\mu = k/h^2$ we deduce that

$$\left|T_j^m\right| \le \frac{1}{2}k\left[M_{tt} + \frac{1}{6\mu}M_{xxxx}\right].$$

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This is first order consistency in time. ♡

- Is the explicit approximation stable?

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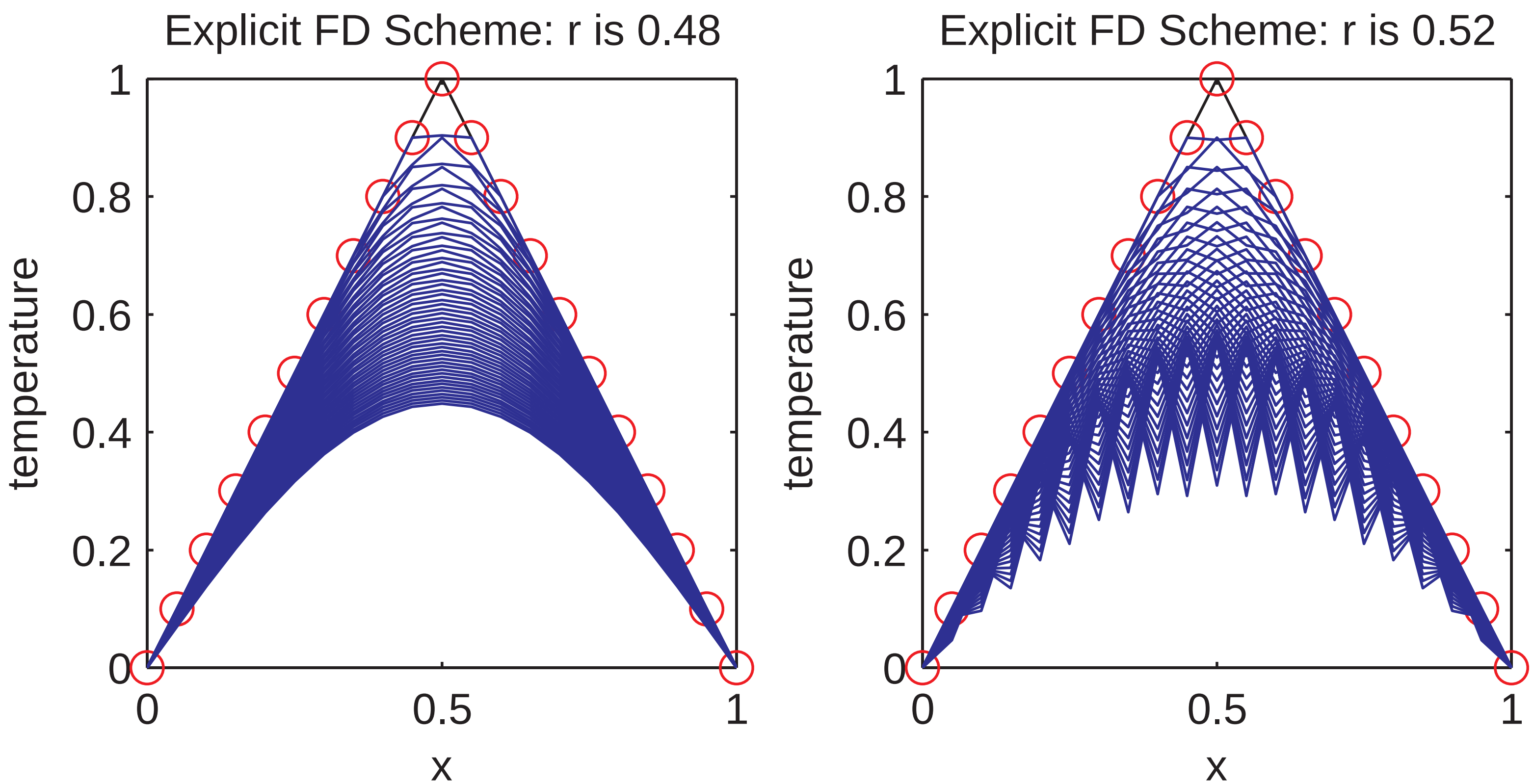
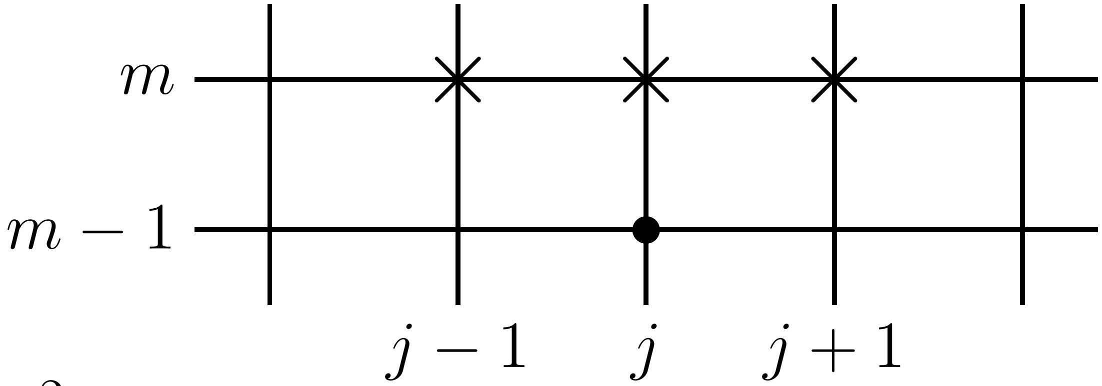
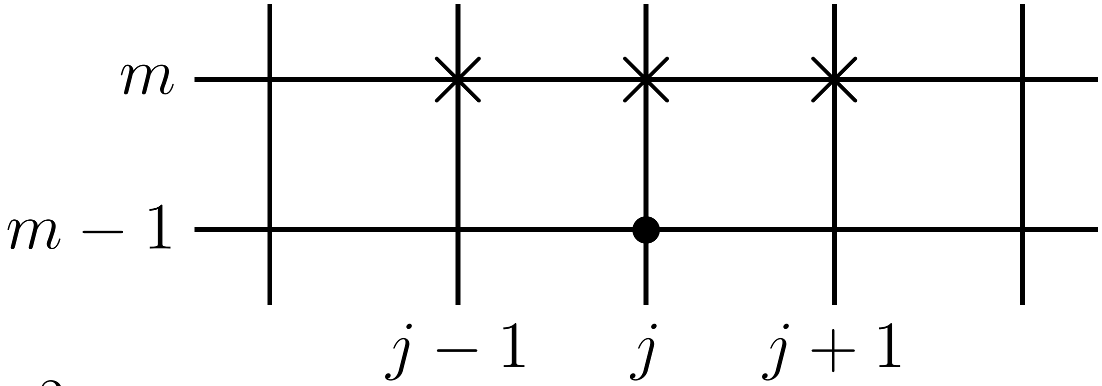
It all depends on $\mu = r$





For $\mu = k/h^2$

$$-\mu u_{j-1}^{m} + (1+2\mu)u_j^m - \mu u_{j+1}^m = u_j^{m-1} \qquad \begin{array}{l} j = 1, 2, \ldots, n-1 \\ m = 0, 1, 2, \ldots \end{array}$$



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This is a tridiagonal matrix solve at each time level:

$$\begin{bmatrix} 1+2\mu & -\mu & 0 & & \\ & \ddots & \ddots & & \\ & -\mu & 1+2\mu & -\mu & \\ & & \ddots & \ddots & \\ & & 0 & -\mu & 1+2\mu \end{bmatrix} \begin{bmatrix} u_1^m \\ \vdots \\ u_j^m \\ \vdots \\ u_{n-1}^m \end{bmatrix} = \begin{bmatrix} u_1^{m-1} \\ \vdots \\ u_j^{m-1} \\ \vdots \\ u_{n-1}^{m-1} \end{bmatrix}$$

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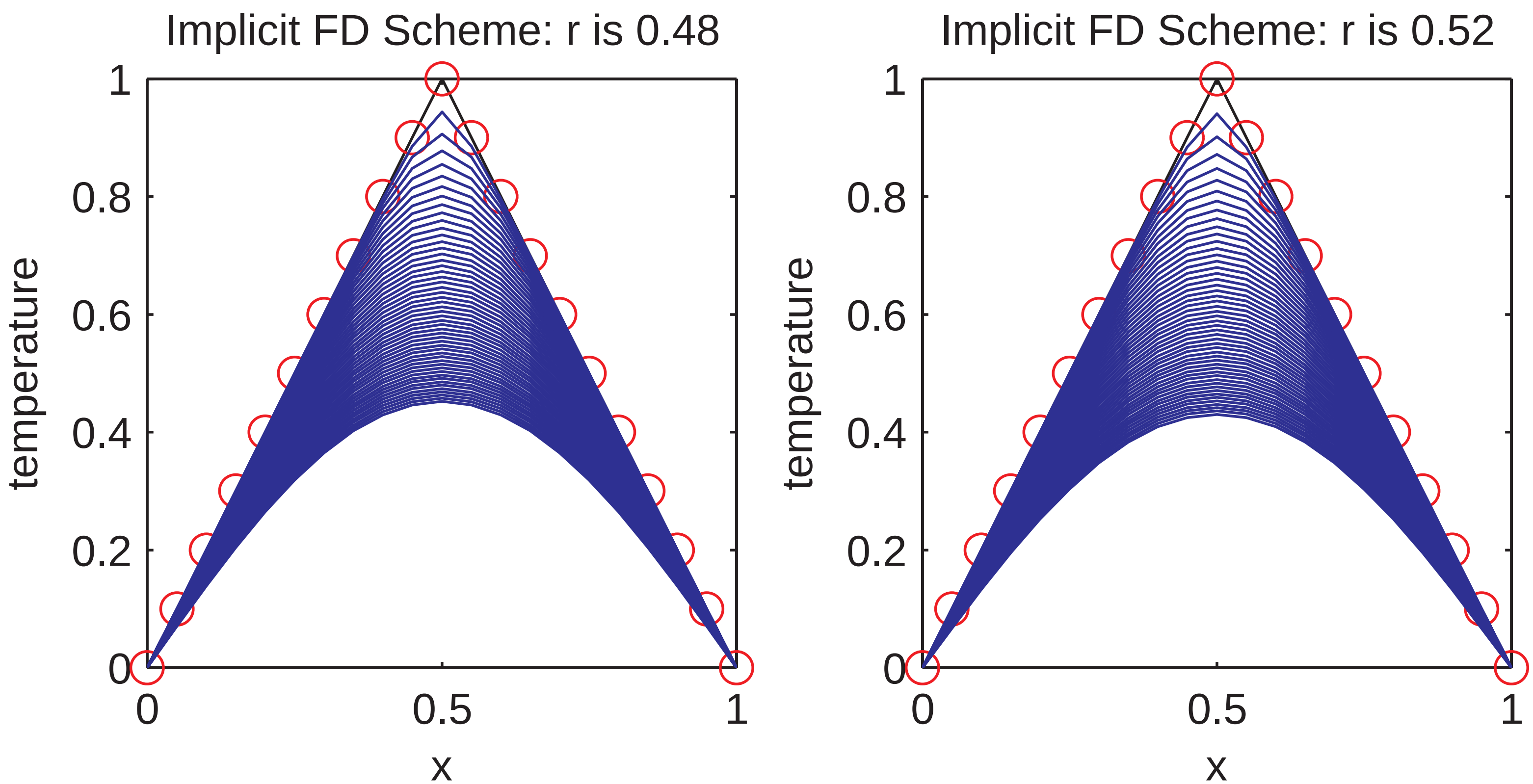
Yes for any $\mu = r$



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$$\left|T_j^m\right| \le \frac{1}{2}kM_{tt} + \frac{1}{12}h^2M_{xxxx} \qquad \begin{array}{l} j = 1, 2, \ldots, n-1 \\ m = 1, 2, \ldots \end{array},$$

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Moreover, for a fixed ratio $\mu = k/h^2$ we deduce that

$$\left|T_j^m\right| \le \frac{1}{2}k\left[M_{tt} + \frac{1}{6\mu}M_{xxxx}\right].$$

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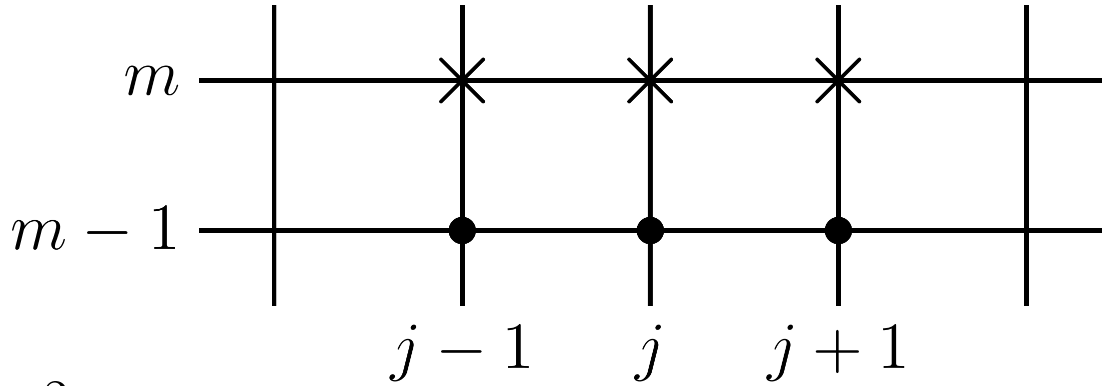
This is first order consistency in time. ♡

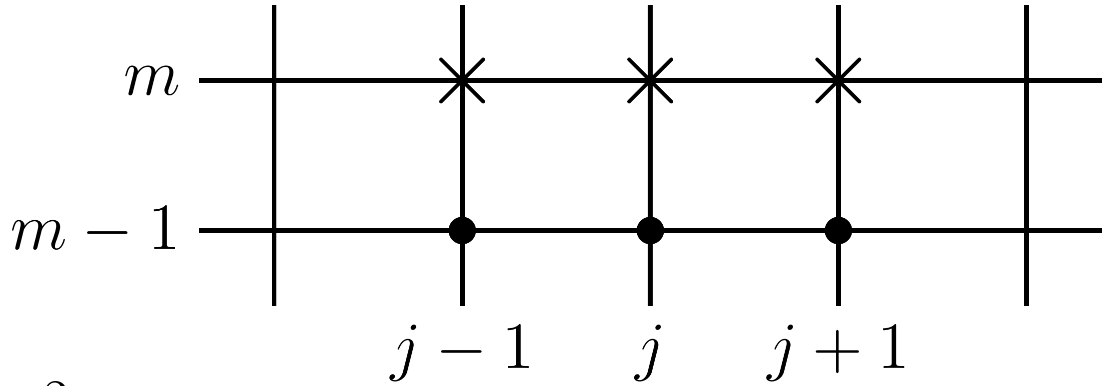
Crank–Nicolson approximation



For $\mu = k/h^2$

$$-\frac{\mu}{2}u_{j-1}^{m} + (1+\mu)u_j^m - \frac{\mu}{2}u_{j+1}^{m} = \frac{\mu}{2}u_{j-1}^{m-1} + (1-\mu)u_j^{m-1} + \frac{\mu}{2}u_{j+1}^{m-1}.$$

Crank–Nicolson approximation



For $\mu = k/h^2$

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This is a tridiagonal matrix solve at each time level:

$$\begin{bmatrix} \ddots & \ddots & \\ -\frac{\mu}{2} & 1+\mu & -\frac{\mu}{2} \\ & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ u_j^m \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \ddots & \\ \frac{\mu}{2} & 1-\mu & \frac{\mu}{2} \\ & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ u_j^{m-1} \\ \vdots \end{bmatrix}.$$

- Is the CN approximation consistent?

Yes ...

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$$\left|T_j^{m+\frac{1}{2}}\right| \leq \frac{1}{12}h^2 M_{xxxx} + \frac{1}{12}k^2 M_{ttt} \qquad \begin{array}{l} j = 1, 2, \ldots, n-1 \\ m = 1, 2, \ldots \end{array} .$$

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Moreover, for a fixed ratio $r = k/h$ we deduce that

$$\left|T_j^{m+\frac{1}{2}}\right| \le \frac{1}{12}k^2 \left[\frac{1}{r^2} M_{xxxx} + M_{ttt}\right].$$

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Moreover, for a fixed ratio $r = k/h$ we deduce that

$$\left|T_j^{m+\frac{1}{2}}\right| \leq \frac{1}{12}k^2 \left[\frac{1}{r^2} M_{xxxx} + M_{ttt}\right].$$

This is second order consistency in time. ♡♡