

This homework contributes 20% of the overall assessment of the course.

All problems are worth **2** marks, except for * which are worth double.

We want to solve an *anisotropic* diffusion equation defined on a square domain $\Omega = (0, 1) \times (0, 1)$ with boundary $\partial\Omega$ consisting of two nonoverlapping pieces $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$. Thus, given a diagonal conductivity matrix $K = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}$ with constant coefficients, $0 < k_{11} \leq k_{22} < \infty$, we seek $u : \Omega \rightarrow \mathbb{R}$ satisfying

$$\left. \begin{aligned} -\nabla \cdot (K \nabla u) &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega_D, \\ K \nabla u \cdot \vec{n} &= 1 && \text{on } \partial\Omega_N, \end{aligned} \right\} (D)$$

where \vec{n} is the outward pointing normal. We also assume that $\partial\Omega_D$ has nonzero length, so that $u = 0$ on some *part* of the boundary.

You may use the Cauchy–Schwarz inequality or the Poincaré–Friedrichs inequality in answering any of the following questions without giving a proof.

1. Explain the difference between a *classical solution* satisfying (D) and a *strong* solution to the boundary value problem.
2. Given the test space $X := \{v | v \in \mathcal{H}^1(\Omega), v = 0 \text{ on } \partial\Omega_D\}$, where $\mathcal{H}^1(\Omega)$ is the standard Sobolev space, use the product differentiation formula

$$\nabla \cdot (v K \nabla u) = K \nabla u \cdot \nabla v + v \nabla \cdot (K \nabla u)$$

to show that u solving (D) also satisfies the variational formulation: find $u \in X$ such that

$$\int_{\Omega} K \nabla u \cdot \nabla v = \int_{\partial\Omega_N} v \, ds \quad \forall v \in X. \tag{V}$$

3. Define the Galerkin approximation $u_h \in X_h = \text{span}\{\phi_j\}_{j=1}^n \subset X$ and identify explicitly the entries A_{ij} and f_i of the $n \times n$ Galerkin system

$$A \mathbf{x} = \mathbf{f}.$$

4. Prove that the Galerkin solution u_h exists and is unique.

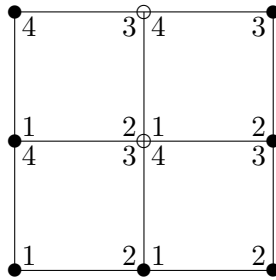
For the following questions, consider the specific case with $\partial\Omega_N$ given by the set of boundary points (x, y) with $0 < x < 1$ and $y = 1$ (so that $u = 0$ on three sides of the domain).

5. * Let $u_h \in X_h$ be the piecewise linear approximation to the solution of (V) with $k_{11} = 1$ and $k_{22} = 2$ that is defined on a uniform triangulation of a grid of squares of size h that is formed by constructing a diagonal from the bottom left to the top right corner of each square.

Assuming that the three nodal basis functions are numbered anticlockwise around the right-angle of each triangle, compute the 3×3 element matrix that contributes to the Galerkin matrix A .

Use this information to compute the Galerkin solution u_h in the case that $h = 1/2$ (that is, for a mesh of 8 triangles).

6. Suppose further that $u_h \in X_h$ is the piecewise bilinear approximation to u satisfying (V) with that is associated with the uniform grid of square elements shown below.



Consider a general square element \square_k , with edge length h with nodal basis functions numbered anticlockwise (as shown). Show that the Jacobian matrix associated with the mapping to \square_k from the reference element $\square_\star = [-1, 1] \times [-1, 1]$ is the diagonal matrix

$$J_k = \frac{1}{2} \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}.$$

7. * By mapping the derivatives of the reference element basis functions, show that the 4×4 element matrix associated with the negative Laplacian operator is given by

$$A^{\otimes} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{bmatrix}.$$

Note: you only need to work out three distinct entries of the matrix.

8. Assemble the entries in the 2×2 Galerkin system associated with approximating the solution of (V) in the case $k_{11} = 1$ and $k_{22} = 1$, using the grid shown above. Solve the associated linear equation system to generate the bilinear finite element solution.

Any handwritten submission should include your student registration number and must be submitted electronically (*via Canvas*) before 12pm on Friday 24th April 2026. All work submitted *must be your own*.

I plan to return your mark (broken down question by question) before the end of the April.