This homework contributes 20% of the overall assessment of the course.
Each problem is worth **2** marks and computational exercises are worth **3** marks.

Given an open bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ consisting of two nonoverlapping pieces $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, let $u : \Omega \to \mathbb{R}$ be the solution of the boundary value problem:

$$\left.\begin{aligned} -\nabla^2 u = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega_D, \quad \frac{\partial u}{\partial n} + u = 1 \text{ on } \partial\Omega_N. \end{aligned}\right\} \quad (D)$$

You can assume that $\partial\Omega_N$ has nonzero length so that the *Robin* boundary condition holds on *some part* of the boundary. You may use the Cauchy–Schwarz inequality or the Poincaré–Friedrichs inequality in answering any of the following questions without giving a proof.

1. Explain what is meant by a *classical solution* of (D). What is the physical relevance of the boundary condition on $\partial\Omega_N$?

2. Given the test space $X := \left\{ v | v \in \mathcal{H}^1(\Omega), v = 0 \text{ on } \partial\Omega_D \right\}$, where $\mathcal{H}^1(\Omega)$ is the standard Sobolev space, show that u solving (D) also satisfies the variational formulation: find $u \in X$ such that

$$\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega_N} uv\, ds = \int_{\partial\Omega_N} v\, ds \qquad \forall v \in X. \qquad (\star)$$

   Define the Galerkin approximation $u_h \in X_h = \operatorname{span}\{\phi_j\}_{j=1}^k \subset X$ and show that the approximation method leads to a $k \times k$ matrix system

$$A\,\mathbf{x} = \mathbf{f}.$$

   Identify explicitly the entries A_{ij} of the matrix A and f_i of the vector $\mathbf{f}$.

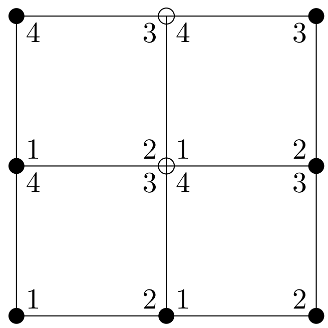
3. Discuss whether or not a solution of $(\star)$ is *unique* in the special case of $\partial\Omega = \partial\Omega_N$.

4. Prove that the Galerkin solution u_h is the *best approximation* to $u \in X$ when measured in the energy norm $\|u\|_E$, that is

$$\|u - u_h\|_E \le \|u - v_h\|_E \quad \forall v_h \in X_h,$$

   where $\|u\|_E^2 := \int_\Omega \nabla u \cdot \nabla u + \int_{\partial\Omega_N} u^2\, ds$.

Suppose that Ω is the square domain $(-1,1)\times(-1,1)$ and that $\partial\Omega_N$ is the top boundary: that is $y=1$ with $-1<x<1$. Suppose further that $u_h\in X_h$ is the piecewise bilinear approximation to u satisfying $(\star)$ that is associated with the uniform grid of square elements with $h=1$ shown below.



5. Consider a general square element $\square_k$, with edge length h with nodal basis functions numbered anticlockwise (as shown). Show that the Jacobian matrix associated with the mapping to $\square_k$ from the reference element $\square_\star=[-1,1]\times[-1,1]$ is the diagonal matrix

$$J_k=\frac{1}{2}\begin{pmatrix} h & 0\\ 0 & h\end{pmatrix}.$$

By mapping the derivatives of the reference element basis functions, show that the 4×4 element matrix associated with the negative Laplacian operator is given by

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6}\\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3}\\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6}\\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3}\end{bmatrix}.$$

Note: you only need to work out three entries of the matrix — you can appeal to symmetry for the remaining entries.

6. Assemble the entries in the 2×2 Galerkin system associated with solving $(\star)$ using the grid shown above. Then solve the linear equation system to generate the bilinear finite element solution.

7. Show that the function $u(r,\theta) = r^{2/3}\sin((2\theta+\pi)/3)$ satisfies Laplace's equation expressed in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Given that $u \in \mathcal{H}^s(\Omega) \iff \|D^s u\| < \infty$, show that $u(r,\theta)$ defined on the pie-shaped domain Ω where $0 \le r \le 1$ and $-\pi/2 \le \theta \le \pi$ is in $\mathcal{H}^1(\Omega)$, but is not in $\mathcal{H}^2(\Omega)$. (Hint: work in polar coordinates, and note that $dx\,dy = r dr\,d\theta$.)

Computational Exercises. The T-IFISS software package offers a choice of two-dimensional domains on which anisotropic diffusion problems can be posed, along with boundary conditions and a choice of finite element approximation on a structured or unstructured triangular mesh.

8. The aim of this exercise is to assess the effectiveness of the adaptive refinement strategy that is built into T-IFISS by looking at a problem with a singular solution. The test problem can be set up in T-IFISS by setting `tolerance=5e-3` and then running the driver `Run8DigitChallenge`.

   You should discover that the adaptive algorithm converges in 15 steps and that the number of vertices (degrees of freedom) on the final mesh is 1901. Save the plots of the refinement path (Figure 3) and of the final mesh (Figure 2). (Hint: use the command `savefig`.)

   Next, run the driver `ell_adiff` with linear approximation with grid parameter set to `5,6,7` and `8` so as to estimate the order of convergence of the uniform grid finite element approximation in the energy norm (as a power of h). You might like to add this data to the previously saved plot to facilitate a direct comparison. (The number of vertices is given by `length(x_gal)` and the energy error estimate is given by `err_p`.)

   Finally, repeat the experiment, this time by running `ell_adiff` using quadratic approximation with the grid parameter set to `4,5,6` and `7`. (The number of degrees of freedom is given by `length(x_gal)` and the energy error estimate is given by `norm(elerr_p_p4)`.) The key point here is that while the uniform grid quadratic approximation is better than the linear approximation the order of convergence is exactly the same. Why is this behaviour to be expected?

9. Write a MATLAB function that calls `eigs` and generates an estimate of the three smallest eigenvalues and associated eigenfunctions of the negative Laplacian operator on the L-shaped domain featured in the previous exercise. (Hint: look at question 3 on Tutorial sheet 3 first.) Use your function to generate a table of estimated eigenvalues for a set of three increasingly fine meshes using linear approximation. Generate plots of the corresponding eigenfunctions computed on the finest mesh.

Any handwritten submission should include your student registration number and must be submitted electronically (*via Blackboard*) before 2pm on Thursday 15th April 2021. All work submitted *must be your own*.

Electronic submissions should consist of a single file (in .pdf format). This file should contain a concise discussion, tables of numbers, graphical output and a list of the MATLAB commands used to generate results on the finest subdivision.

I expect to return your mark (broken down question by question) by sending you an email before 11pm on Wednesday 21st April. The overall mark will be visible in the mark centre on Blackboard soon after that.