This homework contributes 20% of the overall assessment of the course.
Each problem is worth **2** marks and computational exercises are worth **3** marks.

Given an open bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ consisting of two nonoverlapping pieces $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, let $u : \Omega \to \mathbb{R}$ be the solution of the boundary value problem:

$$\left.\begin{aligned} -\nabla^2 u = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega_D, \quad \frac{\partial u}{\partial n} + u = 1 \text{ on } \partial\Omega_N. \end{aligned}\right\} \quad (D)$$

You can assume that $\partial\Omega_N$ has nonzero length so that the *Robin* boundary condition holds on *some part* of the boundary. You may use the Cauchy–Schwarz inequality or the Poincaré–Friedrichs inequality in answering any of the following questions without giving a proof.

1. Explain what is meant by a *classical solution* of (D). What is the physical relevance of the boundary condition on $\partial\Omega_N$?

2. Given the test space $X := \left\{v | v \in \mathcal{H}^1(\Omega), v = 0 \text{ on } \partial\Omega_D\right\}$, where $\mathcal{H}^1(\Omega)$ is the standard Sobolev space, show that u solving (D) also satisfies the variational formulation: find $u \in X$ such that

$$\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega_N} uv\, ds = \int_{\partial\Omega_N} v\, ds \qquad \forall v \in X. \qquad (\star)$$

   Define the Galerkin approximation $u_h \in X_h = \text{span}\{\phi_j\}_{j=1}^k \subset X$ and show that the approximation method leads to a $k \times k$ matrix system

$$A\,\mathbf{x} = \mathbf{f}.$$

   Identify explicitly the entries $A_{i\,j}$ of the matrix A and f_i of the vector $\mathbf{f}$.

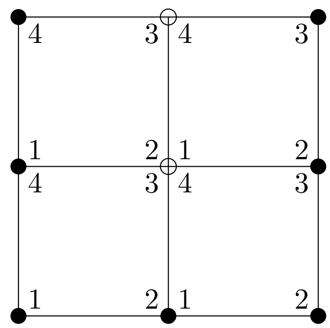
3. Discuss whether or not a solution of $(\star)$ is *unique* in the special case of $\partial\Omega = \partial\Omega_N$.

4. Prove that the Galerkin solution u_h is the *best approximation* to $u \in X$ when measured in the energy norm $\|u\|_E$, that is

$$\|u - u_h\|_E \le \|u - v_h\|_E \quad \forall v_h \in X_h,$$

   where $\|u\|_E^2 := \int_\Omega \nabla u \cdot \nabla u + \int_{\partial\Omega_N} u^2\, ds$.

Suppose that Ω is the square domain $(-1,1)\times(-1,1)$ and that $\partial\Omega_N$ is the top boundary: that is $y=1$ with $-1<x<1$. Suppose further that $u_h \in X_h$ is the piecewise bilinear approximation to u satisfying $(\star)$ that is associated with the uniform grid of square elements with $h=1$ shown below.



5. Consider a general square element $\square_k$, with edge length h with nodal basis functions numbered anticlockwise (as shown). Show that the Jacobian matrix associated with the mapping to $\square_k$ from the reference element $\square_\star = [-1,1]\times[-1,1]$ is the diagonal matrix

$$J_k = \frac{1}{2}\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}.$$

By mapping the derivatives of the reference element basis functions, show that the 4×4 element matrix associated with the negative Laplacian operator is given by

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{bmatrix}.$$

6. Assemble the entries in the 2×2 Galerkin system associated with solving $(\star)$ using the grid shown above. Then solve the linear equation system (for example, using `matlab`) to generate the bilinear finite element solution.

7. Suppose that X_h corresponds to a piecewise bilinear finite element approximation space and let $\pi_h u$ represent the piecewise bilinear interpolant of $u \in X$. Given that the following interpolation error bounds hold,

$$\|\nabla(u - \pi_h u)\|_{L^2(\Box_k)} \le C_1 h_k, \quad \|u - \pi_h u\|_{L^2(E_k)} \le C_2 h_k, \tag{‡}$$

where C_1 and C_2 are constants, h_k is the longest edge of element $\Box_k$ and E_k is any edge of $\Box_k$ that lies on $\partial\Omega$, show that the finite element approximation converges,

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \to 0 \text{ as } h \to 0,$$

where $h = \max_k h_k$. Note that you do not need to establish (‡).

Computational Exercises. The T-IFISS software package offers a choice of two-dimensional domains on which anisotropic diffusion problems can be posed, along with boundary conditions and a choice of finite element approximation on a structured or unstructured triangular mesh.

8. We want to investigate Example 1.1 in [ESW] with a typical solution illustrated in Figure 1.1 in [ESW]. By running the driver `diff_testproblem` and choosing problem 1, tabulate the error estimate η that is generated using linear approximation on a sequence of uniform 16×16, 32×32 and 64×64 triangular meshes. From these, estimate the order of convergence of the finite element approximation in the energy norm. Then, repeat the experiment using quadratic approximation. You should find that the experimental order of convergence is increased.

   One way of estimating the exact energy error is to compute a *reference solution* using a fine grid and then to substitute it into the error representation formula $\|\nabla(u - u_h)\|^2_{L^2(\Omega)} = \|\nabla u\|^2_{L^2(\Omega)} - \|\nabla u_h\|^2_{L^2(\Omega)}$. Apply this strategy to assess the quality of the error estimate η by repeating the computations made earlier and comparing with a reference quadratic solution computed on a 128×128 grid.

9. The aim of this exercise is to assess the effectiveness of the adaptive refinement strategy that is built into T-IFISS by running Example 1.4 in [ESW]. The analytic test problem can be set up in T-IFISS by running the driver `diff_testproblem`, choosing problem 5, and editing the generated data files `specific_rhs` (you need to set f to zero), `specific_adiff` (you need to set k_x and k_y to one) and `specific_gradcoeff` (you need to set the coefficient derivatives to zero).

   Next, set `sn=5` and `dom_type=2`. Run `adiff_adaptive_main` with default parameters and with the error tolerance set to `1e-2`. You should discover that the adaptive algorithm converges in 21 steps and that the number of vertices (degrees of freedom) on the final mesh is 8030. Save the plots of the refinement path (Figure 3) and of the final mesh (Figure 2). (Hint: use the command `savefig`.)

Next, run the driver `ell_adiff` with linear approximation with grid parameter set to `5,6,7` and `8` so as to estimate the order of convergence of the finite element approximation in the energy norm. You might like to add this data to the previously saved plot to facilitate a direct comparison.

Finally, repeat the experiment, this time by running `ell_adiff` using quadratic approximation with the grid parameter set to `4,5,6` and `7` . You should find that the experimental order of convergence is not improved!

Any handwritten submission should include your student registration number and must be handed in before 10am on Thursday 2nd May 2019. All work submitted *must be your own*.

Solutions to the computational exercises should be submitted by sending an email to `d.silvester@manchester.ac.uk` before the deadline. Note that only ONE electronic submission per person is allowed. The subject field of the email must include your student registration number. Electronic submissions should consist of a single file (in .pdf format). This file should be prepared using LaTeX or Word and contain a concise discussion, tables of numbers, graphical output and a list of the MATLAB commands used to generate results on the finest subdivision.

I will acknowledge receipt of all such electronic submissions and I will return your mark by sending you an email before 11pm on Sunday 5th May.