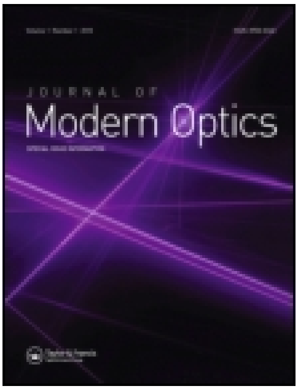

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Visual Pattern Recognition by Assignment of Invariant Features and Feature-relations

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Visual pattern recognition by assignment of invariant features and feature-relations

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Abstract. The capacity of the visual system to recognize patterns that have undergone various transformations may be explained by a scheme in which the system associates with each pattern certain features and feature-relations that remain constant under the action of the given transformations. Some of the characteristics of such a recognition scheme are examined and an approach is proposed for the analysis of the pattern structure determined by these feature and feature-relation assignments. Data are presented relating to the stability of these assignments under changes in recognition criteria.

1. Introduction

A large class of models for visual pattern recognition involve the representation of the stimulus in terms of properties or features which remain constant under the action of certain admissible transformations, for example, translations, rotations, and dilatations. In such models, two patterns are recognized as being 'the same' if their feature representations coincide. Alternatively, if recognition is regarded as the assignment of a pattern to a specified class, then a pattern is recognized if its feature representation fits the class specification.

One of the simplest methods for the visual extraction of invariant features of a stimulus was proposed by Pitts and McCulloch [1]. The procedure is that of 'averaging over the group'; that is, for a finite collection G of transformations τ forming a group under composition, an invariant a of an object A is obtained thus:

$$a = \frac{1}{N} \sum_{\tau \in G} f(\tau(A)),$$

where N is the number of members of G, and f is a suitable function. Some models (e.g. those by Deutsch [2], Sutherland [3] and Dodwell [4, 5]) have made use of 'encoding' procedures to explain recognition (see also [6]). For example, in the model by Sutherland [3], patterns are projected onto a rectangular array and then processed in terms of their horizontal and vertical components. A general method for feature abstraction based on autocorrelation has been developed by Moore and Parker [7] and Moore *et al.* [8]. Amongst other pattern properties the characteristic distributions of chord lengths and angles are computed. Other recognition schemes have made use of techniques similar to those employed in optical pattern recognition (for review of the latter techniques, see [9]). For

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example, the work of Campbell and Robson [10], Blakemore and Campbell [11], and others has led to the suggestion that patterns are subjected to a Fourier analysis by a system of linear mechanisms, each selectively sensitive to a different band of spatial frequencies (but see [12] and [13]).

Methods in which essentially global invariants are computed have not, however, been successful in predicting precisely all the ' constancies ' of recognition shown by the human and animal visual systems. In particular, the classical procedures [14–16] involving transformations like the Fourier transformation, Hadamard transformation, and Karhunen–Loeve transformation, give rise to representations of the stimulus space in which the canonically associated distance functions fail to generate ' natural ' object classifications [17].

An alternative to pattern analysis based on the abstraction of global properties is analysis based on the abstraction of local properties (local in the sense that these properties do not depend on the structure of the whole pattern). The role of such properties (or ' critical features ') in letter discrimination has been investigated by Gibson *et al.* [18] and Gibson *et al.* [19] (see [20]). The features they considered included horizontal, vertical and oblique straight lines and vertically open, horizontally open and closed curves. (Blesser *et al.* [21] have instead proposed that letters are identified by certain underlying ' functional attributes ', which are strongly context-dependent. A different transformational approach to letter and word recognition has also been described by Kolers and Perkins [22–24].) More generally, units for detecting features have been incorporated into complex hierarchical models for visual information processing [25, 26] (see also [20]). Barlow *et al.* [27] have reviewed some of the physiological evidence for the existence of feature detectors at the local level.

Although pattern representation in terms of features is suitable for much classificatory recognition, such a system is not appropriate for the specification or comparison of patterns in which the spatial arrangement of features is important. The approach to recognition in which various kinds of relations are set up between features is usually referred to as the structural or syntactic approach [28–30]. For example, with the directed lines shown in figure 1 (i) as local features or *primitives* and ' head-to-tail concatenation ' as the only relation, the triangle in figure 1 (ii) would be represented as *abc* [30]. Other relations, such as ' right ', ' above ', and ' inside ', enable the generation of more complicated objects from suitable pattern primitives [31]. Structural schemes of this kind for visual

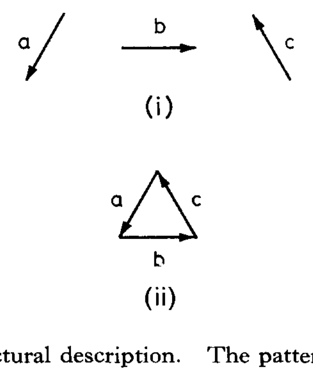

Figure 1. Example of a structural description. The pattern primitives (i) are composed under the relation 'head-to-tail concatenation' to yield the triangle (ii). Based on an example by Fu [30].

pattern recognition have been described by, amongst others, Sutherland [32], Leeuwenberg [33, 34] and Reed [35].

Little is known, however, of the exact nature of the higher-order relations that might be used by the visual system in establishing a structural description of a scene [36] (see also [27]). It seems unlikely that any of the syntax analysis or parsing procedures developed for automatic recognition [30] can be used to investigate these relations, although, if some relations can be evaluated, further examples might be generated by algebraic procedures [37].

The present study is concerned with some of the functional properties displayed by recognition systems based on feature and feature-relation assignment. An approach is described for the analysis of the pattern structure determined by such systems and data presented relating to the stability of this pattern structure under changes in recognition criteria.

2. Generalized pattern recognition

The sense in which two objects are identified as being 'the same' is not given a consistent interpretation in the literature on visual recognition (for reviews, see [38–40]). The two most common identification criteria used in studies involving the comparisons of patterns are (i) if one pattern can be obtained from the other by a planar rotation (e.g. [41, 42]) and (ii) if one pattern can be obtained from the other by a rotation in depth (e.g. [43, 44]). In studies of pattern classification, however, quite general non-linear transformations of the stimuli are permitted (see [40]). The formulation of a definition for 'sameness' clearly requires some scheme for the specification of an admissible set of transformations. In this section a generalized form of visual recognition is introduced based on the notion of a mathematical structure. The main result is that the collection of admissible transformations associated with such a structure constitutes a group under composition of mappings.

We first establish some notations and definitions. An object or pattern is simply a particular luminance distribution either in ordinary Euclidean 3-space R^3 (R denotes the real line) or in 2-space R^2. We shall not be explicitly concerned with stereoscopic vision, so for simplicity we shall work with the plane R^2, which may be assumed fixed and perpendicular to the line of gaze. A pattern A is then a mapping (also denoted by A) of a non-empty subset U_A of R^2 into R such that $A(p) \geqslant 0$ is the luminance of the object at the point $p \in U_A$. The set U_A is sometimes called the *domain* of the object. (If colour vision is relevant, then $A(p)$ is a point in R^3, the coordinates specifying the 'red', 'green' and 'blue' components of the stimulus at the point p.) The remainder of the plane may be supposed to have a uniform luminance distribution defined on it.

The action on a pattern of any suitable locally defined one-to-one (*injective*) transformation τ of R^2 into R^2 is defined thus:

$$(\tau(A))(p) = A(\tau^{-1}(p)) \quad \text{for all} \quad p \in \tau(U_A).$$

Each point p in the domain of the transformed image $\tau(A)$ is assigned the luminance at its pre-image.

It is possible to endow R^2, as a point set, with various mathematical structures; for example, (i) the usual metric structure given by the distance function

$$d(x, y) = \sqrt{[(x_1 - y_1)^2 + (x_2 - y_2)^2]},$$

where $x=(x_1, x_2)$ and $y=(y_1, y_2)$, or (ii) the usual topological structure generated by the open discs $B(x, a)$, where, for $x \in R^2$ and $a \in R$, $B(x, a)$ consists of all points y in R^2 such that $d(x, y) < a$. In general, a mathematical structure on a set may be thought of as a certain collection of various-order relations and scalar-valued functions defined on that set. Axiomatizations of the notion of structure have been given by Bourbaki [45] and by Hedrlín *et al.* [46], and Pultr [47]. The Bourbaki theory is expressed within the theory of sets, whereas Hedrlín *et al.* use the category theory approach. The essential equivalence of the two systems has, however, been proved by Blanchard [48].

Given a mathematical structure S on R^2, a one-to-one and onto mapping (a *bijection*) τ of R^2 into R^2 that ' preserves ' the structure S is said to be an *isomorphism* (strictly an *automorphism*) of R^2 endowed with S [45]. It may be shown that the inverse τ^{-1} is also an isomorphism of R^2 endowed with S [45]. For example, the isomorphisms of the usual metric structure on R^2 (the *isometries*) preserve distances, i.e. $d(\tau(x), \tau(y))=d(x, y)$, and the isomorphisms of the usual topological structure on R^2 (the *homeomorphisms*) preserve the open sets, i.e. $\tau(U)$ is open if (and only if) U is open. The set of isomorphisms of a structure S is denoted by Iso(S). For a given structure S on R^2, the composition of isomorphisms of S is always defined. Under this composition the set of all isomorphisms of S thus forms a group.

Provided certain conditions are satisfied, a structure on R^2 may give rise to an induced structure on a subset of R^2, in particular, on the domain of an object. For example, an object A acquires a metric structure by restriction of the distance function to U_A or a topological structure by intersection of the open sets with U_A. The isomorphisms of structures obtained by restriction are called *local isomorphisms*. A structure assigned to a subset of R^2 need not, however, always arise in this way.

Given a structure S on R^2 that gives induced structures on the domains of a suitable class of objects, we can then define visual recognition with respect to S in the following way [49]. (All objects, mappings, etc. defined on R^2 should be understood to be specified only to within visual acuity [50].) For objects A and B on R^2, visual recognition of A as B, with respect to S, is the visual association of A and B by means of a local isomorphism of S. Suppose object B may be written as $\tau(A)$ for some local isomorphism τ of S. Experimentally, recognition or identification of A and B with respect to S is then inferred if B can be visually distinguished from other objects which cannot be written as $\tau(A)$. Thus, if S is the usual metric structure on R^2, recognition of A as B with respect to S is judged to be effected if B may be visually determined as being the result of the application to A of (the restriction of) a translation, rotation or reflection. The simplest case is when S is the *structure of a set* [45], characterized by the number of elements in the set or the *cardinality* of the set. Recognition of A as B with respect to S is then judged to be effected if B may be visually determined as being the result of the application to A of a (local) bijection. The two patterns are judged to be ' the same ' in this second case if they have the same number of points.

The above definition of recognition applies to the simultaneous or sequential comparison of stimulus objects. It may be extended to the classificatory situation providing there is a representative or ' prototype ' (see [40]) available for the class. Because the set of admissible transformations forms a group, the class of patterns formed by application of these transformations to the prototype must be an equivalence class. Note that although recognition is defined in terms of admissible transformations τ this does not necessarily mean that these associations are set up by continuous 1-parameter families of transformations ψ_t, $0 \leqslant t \leqslant T$, for which $\psi_0 =$ identity and $\psi_T = \tau$. For example, a rotation ρ_θ through θ need not be effected by application of $\rho_{t\theta}$ with t going from 0 to 1. (Dynamical schemes for recognition involving families of transformations, both continuous and discrete, have been described by Pitts and McCulloch [1], Marko [17], and Kolers and Perkins [24].)

3. Structure associated with a set of features and feature-relations

Suppose that there is a certain repertoire of features and feature-relations that can be drawn upon by the visual system in setting up a representation of a stimulus object. Thus, according to some scheme F, a pattern A is given the representation $F(A)$ consisting of the features $f_1(A)$, ..., $f_m(A)$ and relations $r_1(f_1(A), \ldots, f_m(A)), \ldots, r_n(f_1(A), \ldots, f_m(A))$. Let this representation correspond to some structure S_0. If the visual recognition of two objects A and B is achieved by the equating of their representations $F(A)$ and $F(B)$, then the experimental determination of S_0 is straightforward. One selects relevant structures S, and for each S one determines the set of isomorphisms τ of S that may be visually effected by virtue of the visual identification with respect to S of patterns A and $\tau(A)$. Let M be the union of these sets of isomorphisms. Since the visual system cannot set up associations which do not belong to the set $\text{Iso}(S_0)$ of isomorphisms of S_0, and the selection of reference structures S is, we suppose, exhaustive, the union M must coincide with $\text{Iso}(S_0)$. It does not necessarily follow that S_0 is determined uniquely by the set M, for different structures may give rise to the same sets of isomorphisms. Nevertheless, it may be argued that there is no need to distinguish between structures which operationally give the same recognition characteristics. This redundancy in structure specification can be resolved with the introduction of an equivalence relation $\sim$ on the set of all structures thus:

$$S_1 \sim S_2 \quad \text{if and only if} \quad \text{Iso}(S_1) = \text{Iso}(S_2).$$

In the following we shall assume that the set of all structures has in this way been ' reduced '. Given a family of mappings, the actual construction of a structure S, such that the set of isomorphisms $\text{Iso}(S)$ of S coincides with the given family, has been examined by Hedrlín and Pultr [51], Pultr and Hedrlín [52], and Pultr [53].

It seems unlikely, however, that the visual recognition of two objects A and B is indeed achieved by the precise equating of their representations $F(A)$ and $F(B)$. This follows from the fact that for certain structures on R^2, it is possible to visually identify an object A with its transform $\tau(A)$ for an isomorphism τ ' close ' to the identity, although such an identification may not be possible for an isomorphism ' far ' from the identity. (There are several ways in which metrics may be defined on spaces of mappings [54].) Consider the case of the standard metric structure. Patterns may be recognized under small rotations but not under large rotations [41, 42]. But if the representations of a pattern A and its rotated transform $\rho_\theta(A)$ coincide for small θ, i.e.

$$F(A) = F(\rho_\theta(A)),$$

then they must coincide for large θ, since by iteration

$$F(A) = F(\rho_\theta{}^n(A)),$$

where $\rho_\theta{}^n = \rho_\theta \circ \rho_\theta \circ \ldots \circ \rho_\theta$ (n times), and $\rho_\theta{}^n = \rho_{n\theta}$.

A consequence of this is that there may be relatively few true ' perceptual invariants ', i.e. visual quantities that remain exactly the same independent of the transformations applied (see [38]).

In another study by the author [49], it is assumed that the visual system might assign an underlying structure to stimulus objects, although not necessarily be capable of effecting all the isomorphisms of that structure in visual recognition. This underlying structure, corresponding here to S_0, would be the structure with the smallest set of isomorphisms that includes the experimentally determined set M defined above. Formally, one structure S_1 is said to be richer than another S_2, written $S_1 \prec S_2$, if Iso(S_1) is a subset of Iso(S_2). (The relation $\prec$ is a partial order on the set of all structures.) Thus, S_0 would be the richest structure such that Iso(S_0) includes M. This approach is inappropriate here because of the method by which the visual recognition of patterns is assumed to be effected. Given an arbitrary isomorphism τ of S_0, any collection of features and relations on an object A which serves to define S_0 (for example, the distances between pairs of points giving a metric structure) must be precisely the same on the transformed object $\tau(A)$. But the coincidence of the representations $F(A)$ and $F(\tau(A))$ is equivalent to the transformation τ being such that it may be visually effected. The visual system would consequently be capable of setting up *all* the isomorphisms of S_0, in contradiction to the hypothesis.

An alternative approach to the problem is to suppose that the coincidence of the representations $F(A)$ and $F(\tau(A))$ is only ' approximate ', with each determining a different structure of the same species. For example, in figure 2, the set of points a, b, c, d, e, f, g (which might correspond to particular local features) has been endowed with two different partial order structures (i) and (ii). The two structures are ' close ', in that the identity mapping of (i) into (ii) leaves almost all the relations between the elements intact. If $\approx$ represents a visually

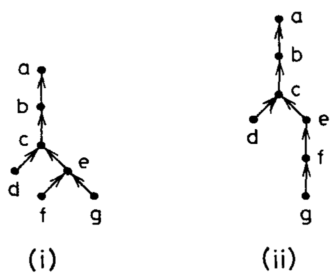

Figure 2. Two partial order relations on the sets of elements a–g. In each case, the relation 'x is greater than y' holds if and only if there is an unbroken succession of arrows running from the point y to the point x.

acceptable level of concurrence of pattern representations, then $F(A) \approx F(B)$ and $F(B) \approx F(C)$ does not necessarily imply $F(A) \approx F(C)$; that is, transitivity of recognition is not a property of this approximation scheme. Thus, for recognition with respect to the standard metric structure on R^2, pattern A may be identified with pattern $\rho_\theta(A)$, and pattern $\rho_\theta(A)$ with pattern $\rho_{2\theta}(A)$, but not pattern A with pattern $\rho_{2\theta}(A)$. Theories of such 'fuzzy' relationships have been constructed by Zeeman [50] and Zadeh [55] and subsequently developed by Poston [56], Arbib [57], Zadeh [58–60], Tamura *et al.* [61], Goguen [62], and others (see [63]). The actual method of visual assignment and comparison of pattern structures may follow a probabilistic scheme, such as that proposed by Vitz and Todd [64] for the analysis of geometrical figures. In this scheme the probability of sampling a particular element at a given level depends on the relative size of that element. For example, the probability of sampling a line depends on the length of the line in relation to other lines in the figure.

The collection M of all isomorphisms that can be visually effected may still be used to determine the representation structure S_0 in the case in which the comparison of patterns is 'fuzzy'. The procedure is the opposite of that previously alluded to in [49]. Instead of determining the richest structure S for which Iso (S) includes the set M, one determines the poorest structure S for which the set M includes Iso(S). (Recall that the poorer the structure, the more isomorphisms it has.) The reason is as follows. For a pattern A, the features and relations $F(A)$ assigned to A define the structure S_0. Hence, if τ is an isomorphism of S_0, the representation $F(\tau(A))$ coincides with $F(A)$. But because the comparison of the representations $F(A)$ and $F(\tau(A))$ is only approximate, it is possible to replace the isomorphism τ by a close non-isomorphism τ', so $F(A) \approx F(\tau'(A))$. What operationally distinguishes the isomorphisms τ from the non-isomorphisms τ' is that the former constitute a group, since they arise from the structure S_0, whereas the latter do not. The poorest structure S whose isomorphisms belong to M precisely defines this group, and, up to the equivalence defined earlier, coincides with S_0.

4. An experiment on visual recognition with respect to two different structures

An essential requirement of the investigative scheme outlined at the end of the previous section is that the representation of a pattern formed by the visual system is stable and independent of the particular structure with respect to which recognition is evaluated. That is, if pattern A is compared with pattern B, first with respect to structure S_1 and then with respect to structure S_2, the features and feature-relations $F(A)$ and $F(B)$ assigned to the two should be the same for S_1 and S_2. (It is supposed that the task is one involving perceptual not cognitive processes, see [65].) More specifically, if $B = \tau(A)$, where τ is an isomorphism both with respect to S_1 and with respect to S_2, then A and B should be visually identifiable with respect to S_1 if and only if they are visually identifiable with respect to S_2.

This hypothesis was tested experimentally for the case in which S_1 is the usual metric structure on R^2 and S_2 is the structure of a set†. Since S_2 is the poorest of all structures (every bijection is an isomorphism), every isomorphism of S_1 is an isomorphism of S_2. Hence, for pattern pairs $A, \tau(A)$, with τ an isometry,

† A full account of this experiment by the author is to be published elsewhere.

the effect of systematically varying τ on the visual identification of A and $\tau(A)$ determined with respect to the metric structure should be the same as that on the visual identification of A and $\tau(A)$ determined with respect to the structure of a set. Subjects (48 in all) were presented with different pairs of random-dot patterns like those in figure 3. Half the subjects were required to indicate whether one pattern could be obtained from the other by some combination of

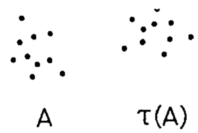

Figure 3. Example of random-dot patterns A, $\tau(A)$ used in recognition experiment.

translation, rotation, and reflection; the other half were required to indicate whether the two patterns had the same number of dots. (The 200-msec stimulus exposure times were too short to permit counting.) The same stimulus figures were used for each group of subjects. Figure 4 shows results for the case in which $\tau = \rho_\theta$, a rotation in the plane about angle θ (with a fixed translation). The proportion of correctly identified pairs is plotted against θ. The variation in recognition performance determined with respect to the structure of a set (continuous line) is seen to be very similar to the variation in recognition performance determined with respect to the metric structure (broken line). (Differences between the two sets of data, measured by procedures given in [66], are not significant at the 10 per cent significance level.) The reduction in ' shape ' identification with rotation is well known [38].

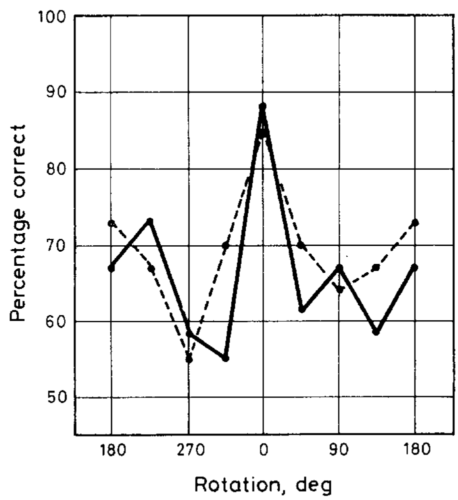

Figure 4. Recognition performance determined for two different recognition criteria. Percentage of correctly identified pattern pairs is plotted against angle of pattern rotation. Data indicated by continuous line correspond to recognition evaluated with respect to the structure of a set ('number of elements') and data shown by broken line correspond to recognition evaluated with respect to metric structure ('distance between points'). Number of readings per point is 33.

5. Conclusion

Feature-extraction theories are applicable to the analysis of visual classificatory recognition when a relatively small number of pattern classes are involved [67–71]. For general pattern recognition, however, it has been suggested by Sutherland [32, 36] and others that a structural approach involving the assignment to a pattern of various features and feature-relations offers the most comprehensive description. Nevertheless, such a system is unlikely to be of the strictly linguistic kind, in which patterns are generated according to a system of production rules [30]. As Kolers and Perkins [24] have stressed, many naturally occurring objects and pictures have no underlying regulations governing their detailed composition.

In the present study some of the characteristics of a structural theory of recognition have been examined. It has been shown that the visual comparison of a pattern A with some transformed pattern $\tau(A)$, in terms of structural representations, may be effected on a ' fuzzy ' basis, with recognition not necessarily resulting from the assignment of features and feature-relations which are strictly invariant under the action of the transformation τ. A procedure for the analysis of the structure determined by these features and feature-relation assignments has been suggested here. Such a procedure requires that these assignments remain stable under changes in externally imposed recognition criteria. For two such criteria, namely, recognition with respect to a metric structure, where distance is to be preserved, and recognition with respect to the structure of a set, where the number of elements is to be preserved, the data presented show that these assignments are indeed stable.

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References

[1] Pitts, W., and McCulloch, W. S., 1947, *Bull. math. Biophys.*, **9**, 127.
[2] Deutsch, J. A., 1955, *Br. J. Psychol.*, **46**, 30.
[3] Sutherland, N. S., 1957, *Nature, Lond.*, **179**, 11.
[4] Dodwell, P. C., 1957, *Br. J. Psychol.*, **48**, 221.
[5] Dodwell, P. C., 1964, *Psychol. Rev.*, **71**, 148.
[6] Dodwell, P. C., 1970, *Visual Pattern Recognition* (New York: Holt).
[7] Moore, D. J. H., and Parker, D. J., 1974, *Pattern Recognition*, **6**, 149.
[8] Moore, D. J. H., Seidl, R. A., and Parker, D. J., 1975, *Int. J. Man-Machine Studies*, **7**, 449.
[9] Ullman, J. R., 1974, *Opto-electronics*, **6**, 319.
[10] Campbell, F. W., and Robson, J. G., 1968, *J. Physiol.*, **197**, 551.
[11] Blakemore, C., and Campbell, F. W., 1969, *J. Physiol.*, **203**, 237.
[12] Nachmias, J., Sansbury, R., Vassilev, A., and Weber, A., 1973, *Vision Res.*, **13**, 1335.
[13] Legéndy, C. R., 1975, *Biol. Cybernet.*, **17**, 157.
[14] Nagy, G., 1968, *Proc. I.E.E.E.*, **56**, 836.
[15] Andrews, H. C., 1972, *Introduction to Mathematical Techniques in Pattern Recognition* (New York: Wiley–Interscience).
[16] Kittler, J., 1975, *Int. J. Man-Machine Studies*, **7**, 609.
[17] Marko, H., 1973, *Kybernetik*, **13**, 132.
[18] Gibson, E. J., Osser, H., Schiff, W., and Smith, J., 1963, *Final Report on Cooperative Research, Project No.* 639, (U.S. Office of Education).

[19] GIBSON, E. J., SCHAPIRO, F., and YONAS, A., 1968, *Final Report, Project No.* 5–1213, (Cornell University and U.S.O.E.), p. 76.
[20] GIBSON, E. J., 1969, *Principles of Perceptual Learning and Development* (New York: Appleton).
[21] BLESSER, B., SHILLMAN, R., COX, C., KUKLINSKI, T., VENTURA, J., and EDEN, M., 1973, *Visible Language*, **7,** 209.
[22] KOLERS, P. A., and PERKINS, D. N., 1969, *Perception Psychophys.*, **5,** 265.
[23] KOLERS, P. A., and PERKINS, D. N., 1969, *Perception Psychophys.*, **5,** 275.
[24] KOLERS, P. A., and PERKINS, D. N., 1975, *Cognitive Psychol.*, **7,** 228.
[25] SELFRIDGE, O., and NEISSER, U., 1960, *Scient. Am.*, **203,** 60.
[26] NEISSER, U., 1967, *Cognitive Psychology* (New York: Appleton).
[27] BARLOW, H. B., NARASIMHAN, R., and ROSENFELD, A., 1972, *Science, N.Y.*, **177,** 567.
[28] NARASIMHAN, R., 1962, Rep. 121, Digital Computer Laboratory, University of Illinois, Urbana.
[29] CLOWES, M. B., 1969, *Automatic Interpretation and Classification of Images*, edited by A. Grasselli (London: Academic Press), p. 43.
[30] FU, K. S., 1974, *Syntactic Methods in Pattern Recognition* (New York: Academic Press).
[31] LEDLEY, R. S., 1962, *Programming and Utilizing Digital Computers* (New York: McGraw-Hill).
[32] SUTHERLAND, N. S., 1968, *Proc. R. Soc.* B, **171,** 297.
[33] LEEUWENBERG, E. L. J., 1968, *Structural Information of Visual Patterns* (The Hague: Mouton).
[34] LEEUWENBERG, E. L. J., 1971, *Am. J. Psychol.*, **84,** 307.
[35] REED, S. K., 1971, *Structural Models and the Encoding of Line Patterns* (Laboratory of Experimental Psychology, University of Sussex).
[36] SUTHERLAND, N. S., 1973, *Handbook of Perception*, edited by E. C. Carterette and M. P. Friedman (New York: Academic Press), p. 157.
[37] KULIKOWSKI, J., 1972, *Algebraic Methods in Pattern Recognition* (Vienna: Springer).
[38] HAKE, H. W., 1966, *Pattern Recognition*, edited by L. Uhr (New York: Wiley), p. 142.
[39] ZUSNE, L., 1970, *Visual Perception of Form* (New York: Academic Press).
[40] REED, S. K., 1973, *Psychological Processes in Pattern Recognition* (New York: Academic Press).
[41] DEARBORN, G. V. N., 1899, *Psychol. Rev.*, **6,** 395.
[42] ARNOULT, M. D., 1954, *J. exp. Psychol.*, **47,** 323.
[43] SHEPARD, R. N., and METZLER, J., 1971, *Science, N.Y.*, **171,** 701.
[44] MASSARO, D. W., 1973, *Perception Psychophys.*, **13,** 413.
[45] BOURBAKI, N., 1968, *Theory of Sets* (Paris: Hermann; Reading, Mass.: Addison-Wesley), Chap. 4.
[46] HEDRLÍN, Z., PULTR, A., and TRNKOVA, V., 1967, *General Topology and its Relations to Modern Analysis and Algebra, II, Proc. Second Prague Topological Symposium*, 1966 (Prague: Academia).
[47] PULTR, A., 1968, *Commentat. math. Univ. Carol.*, **9,** 281.
[48] BLANCHARD, A. A., 1974, *Can. J. Math.*, **26,** 1217.
[49] FOSTER, D. H., 1975, *Biol. Cybernet.*, **17,** 77.
[50] ZEEMAN, E. C., 1962, *The Topology of 3-Manifolds and Related Topics*, edited by M. K. Fort, Jr. (New Jersey: Prentice-Hall), p. 240.
[51] HEDRLÍN, Z., and PULTR, A., 1964, *Mh. Math.*, **68,** 213.
[52] PULTR, A., and HEDRLÍN, Z., 1964, *Mh. Math.*, **68,** 421.
[53] PULTR, A., 1967, *Commentat. math. Univ. Carol.*, **8,** 53.
[54] EELLS, J., 1966, *Bull. Am. math. Soc.*, **72,** 751.
[55] ZADEH, L. A., 1965, *Inf. Control*, **8,** 338.
[56] POSTON, T., 1971, *Fuzzy Geometry* (Institute of Mathematics, University of Warwick).
[57] ARBIB, M. A., 1967, *Kybernetik.*, **3,** 223.
[58] ZADEH, L. A., 1968, *Inf. Control*, **12,** 99.
[59] ZADEH, L. A., 1970, *E.R.L. memo M*277 (Electrical Research Laboratory, University of California, Berkeley).
[60] ZADEH, L. A., 1974, *E.R.L. memo M*479 (University of California, Berkeley).
[61] TAMURA, S., HIGUCHI, S., and TANAKA, K., 1971, *I.E.E.E. Trans. Syst. Sci. Man Cybernet.*, **1,** 61.

[62] GOGUEN, J. A., 1974, *Int. J. Man–Machine Studies*, **6,** 513.
[63] KAUFMANN, A., 1975, *Theory of Fuzzy Subsets* (New York: Academic Press).
[64] VITZ, P. C., and TODD, T. C., 1971, *Psychol. Rev.*, **78,** 207.
[65] JULESZ, B., 1975, *Scient. Amer.*, **232,** 34.
[66] COX, D. R., 1970, *The Analysis of Binary Data* (London: Methuen), §§ 5.3, 6.3.
[67] RABBITT, P. M., 1967, *Am. J. Psychol.*, **80,** 1.
[68] YONAS, A., 1969, *The Acquisition of Information Processing Strategies in a Time-Dependent Task* (Cornell University).
[69] MAVRIDES, C. M., and BROWN, D. R., 1969, *Perception Psychophys.*, **6,** 276.
[70] AIKEN, L. S., and BROWN, D. R., 1971, *Perception Psychophys.*, **9,** 145.
[71] AIKEN, L. S., and BROWN, D. R., 1971, *Perception Psychophys.*, **9,** 279.