## Turbulence Modelling in STREAM

### 1. List of Turbulence Models

| Model | Code | Low-Re? |
|---|---|---|
| **One-equation models** | | |
| Wolfshtein (1969) | WO | ✓ |
| Norris and Reynolds (1975) | NR | ✓ |
| **Linear *k*-ε models** | | |
| Launder and Spalding (1974) – Standard *k*-ε | HR | × |
| Yakhot et al. (1992) – RNG *k*-ε | RG | × |
| Shih, Liou et al. – realisable model (1995) | RE | × |
| Launder and Sharma (1974) | LS | ✓ |
| Lam and Bremhorst (1981) | LB | ✓ |
| Chien (1982) | CH | ✓ |
| Lien and Leschziner (1993) | LL | ✓ |
| **Linear k-ω models** | | |
| Wilcox (1988) | WX | ✓ |
| Wilcox (1994) | W2 | ✓ |
| Menter (1994) – Baseline model | BL | ✓ |
| Menter (1994) – SST model | FM | ✓ |
| **Non-linear k-ε models** | | |
| Speziale (1987) | SP | × |
| Rubinstein and Barton (1992) | RB | × |
| Shih, Zhu and Lumley (1995) – quadratic realisable model | SH | × |
| Gatski and Speziale (1993) | GS | × |
| Lien, Chien and Leschziner (1996) | CU | ✓ |
| Craft, Launder and Suga (1996) | KS | ✓ |
| Apsley and Leschziner (1998) | DA | ✓ |
| **Differential stress models** | | |
| Gibson and Launder (1978) | GL | × |
| Craft and Launder (1992) | CL | × |
| Speziale, Sarkar and Gatski (1991) – SSG model | SG | × |
| Shima (1998) | NS | ✓ |
| Hanjalić and Jakirlić (1995) | HJ | ✓ |
| Wilcox (1988b) – multiscale *k*-ω model | WM | ✓ |

Most models have been tested only in incompressible flow. In compressible flow the Favre (density-weighted) average is assumed to replace the Reynolds average in the specifications that follow.

## 2. Eddy-Viscosity Models (Linear and Non-Linear)

### 2.1 Constitutive (Stress-Strain) Relationship

For *linear* eddy-viscosity models the stress-strain relationship is

$$-\overline{u_i u_j} = -\frac{2}{3}k\delta_{ij} + \nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3}\frac{\partial U_k}{\partial x_k}\delta_{ij}\right)$$

where

$$\nu_t \equiv \frac{\mu_t}{\rho} = \begin{cases} C_\mu f_\mu \dfrac{k^2}{\varepsilon} & (k-\varepsilon \text{ models}) \\ \alpha^* \dfrac{k}{\omega} & (k-\omega \text{ models}) \\ C_\mu^{1/4} k^{1/2} l_\mu & (\text{one}-\text{equation models}) \end{cases}$$

In the one-equation models in STREAM a transport equation is solved for $k$ and the lengthscale $l_\mu$ is specified algebraically.

Between $k$-$\varepsilon$ and $k$-$\omega$ models there is a rough correspondence:

$$\omega \approx \frac{\varepsilon}{C_\mu k}, \qquad \alpha^* \approx f_\mu, \qquad \beta^* \approx C_\mu f_\mu$$

For *nonlinear* eddy-viscosity models up to cubic order the stress-strain relationship in incompressible flow may be written in the following form:

$$\begin{aligned}\mathbf{a} = &-2 f_\mu C_\mu \mathbf{s} \\ &+\beta_1(\mathbf{s}^2 - \tfrac{1}{3}\{\mathbf{s}^2\}) + \beta_2(\mathbf{ws} - \mathbf{sw}) + \beta_3(\mathbf{w}^2 - \tfrac{1}{3}\{\mathbf{w}^2\}) \\ &-(\gamma_1\{\mathbf{s}^2\} + \gamma_2\{\mathbf{w}^2\})\mathbf{s} - \gamma_3(\mathbf{w}^2\mathbf{s} + \mathbf{sw}^2 - \{\mathbf{w}^2\}\mathbf{s} - \tfrac{2}{3}\{\mathbf{wsw}\}\mathbf{I}) - \gamma_4(\mathbf{ws}^2 - \mathbf{s}^2\mathbf{w})\end{aligned}$$

where the following general notation is used for second-rank tensors:

$$\mathbf{T} \equiv (T_{ij}), \qquad \{\mathbf{T}\} \equiv trace(\mathbf{T}) = T_{kk}, \qquad T_n \equiv trace(\mathbf{T}^n), \qquad \mathbf{I} \equiv (\delta_{ij})$$

The *dimensional* mean strain and mean vorticity tensors are denoted in upper case by

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$$

whilst *dimensionless* quantities – anisotropy $\mathbf{a}$, mean strain $\mathbf{s}$ and mean vorticity $\mathbf{w}$ – are written in lower case and defined by

$$a_{ij} = \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij}, \qquad s_{ij} = \tau S_{ij}, \qquad w_{ij} = \tau W_{ij}$$

The turbulent timescale $\tau$ is given by

$$\tau = \begin{cases} \dfrac{k}{\varepsilon} & (k-\varepsilon \text{ models}) \\ \dfrac{1}{\beta^*\omega} & (k-\omega \text{ models}) \\ \dfrac{l_\varepsilon}{C_\mu^{3/4} k^{1/2}} & (\text{one}-\text{equation models}) \end{cases}$$

For compressible flows, $S_{ij}$ is replaced in constitutive relations by its deviatoric form

$$S_{ij}^{*} = S_{ij} - \tfrac{1}{3} S_{kk} \delta_{ij}$$

and for system rotation $\mathbf{\Omega}$, $W_{ij}$ is replaced by

$$W_{ij}^{*} = W_{ij} - \varepsilon_{ijk} \Omega_k$$

The following dimensionless shear parameters may be defined:

$$\bar{s} = \sqrt{2 s_{ij} s_{ij}} = \sqrt{2 s_2}\ , \qquad \bar{w} = \sqrt{2 w_{ij} w_{ij}} = \sqrt{2(-w_2)}$$

(Both reduce to $(k/\varepsilon)(\partial U/\partial y)$ in simple shear flow.)

The rate of production of turbulent kinetic energy (per unit mass) by mean shear is

$$P^{(k)} = -\overline{u_i u_j} \frac{\partial U_i}{\partial x_j}$$

## 2.2 Transport Equations For Turbulence Variables

<u>$k$ equation</u>

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho U_j k) = \frac{\partial}{\partial x_j}\left[ (\mu + \frac{\mu_t}{\sigma^{(k)}}) \frac{\partial k}{\partial x_j} \right] + \rho(P^{(k)} - \varepsilon - D)$$

$D$ is only non-zero for models which distinguish homogeneous and inhomogeneous dissipation rates. In many original references (but not here) ε is often written as $\tilde{\varepsilon}$.

For other models ε is determined in the $k$ equation by

$$\varepsilon = \begin{cases} \beta^* \omega k & k-\omega \text{ models} \\ \dfrac{C_\mu^{3/4} k^{3/2}}{l_\varepsilon} & \text{one-equation models} \end{cases}$$

<u>ε equation</u>

$$\frac{\partial}{\partial t}(\rho \varepsilon) + \frac{\partial}{\partial x_j}(\rho U_j \varepsilon) = \frac{\partial}{\partial x_j}\left[ (\mu + \frac{\mu_t}{\sigma^{(\varepsilon)}}) \frac{\partial \varepsilon}{\partial x_j} \right] + \rho(C_{\varepsilon 1} f_1 P^{(k)} - C_{\varepsilon 2} f_2 \varepsilon)\frac{\varepsilon}{k} + \rho(S_l + S_\varepsilon)$$

The additional source terms $S_l$ and $S_\varepsilon$ are used to control the growth of the turbulent length scale and correct near-wall viscous sublayer behaviour, respectively. $S_l$ can be also used to incorporate any non-conventional model for ε production, such as in the realisable model of Shih, Liou et al. (1995); in that model $C_{\varepsilon 1}$ is set to 0 and all production of ε put in $S_l$.

<u>ω equation</u>

$$\frac{\partial}{\partial t}(\rho \omega) + \frac{\partial}{\partial x_j}(\rho U_j \omega) = \frac{\partial}{\partial x_j}\left[ (\mu + \frac{\mu_t}{\sigma^{(\omega)}}) \frac{\partial \omega}{\partial x_j} \right] + \rho(\frac{\alpha}{\nu_t} P^{(k)} - \beta \omega^2) + \rho S_\omega$$

## 2.3 Viscosity-Dependent Parameters

Relevant non-dimensional lengths are defined by

$$y^+ = \frac{u_\tau y_n}{\nu}$$

$$y^* = \frac{k^{1/2} y_n}{\nu}$$

where $y_n$ denotes the distance to the nearest wall, $u_\tau = \sqrt{\tau_w / \rho}$ and $\tau_w$ is wall stress.

Turbulent Reynolds numbers are

$$R_t = \frac{k^2}{\nu\varepsilon}, \qquad R_w = \frac{k}{\nu\omega}$$

## 2.4 Length Scales

These are used directly in one-equation models and indirectly in some two-equation models. They are also used in a two-layer treatment of wall boundaries.

Wolfshtein (1969):

$$l_\mu = \kappa y(1 - \mathrm{e}^{-0.016 y^*}), \qquad l_\varepsilon = \kappa y(1 - \mathrm{e}^{-0.263 y^*})$$

Norris and Reynolds (1975):

$$l_\mu = \kappa y(1 - \mathrm{e}^{-0.0198 y^*}), \qquad l_\varepsilon = \kappa y \frac{y^*}{y^* + 2\kappa / C_\mu^{3/4}}$$

The following values are assumed:

$C_\mu = 0.09, \quad \kappa = 0.41$

whilst in the $k$ equation:

$\sigma^{(k)} = 1.0$

## 2.5 Coefficients in Linear *k*-ε Models

*Stress-strain relationship and transport equations*

| Model | Code | $C_\mu$ | $C_{\varepsilon 1}$ | $C_{\varepsilon 2}$ | $\sigma^{(k)}$ | $\sigma^{(\varepsilon)}$ | $S_l$ |
|---|---|---|---|---|---|---|---|
| Launder and Spalding (1974) | HR | 0.09 | 1.44 | 1.92 | 1.0 | 1.3 | 0 |
| Yakhot el al. (1992) | RG | 0.085 | 1.42 | 1.68 | 0.719 | 0.719 | 0 |
| Shih, Liou et al. (1995) | RE | $\frac{1}{A_0 + A_s^* \sigma^*}$ | 0 | $1.9\frac{k}{k+\sqrt{\nu\varepsilon}}$ | 1.0 | 1.2 | $\max(0.43, \frac{\bar{s}}{5+\bar{s}})S\varepsilon$ |
| Launder and Sharma (1974) | LS | 0.09 | 1.44 | 1.92 | 1.0 | 1.3 | 0 |
| Lam and Bremhorst (1981) | LB | 0.09 | 1.44 | 1.92 | 1.0 | 1.3 | 0 |
| Chien (1982) | CH | 0.09 | 1.35 | 1.80 | 1.0 | 1.3 | 0 |
| Lien and Leschziner (1993) | LL | 0.09 | 1.44 | 1.92 | 1.0 | 1.3 | 0 |

*Viscous terms*

| Model | Code | $f_\mu$ | $D$ | $f_1$ | $f_2$ | $S_\varepsilon$ |
|---|---|---|---|---|---|---|
| Launder and Sharma (1974) | LS | $\exp(\frac{-3.4}{(1+R_t/50)^2})$ | $2\nu(\frac{\partial k^{1/2}}{\partial x_i})^2$ | 1 | $1-0.03\mathrm{e}^{-R_t^2}$ | $2\nu\nu_t\left(\frac{\partial U_i}{\partial x_j \partial x_k}\right)^2$ |
| Lam and Bremhorst (1981) | LB | $(1-\mathrm{e}^{-0.0165y^*})^2 \times (1+\frac{20.5}{R_t})$ | 0 | $1+(\frac{0.05}{f_\mu})^3$ | $1-\mathrm{e}^{-R_t^2}$ | 0 |
| Chien (1982) | CH | $1-\mathrm{e}^{-0.0115y+}$ | $\frac{2\nu k}{y_n^2}$ | 1 | $1-0.22\mathrm{e}^{-(R_t/6)^2}$ | $-\frac{2\nu\varepsilon}{y_n^2}\mathrm{e}^{-y^+/2}$ |
| Lien and Leschziner (1993) | LL | $\frac{l_\mu^{(1)}}{l_\varepsilon^{(1)}}$ | 0 | 1 | $1-0.03\mathrm{e}^{-R_t^2}$ | $C_{\varepsilon 2} f_2 \frac{\varepsilon^{(1)}\varepsilon}{k}\mathrm{e}^{-0.0022y^{*2}}$ |

Notes

- The Lien and Leschziner (1993) model uses the one-equation model of Wolfshtein (1969) to supply lengthscales $l_\mu^{(1)}$ and $l_\varepsilon^{(1)}$. The original model actually modifies $f_1$ rather than $S_\varepsilon$, but the two formulations are equivalent.

- In the Shih, Liou et al. (1995) realisable linear model, $A_0 = 4.0$, the constant $A_s*$, which is derived as the positive root of a cubic equation, is given by

$$A_s* = \sqrt{6}\cos\phi, \qquad \phi = \frac{1}{3}\cos^{-1}(\sqrt{6}\frac{s_3}{s_2^{3/2}})$$

  and the shear parameter $\sigma^*$ is

$$\sigma* = (s_{ij}s_{ij} + w_{ij}w_{ij})^{1/2}$$

  The unfamiliar ε production term has been transferred to $S_l$.

  Also, although this is not strictly a low-Re model, the ε removal term in that model contains

$$\frac{\varepsilon^2}{k + \sqrt{\nu\varepsilon}}$$

rather than the more common $\varepsilon^2/k$. This is implemented in the common form by setting $C_{\varepsilon 2}$ as given in the table.

## 2.6 Coefficients in Linear *k*-ω Models

| Model | Code | α* | β* | α | β | $\sigma^{(k)}$ | $\sigma^{(\omega)}$ | $S_\omega$ |
|---|---|---|---|---|---|---|---|---|
| Wilcox (1988a) | WX | 1 | $\frac{9}{100}$ | $\frac{5}{9}$ | $\frac{3}{40}$ | 2 | 2 | 0 |
| Wilcox (1994a) | W2 | $\frac{\frac{1}{40}+\frac{R_\omega}{6}}{1+\frac{R_\omega}{6}}$ | $\frac{9}{100}\frac{\frac{5}{18}+(\frac{R_\omega}{8})^4}{1+(\frac{R_\omega}{8})^4}$ | $\frac{5}{9}\frac{\frac{1}{10}+\frac{R_\omega}{2.7}}{1+\frac{R_\omega}{2.7}}$ | $\frac{3}{40}$ | 2 | 2 | 0 |
| Menter (1994) – Baseline | BL | 1 | 0.09 | $B\begin{pmatrix}0.553\\0.440\end{pmatrix}$ | $B\begin{pmatrix}0.075\\0.083\end{pmatrix}$ | $\frac{1}{B\begin{pmatrix}0.5\\1.0\end{pmatrix}}$ | $\frac{1}{B\begin{pmatrix}0.5\\0.856\end{pmatrix}}$ | $B\begin{pmatrix}0\\\frac{1.71}{\omega}\nabla k\bullet\nabla\omega\end{pmatrix}$ |
| Menter (1994) – SST | FM | $\min(1,\frac{0.031}{F_2}\frac{\omega}{\overline{\omega}})$ | 0.09 | $B\begin{pmatrix}0.553\\0.440\end{pmatrix}$ | $B\begin{pmatrix}0.075\\0.083\end{pmatrix}$ | $\frac{1}{B\begin{pmatrix}0.5\\1.0\end{pmatrix}}$ | $\frac{1}{B\begin{pmatrix}0.5\\0.856\end{pmatrix}}$ | $B\begin{pmatrix}0\\\frac{1.71}{\omega}\nabla k\bullet\nabla\omega\end{pmatrix}$ |

Notes

Menter's models are constructed as a blend of *k*-ω and *k*-ε models, phrased in *k*-ω form:

$$B\begin{pmatrix}a\\b\end{pmatrix}=F_1 a+(1-F_1)b$$

where:

$$r_\omega=\frac{k^{1/2}}{\beta^*\omega y_n},\qquad r_\nu=\frac{\nu}{y_n^2\omega},\qquad r_g=\frac{k\omega}{y_n^2\max(\nabla k\bullet\nabla\omega,0.0)}$$

$$F_1=\tanh(arg_1^4),\qquad arg_1=\min(\max(r_\omega,500r_\nu),2r_g)$$

$$F_2=\tanh(arg_2^2),\qquad arg_2=\max(2r_\omega,500r_\nu)$$

## 2.7 Coefficients in Non-Linear *k*-ω Models

*Non-linear stress-strain relationship*

| Model | Code | $C_\mu$ | $(\beta_1, \beta_2, \beta_3)$ | $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ |
|---|---|---|---|---|
| Speziale (1987) | SP | 0.09 | (0.054, 0.054, 0) | (0, 0, 0, 0) |
| Rubinstein and Barton (1992) | RB | 0.085 | (0.230, 0.047, 0.189) | (0, 0, 0, 0) |
| Shih, Zhu and Lumley (1995) | SH | $\frac{1}{A_0 + A_s^* \sigma^*}$ | $\frac{(1-9C_\mu^2 s_2)^{1/2}}{1+6[s_2(-w_2)]^{1/2}}(0, 2, 0)$ | (0, 0, 0, 0) |
| Gatski and Speziale (1993) | GS | $\frac{1}{2}\alpha_1 C_1^*$ | $(\alpha_3 C_3^*, \alpha_2 C_2^*, 0)$, | (0, 0, 0, 0) |
| Lien, Chen and Leschziner (1996) | CU | $\frac{2/3}{1.25+\bar{s}+0.9\overline{w}}$ | $\frac{f_\mu}{1000+\bar{s}^3}(3, 15, -19)$ | $C_\mu^3 f_\mu (16, 16, 0, -80)$ |
| Craft, Launder and Suga (1996) | KS | $\frac{0.3(1-\exp(-0.36e^{0.75\eta})}{1+0.35\eta^{3/2}}$ | $C_\mu f_\mu (-0.4, 0.4, -1.04)$ | $C_\mu^3 f_\mu (40, 40, 0, -80)$ |
| Apsley and Leschziner (1998) | DA | $\frac{(-a_{12}^*)}{1+\frac{1}{3}\overline{\beta}^2-\overline{\gamma}^2}\frac{f_P}{\sigma^*}$ | $(\frac{f_P}{\sigma^*})^2(6(a_{11}^*+a_{22}^*), a_{11}^*-a_{22}^*, 0)$ | $4C_\mu(\frac{f_P}{\sigma^*})^2(\frac{1}{3}\overline{\beta}^2, \overline{\gamma}^2, \frac{3}{2}\overline{\gamma}^2, \frac{3}{2}\overline{\beta}\overline{\gamma})$ |

*Turbulence transport equations*

| Model | Code | $C_{\varepsilon 1}$ | $C_{\varepsilon 2}$ | $\sigma^{(k)}$ | $\sigma^{(\varepsilon)}$ | $S_l$ |
|---|---|---|---|---|---|---|
| Speziale (1987) | SP | 1.44 | 1.92 | 1.0 | 1.3 | 0 |
| Rubinstein and Barton (1992) | RB | 1.42 | 1.68 | 0.719 | 0.719 | $C_\mu \frac{\bar{s}^3(\bar{s}/4.38-1)}{1+0.012\bar{s}^3}\frac{\varepsilon^2}{k}$ |
| Shih, Zhu and Lumley (1995) | SH | 1.44 | 1.92 | 1.0 | 1.3 | 0 |
| Gatski and Speziale (1993) | GS | 1.44 | 1.83 | 1.0 | 1.3 | 0 |
| Lien, Chen and Leschziner (1996) | CU | 1.44 | 1.92 | 1.0 | 1.3 | 0 |
| Craft, Launder and Suga (1996) | KS | 1.44 | 1.92 | 1.0 | 1.3 | YAP |
| Apsley and Leschziner (1998) | DA | 1.44 | 1.83 | $\frac{1.0}{1+\overline{\beta}^2/3-\overline{\gamma}^2}$ | $\frac{1.37}{1+\overline{\beta}^2/3-\overline{\gamma}^2}$ | 0 |

*Viscous Terms*

| Model | Code | $f_\mu$ | $D$ | $f_1$ | $f_2$ | $S_\varepsilon$ |
|---|---|---|---|---|---|---|
| Lien, Chen and Leschziner (1996) | CU | $l_\mu^{(1)}/l_{(\varepsilon)}^{(1)}$ | 0 | 1 | $1-0.3e^{-R_t^2}$ | $C_{\varepsilon 2} f_2 \frac{\varepsilon^{(1)}\varepsilon}{k} e^{-0.00375 y^{*2}}$ |
| Craft, Launder and Suga (1996) | KS | $1-\exp[-(\frac{R_t}{90})^2-\frac{R_t}{400}]$ | $2\nu(\frac{\partial k^{1/2}}{\partial x_i})^2$ | 1 | $1-0.3e^{-R_t^2}$ | $0.0022\frac{\bar{s}\nu_t k^2}{\varepsilon}(\frac{\partial^2 U_i}{\partial x_j \partial x_k})^2 \quad (R_t \le 250)$ |
| Apsley and Leschziner (1998) | DA | 1 | 0 | 1 | 1 | $C_{\varepsilon 2}\frac{\varepsilon^{(1)}\varepsilon}{k} e^{-0.0038 y^{*2}}$ |

Notes

- The original Speziale (1987) model included terms involving $DS_{ij}/Dt$. These have been found to provoke numerical instability and have, therefore, been omitted from the stress-strain relationship.

- In the Shih, Zhu and Lumley (1995) quadratic model, $A_0 = 6.5$, the constant $A_s*$, which is derived as the positive root of a cubic equation, is given by

$$A_s* = \sqrt{6}\cos\phi, \qquad \phi = \frac{1}{3}\cos^{-1}(\sqrt{6}\frac{s_3}{s_2^{3/2}})$$

and the shear parameter $\sigma^*$ is

$$\sigma* = (s_{ij}s_{ij} + w_{ij}w_{ij})^{1/2}$$

- In the Gatski and Speziale (1993) model, $C_i*$ are shear-dependent terms based on the regularisation of the 2-d solution as given by Speziale and Xu (1996):

$$C_1* = \frac{(1+2\zeta^2)(1+6\eta^5)+\frac{5}{3}\eta^2}{(1+2\zeta^2)(1+2\zeta^2+\eta^2+6b_1\eta^6)}, \qquad C_{2,3}* = \frac{(1+2\zeta^2)(1+\eta^4)+\frac{2}{3}\eta^2}{(1+2\zeta^2)(1+2\zeta^2+b_{2,3}\eta^6)}$$

where

$$\eta = \frac{1}{2}\frac{\alpha_3}{\alpha_1}(s_2)^{1/2}, \qquad \zeta = \frac{\alpha_2}{\alpha_1}(-w_2)^{1/2}, \qquad (b_1, b_2, b_3) = (7.0, 6.3, 4.0)$$

$$\alpha_1 = (\tfrac{4}{3} - C_2)g, \qquad \alpha_2 = \tfrac{1}{2}\alpha_1(2 - C_4)g, \qquad \alpha_3 = \alpha_1(2 - C_3)g$$

$$g = \frac{1}{\frac{1}{2}C_1 + (P^{(k)}/\varepsilon)_{eqm} - 1}, \qquad \left(\frac{P^{(k)}}{\varepsilon}\right)_{eqm} = \frac{C_{\varepsilon 2} - 1}{C_{\varepsilon 1} - 1}$$

The constants, which come from the SSG pressure-strain model, are

$C_1 = 6.8, \quad C_2 = 0.36, \quad C_3 = 1.25, \quad C_4 = 0.40$

Note also that, for system rotation in this model only, $w_{ij}$ is alternatively defined by

$$w_{ij} = \tau\left[W_{ij} - \left(\frac{4 - C_4}{2 - C_4}\right)\varepsilon_{ijk}\Omega_k\right]$$

- In the Craft et al. (1996) model,

$$\eta = \max(\bar{s}, \bar{w})$$

The Yap correction (Yap, 1987) in the dissipation equation is

$$YAP = \max(0.83(\gamma - 1)\gamma^2\frac{\varepsilon^2}{k}, 0), \qquad \gamma = \frac{C_{\mu 0}^{3/4}k^{3/2}/(\kappa y_n)}{\varepsilon}, \qquad C_{\mu 0} = 0.09$$

- The low-Re terms in the Lien, Chen and Leschziner (1996) model are based on the one-equation model of Norris and Reynolds (1975) for the mixing and dissipation lengths near the wall. The original model actually modifies $f_1$ rather than $S_\varepsilon$, but the two formulations are equivalent.

- In the Apsley and Leschziner (1998) model, $f_\mu$ is incorporated naturally into $C_\mu$. $a^*_{\alpha\beta}$ and $\sigma^*$ are curve fits to the three independent anisotropy components and shear parameter, respectively, from DNS data for plane channel flow:

$$a^*_{11} = 1 + 0.42\exp(0.296 y^{*1/2} - 0.040 y^*) - \frac{2}{3}$$
$$a^*_{22} = 0.404[1 - \exp(-0.001 y^* - 0.000147 y^{*2})] - \frac{2}{3}$$
$$a^*_{12} = -0.3[1 - \exp(-0.00443 y^{*1/2} - 0.0189 y^*)]$$
$$\sigma^* = 3.33[1 - \exp(-0.45 y^*)][1 + 0.277 y^{*3/2}\exp(-0.088 y^*)]$$

The constants $\bar{\beta}$ and $\bar{\gamma}$ are based on the values of the anisotropy components and shear parameter in the log-law region and are given by

$$\bar{\beta} = 0.222, \qquad \bar{\gamma} = 0.623$$

Modifications to $C_\mu$, $\sigma^{(k)}$ and $\sigma^{(\varepsilon)}$ arise because, unlike the other non-linear models, the first two cubic terms do not cancel out in simple shear flow. The non-equilibrium parameter $f_p$ which accounts for departures of the local shear parameter $\sigma = (k/\varepsilon)\sqrt{(\partial U_i/\partial x_j)^2} = (s_2 - w_2)^{1/2}$ from the calibration value $\sigma^*$ is given by

$$f_P = \frac{2 f_0}{1 + \sqrt{1 + 4 f_0 (f_0 - 1)(\sigma/\sigma^*)^2}}, \qquad f_0 = 1 + 1.25\max(0.09\sigma^{*2}, 1.0)$$

The additional term in the dissipation equation is based upon a curve fit to DNS data for dissipation length:

$$l^{(1)}_\varepsilon = 0.179 y_n (1 + \frac{1.28}{y^*})[1 - \exp(-y^{*2}/279)], \qquad \varepsilon^{(1)} = \frac{C_{\mu 0}^{3/4} k^{3/2}}{l^{(1)}_\varepsilon}$$

*Additional Note*

Some researchers (including Dr Apsley) have noted that numerical implementations of the model can become numerically unstable for extremely fine grids. The root cause appears to be as follows.

In simple shear, the model gives

$$-\frac{\overline{uv}}{k} = a^*_{12}\left(\frac{x - a x^3}{1 - a}\right), \qquad \text{where} \quad a = \bar{\gamma}^2 - \tfrac{1}{3}\bar{\beta}^2, \quad x = f_p \frac{\sigma}{\sigma^*}$$

Unfortunately,

$$\frac{\partial}{\partial x}\left(\frac{-\overline{uv}}{k}\right) = 0 \qquad \text{at} \qquad x = \frac{1}{\sqrt{3a}}$$

With standard values of $\bar{\beta}$ and $\bar{\gamma}$, this occurs at $x = 0.947$ and is uncomfortably close to the "preferred" equilibrium value $x = 1$. The proximity to a stationary point means that $-\overline{uv}/k$ doesn't change much even when shear $\partial U/\partial y$ does, resulting in large fluctuations in $U(y)$ even for a shear stress fixed by the pressure gradient.

This can be eliminated by changing the values of $\bar{\beta}$ and $\bar{\gamma}$ – but that would damage the agreement with experimental results for boundary-layer anisotropy. A better solution (yet to be found!) would be a more stable numerical implementation.

## 3. Differential Stress Models

### 3.1 Transport Equations

$$\frac{\partial}{\partial t}(\rho\overline{u_i u_j}) + \frac{\partial}{\partial x_k}(\rho U_k \overline{u_i u_j}) = \frac{\partial}{\partial x_k} d_{ijk} + \rho(P_{ij} + G_{ij} + \Phi_{ij} - \varepsilon_{ij})$$

$$\frac{\partial}{\partial t}(\rho\varepsilon) + \frac{\partial}{\partial x_j}(\rho U_j \varepsilon) = \frac{\partial d_k^{(\varepsilon)}}{\partial x_j} + \rho(C_{\varepsilon 1} f_1 P^{(k)} - C_{\varepsilon 2} f_2 \varepsilon)\frac{\varepsilon}{k} + \rho(S_l + S_\varepsilon)$$

where

$$P_{ij} = -(\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}), \qquad P^{(k)} = \tfrac{1}{2}P_{ii}$$

$$G_{ij} = \overline{f_i u_j} + \overline{f_j u_i}$$

Modelled terms are given below. Note that:

$$D_{ij} = -(\overline{u_i u_k}\frac{\partial U_k}{\partial x_j} + \overline{u_j u_k}\frac{\partial U_k}{\partial x_i})$$

$$a_{ij} = \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij} \qquad (anisotropy)$$

$$A = 1 - \frac{9}{8}(a_2 - a_3) \qquad (flatness)$$

$$e_{ij} = \frac{\varepsilon_{ij}}{\varepsilon} - \frac{2}{3}\delta_{ij}$$

$$E = 1 - \frac{9}{8}(e_2 - e_3)$$

### 3.2 Diffusion

$$d_{ijk} = (\mu\delta_{kl} + C_s \frac{\rho k \overline{u_k u_l}}{\varepsilon})\frac{\partial}{\partial x_l}(\overline{u_i u_j})$$

$$d_k^{(\varepsilon)} = (\mu\delta_{kl} + C_\varepsilon \frac{\rho k \overline{u_k u_l}}{\varepsilon})\frac{\partial \varepsilon}{\partial x_l}$$

### 3.3 Pressure-Strain Correlation

$$\Phi_{ij} = \Phi_{ij}^{(1)} + \Phi_{ij}^{(2)} + \Phi_{ij}^{(w)}$$

<u>Slow Pressure-Strain</u>

$$\Phi_{ij}^{(1)} = -C_1 \frac{\varepsilon}{k}(\overline{u_i u_j} - \frac{2}{3}k\delta_{ij})$$

or, for a non-linear extension:

$$\frac{\mathbf{\Phi}^{(1)}}{\varepsilon} = -C_1\mathbf{a} - C_1'(\mathbf{a}^2 - \frac{1}{3}\{\mathbf{a}^2\}\mathbf{I})$$

Fast Pressure-Strain

$$\Phi_{ij}^{(2)} = -C_2(P_{ij} - \frac{1}{3}P_{kk}\delta_{ij}) - C_3(D_{ij} - \frac{1}{3}D_{kk}\delta_{ij}) - C_4 k(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij})$$

or

$$\frac{\mathbf{\Phi}^{(2)}}{\varepsilon} = C_{01}(\mathbf{s} - \frac{1}{3}\{\mathbf{s}\}\mathbf{I}) + C_{11}(\mathbf{sa} + \mathbf{as} - \frac{2}{3}\{\mathbf{as}\}\mathbf{I}) + C_{12}(\mathbf{wa} - \mathbf{aw})$$

The two expressions may be interconverted by:

$$C_{01} = \frac{4}{3}(C_2 + C_3) - C_4, \qquad C_{11} = C_2 + C_3, \qquad C_{12} = C_2 - C_3$$

or

$$C_2 = \frac{1}{2}(C_{11} + C_{12}), \qquad C_3 = \frac{1}{2}(C_{11} - C_{12}), \qquad C_4 = \frac{4}{3}C_{11} - C_{01}$$

Wall Reflection Terms

$$\Phi_{ij}^{(w)} = (\widetilde{\Phi}_{kl} n_k n_l \delta_{ij} - \frac{3}{2}\widetilde{\Phi}_{ik} n_j n_k - \frac{3}{2}\widetilde{\Phi}_{jk} n_i n_k) f^{(w)}$$

Here,

$$\widetilde{\Phi}_{ij} = C_1^{(w)} \frac{\varepsilon}{k}\overline{u_i u_j} + C_2^{(w)}\Phi_{ij}^{(2)}$$

except, in the Craft-Launder (1992) model, where:

$$\widetilde{\Phi}_{ij} = C_1^{(w)} \frac{\varepsilon}{k}\overline{u_i u_j} - 0.1\frac{\partial U_i}{\partial x_k}(\overline{u_k u_j} - \frac{2}{3}k\delta_{kj}) + (0.08P^{(k)} - \frac{0.4}{3}k\frac{\partial U_k}{\partial x_l}n_k n_l)\delta_{ij}$$

In all cases, the wall-distance parameter is

$$f^{(w)} = \frac{C_\mu^{3/4} k^{3/2}/\varepsilon}{\kappa y_n}, \qquad C_\mu = 0.09$$

### 3.4 Dissipation

Except in the Hanjalić and Jakirlić model, dissipation is modelled as isotropic:

$$\varepsilon_{ij} = \tfrac{2}{3}\varepsilon\delta_{ij}$$

### 3.5 Individual Models

Gibson and Launder (1978)

Diffusion:

$C_s = 0.22$, $C_\varepsilon = 0.18$

Pressure-strain:

$C_1 = 1.8$, $C_1' = 0$

$C_2 = 0.6$, $C_3 = 0$, $C_4 = 0$

$C_1^{(w)} = 0.5$, $C_2^{(w)} = 0.3$, $C_l = 2.5$

Dissipation equation:

$C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$, $S_l = 0$

$f_1 = 1$, $f_2 = 1$, $S_\varepsilon = 0$

Speziale, Sarkar and Gatski (1991) – SSG Model

Diffusion: not specified in the original paper; taken as

$C_s = 0.22$, $C_\varepsilon = 0.18$

Pressure-strain:

$$C_1 = 1.7 + 0.9\frac{P^{(k)}}{\varepsilon}, \quad C_1' = -1.05$$

$$C_{01} = 0.8 - 0.65 a_2^{1/2}, \quad C_{11} = 0.625, \quad C_{12} = 0.2$$

No separate wall-reflection

Dissipation equation: not specified in the original paper; taken as

$C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.83$, $S_l = 0$

$f_1 = 1$, $f_2 = 1$, $S_\varepsilon = 0$

Shima (1998)

Diffusion:

$C_s = 0.22$, $C_\varepsilon = 0.15$

Pressure-strain:

$$C_1 = 1 + 2.45 a_2^{1/4} A^{3/4} (1 - \mathrm{e}^{-(7A)^2})(1 - \mathrm{e}^{-(R_t/60)^2}), \quad C_1' = 0$$

$$C_2 = 0.7A, \quad C_3 = 0.3A^{1/2}, \quad C_4 = 0.65A(0.23C_1 + C_2 - 1) + 1.3 a_2^{1/4}$$

No separate wall-reflection

Dissipation equation:

$C_{\varepsilon 1} = 1.44 + \beta_1 + \beta_2$, $C_{\varepsilon 2} = 1.92$, $S_l = 0$

$$\beta_1 = 0.25A\min(\frac{\lambda}{2.5} - 1{,}0) - 1.4A\min(\frac{P^{(k)}}{\varepsilon} - 1{,}0)$$

$$\beta_2 = 1.0A\lambda^2 \max(\frac{\lambda}{2.5} - 1{,}0)$$

$$\lambda = \min(\left|\nabla \frac{k^{3/2}}{\varepsilon}\right|{,}4)$$

$$f_1 = 1, \quad f_2 = \frac{\tilde{\varepsilon}}{\varepsilon}, \quad S_\varepsilon = 0$$

$$\tilde{\varepsilon} = \max(\varepsilon - 2\nu(\frac{\partial k^{1/2}}{\partial x_i})^2{,}0)$$

<u>Hanjalić and Jakirlić (1995)</u>

Diffusion:

$$C_s = 0.22, \quad C_\varepsilon = 0.18$$

Pressure-strain:

$$C_1 = C + A^{1/2}E^2, \quad C_1' = 0$$

$$C_2 = 0.8A^{1/2}, \quad C_3 = 0, \quad C_4 = 0$$

$$C_1^{(w)} = \max(1 - 0.7C{,}0.3), \quad C_2^{(w)} = \min(A{,}0.3), \quad C_l = 2.5$$

$$C = 2.5Af_{a_2}^{1/4} f_{R_t}, \quad f_{a_2} = \min(a_2{,}0.6), \quad f_{R_t} = \min[(\frac{R_t}{150})^{3/2}{,}1]$$

$$\text{Here, } f^{(w)} = \min(\frac{k^{3/2}/\varepsilon}{C_l y_n}{,}1.4)$$

Dissipation (anisotropic):

$$\varepsilon_{ij} = f_\varepsilon \tfrac{2}{3}\varepsilon\delta_{ij} + (1 - f_\varepsilon)\varepsilon_{ij}^*$$

$$\varepsilon_{ij}^* = \frac{\varepsilon}{k} \frac{\overline{u_i u_j} + (\overline{u_i u_k} n_j n_k + \overline{u_j u_k} n_i n_k + \overline{u_n^2} n_i n_j) f_d}{1 + \frac{3}{2}\frac{\overline{u_n^2}}{k} f_d}$$

$$f_\varepsilon = A^{1/2}E^2, \quad f_d = \frac{1}{1 + 0.1R_t}$$

Dissipation equation:

$$C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92, \quad S_l = 0$$

$$f_1 = 1, \quad f_2 = \left[1 - (\frac{C_{\varepsilon 2} - 1}{C_{\varepsilon 2}}) e^{-(R_t/6)^2}\right]\frac{\tilde{\varepsilon}}{\varepsilon}, \quad S_\varepsilon = 0.25\nu \frac{k}{\varepsilon} \overline{u_i u_j} \frac{\partial^2 U_k}{\partial x_i \partial x_l} \frac{\partial^2 U_k}{\partial x_j \partial x_l}$$

$$\tilde{\varepsilon} = \max(\varepsilon - 2\nu(\frac{\partial k^{1/2}}{\partial x_i})^2{,}0)$$

### 3.6 Wilcox (1988b) – Multiscale Model

Wilcox's multiscale model is built upon the premise that one can partition the energy spectrum into large-scale, energy-bearing eddies and small-scale, isotropic, dissipative eddies. The formulation is rather different from that of "traditional" Reynolds-stress transport models and consists of transport equations for $k$, ω and the upper-partition stress tensor

$$T_{ij} = -(\overline{u_i u_j} - \frac{2}{3} e \delta_{ij})$$

where $e$ is the energy of the lower-partition eddies, plus a tensor describing the exchange of energy between upper and lower partitions. The most practical way of incorporating this into the STREAM code is to rewrite Wilcox's equations in terms of equations for $\overline{u_i u_j}$, ω and $k^U$ (the upper-partition turbulence energy).

$$\frac{\partial}{\partial t}(\rho \overline{u_i u_j}) + \frac{\partial}{\partial x_k}(\rho U_k \overline{u_i u_j}) = \frac{\partial}{\partial x_k} d_{ijk} + \rho(P_{ij} + G_{ij} + \Phi_{ij} - \varepsilon_{ij})$$

$$\frac{\partial}{\partial t}(\rho \omega) + \frac{\partial}{\partial x_j}(\rho U_j \omega) = \frac{\partial}{\partial x_j}\left[(\mu + \frac{\mu_t}{\sigma^{(\omega)}})\frac{\partial \omega}{\partial x_j}\right] + \rho(\frac{\alpha}{\nu_t} P^{(k)} - \beta \omega^2) + \rho S_\omega$$

$$\frac{\partial}{\partial t}(\rho k^U) + \frac{\partial}{\partial x_j}(\rho U_j k^U) = \rho\left[(1 - C_2 - C_3) P^{(k)} - (\frac{k^U}{k})^{3/2} \varepsilon\right]$$

where

$$k = \frac{1}{2}\overline{u_i u_i}, \qquad \varepsilon = \beta^* \omega k, \qquad \nu_t \equiv \frac{\mu_t}{\rho} = \frac{k}{\omega}$$

<u>Stress-Transport Equation</u>

Diffusion:

$$d_{ijk} = (\mu + \frac{\mu_t}{\sigma^{(k)}})\frac{\partial k}{\partial x_k}\delta_{ij}, \qquad \sigma^{(k)} = 2$$

Pressure-strain:

$$C_1 = 1 + 4(\frac{k^{(U)}}{k})^{3/2}, \quad C_2 = \frac{42}{55}, \quad C_3 = \frac{6}{55}, \quad C_4 = \frac{1}{4}$$

No wall-reflection terms

Dissipation:

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon \delta_{ij}$$

<u>ω-equation</u>

$$\sigma^{(\omega)} = 2, \quad \alpha = \frac{5}{9}, \quad \beta = \frac{3}{40}$$

$$S_\omega = -\beta \omega \overline{w}$$

## 4. Wall Boundary Conditions

There are three approaches.

(i) Low-Re treatment:
- assume that behaviour is adequately resolved right to the boundary;
- apply viscous modifications to turbulence equations and constitutive relations;
- these viscous modifications are applied *throughout the flow*.

(ii) Wall-function treatment:
- if necessary, model what happens between the near-wall node and the boundary;
- depending on variable, may set wall flux, cell-averaged source or fixed value at the near-wall node;
- viscous effects are included *only for the near-wall cell and boundary*.

(iii) Two-layer treatment (eddy-viscosity models only):
- blend a high- or low-Re 2-equation model into a 1-equation model near a wall.

### 4.1 Wall-Boundary Effects on Individual Transport Variables

| **Variable** | **Common** | **Low-$y^+$** | **Wall function (near-wall cell only)** |
|---|---|---|---|
| $U_i$ | Wall value = 0 | Viscous modifications to $\nu_t$, affecting fluxes. | Wall flux via effective viscosity $\mu_w$ |
| $k$ | Wall value = 0<br>Wall flux = 0 | Viscous terms in transport equation. | Cell-averaged production and dissipation. |
| $\overline{u_i u_j}$ | Wall value = 0<br>Wall flux = 0 | Viscous terms in transport equation | Cell-averaged production and dissipation. |
| $\varepsilon$ (if $D = 0$) | Wall flux = 0 | $\varepsilon_w = \varepsilon_P = \dfrac{2\nu k_P}{y_P^2}$<br>Viscous terms in transport equation. | $\varepsilon_w = \varepsilon_P$ = wall-function value at $P$ |
| $\varepsilon$ (if $D \neq 0$) | Wall flux = 0 | $\varepsilon_w = 0$<br>Viscous terms in transport equation. | N/A |
| $\omega$ | Wall flux = 0 | $\omega_w = \omega_P = \omega_{fac} \dfrac{2\nu k_P}{C_\mu y_P^2}$<br>Viscous terms in transport equation | $\omega_w = \omega_P$ = wall-function value at $P$ |

Notes

- Subscript $P$ refers to the near-wall node, subscript $w$ to the value on the boundary.
- Using the treatment suggested by Menter (1994), $\omega_{fac} \times \dfrac{2}{C_\mu} = 800$.

### 4.2 Wall-Function Approach

#### 4.2.1 Basic Profiles

The wall-function formulae are deduced from the following basic assumptions which make a smooth transition from laminar viscous sublayer to fully-turbulent log layer. Here, $y$ denotes the distance from the boundary, subscript $P$ denotes the value of a variable at the near-wall node (centre of the near-wall cell) and $\Delta$ denotes the thickness of the near-wall cell.

- Total stress constant and an effective total viscosity:

$$\tau = \tau_w = \rho \nu_{eff} \frac{\partial U}{\partial y}, \qquad U(0) = 0$$

$$\nu_{eff} = \begin{cases} \nu & (y \le y_\nu) \\ \nu + \kappa u_0 (y - y_\nu) & (y \ge y_\nu) \end{cases}$$

where

$$u_0 = C_\mu^{1/4} k_P^{1/2}$$

- The dissipation rate is given by:

$$\varepsilon = \begin{cases} \varepsilon_w & (y \le y_\varepsilon) \\ \dfrac{u_0^3}{\kappa (y - y_d)} & (y \ge y_\varepsilon) \end{cases}$$

with $\varepsilon_w$ determined so as to make $\varepsilon$ continuous at $y_\varepsilon$.

Note:

- Profile points $y_\nu$, $y_\varepsilon$ and $y_d$ are defined below.
- The definition of $u_0$ is such that it would equal $u_\tau$ in the log layer.
- The implied equivalent one-equation model ($\nu_t = C_\mu^{1/4} k^{1/2} l_\mu$) would have length scale

$$l_\mu = \kappa y (1 - \frac{y_\nu}{y}) \qquad (y \ge y_\nu)$$

#### 4.2.2 Derived Quantities

Assumed Mean-Velocity Profile

By integration:

$$\frac{U}{u_0} = \frac{\tau_w}{\rho u_0^2} \times \begin{cases} \tilde{y}^+ & (\tilde{y}^+ \le y_\nu^+) \\ y_\nu^+ + \dfrac{1}{\kappa} \ln\left[1 + \kappa (\tilde{y}^+ - y_\nu^+)\right] & (\tilde{y}^+ \ge y_\nu^+) \end{cases}$$

where

$$\tilde{y}^+ = \frac{u_0 y}{\nu}$$

*Important*: $\tilde{y}^+$ here is based on $u_0$ rather than $u_\tau$.

Wall Stress and Effective Wall Viscosity

$$\tau_w = \rho \nu_w \frac{U_P}{y_P}$$

where

$$\nu_w = \nu \times \begin{cases} 1 & (\tilde{y}_P^+ \le y_\nu^+) \\ \dfrac{\tilde{y}_P^+}{y_\nu^+ + \frac{1}{\kappa}\ln\left[1 + \kappa(\tilde{y}_P^+ - y_\nu^+)\right]} & (\tilde{y}_P^+ \ge y_\nu^+) \end{cases}$$

Cell-Averaged Production and Dissipation

$$P_{av}^{(k)} = \begin{cases} 0 & (\tilde{\Delta}^+ \le y_\nu^+) \\ \dfrac{(\tau_w/\rho)^2}{\kappa u_0 \Delta}\left\{\ln\left[1 + \kappa(\tilde{\Delta}^+ - y_\nu^+)\right] - \dfrac{\kappa(\tilde{\Delta}^+ - y_\nu^+)}{1 + \kappa(\tilde{\Delta}^+ - y_\nu^+)}\right\} & (\tilde{\Delta}^+ \ge y_\nu^+) \end{cases}$$

$$\varepsilon_{av} = \begin{cases} \varepsilon_w & (\Delta \le y_\nu) \\ \dfrac{u_0^3}{\kappa\Delta}\left[\ln\left(\dfrac{\Delta - y_d}{y_\varepsilon - y_d}\right) + \dfrac{y_\varepsilon}{y_\varepsilon - y_d}\right] & (\Delta \ge y_\nu) \end{cases}$$

Near-Wall Dissipation

$\varepsilon_P$ is given directly from the assumed ε profile at $y = y_P$; i.e.

$$\varepsilon_P = \begin{cases} \varepsilon_w & (y_P \le y_\varepsilon) \\ \dfrac{u_0^3}{\kappa(y_P - y_d)} & (y_P \ge y_\varepsilon) \end{cases}$$

### 4.2.3 Matching Depths

For smooth walls:

$$y_\nu^+ = 7.37$$

$$y_\varepsilon^+ = 27.4, \qquad y_d^+ = 4.9$$

For arbitrarily-rough walls (Apsley, 2007) the viscous sublayer cutoff is given by:

$$y_\nu^+ = f(B - \frac{1}{\kappa}\ln\kappa), \qquad f(x) = \begin{cases} x & (x \ge 0) \\ \dfrac{1}{\kappa}(1 - e^{-\kappa x}) & (x \le 0) \end{cases}$$

and

$$B = B_{rough} - \frac{1}{\kappa}\ln(k_s^+ + C), \qquad C = e^{\kappa(B_{rough} - B_{smooth})}$$

Here, $B_{smooth}$ and $B_{rough}$ are constants in the fully-smooth and fully-rough logarithmic wall profiles, respectively:

$$U^+ = \frac{1}{\kappa}\ln y^+ + B_{smooth} \qquad \text{and} \qquad U^+ = \frac{1}{\kappa}\ln\frac{y}{k_s} + B_{rough}$$

STREAM assumes values $\kappa = 0.41$, $B_{smooth} = 5.2$ and $B_{rough} = 8$, whence $C = 3.152$.

Similarly, the dissipation-related constants are given by:

$$y_d^+ = y_\nu^+ - \frac{1}{\kappa}$$

$$y_\varepsilon^+ = y_d^+ + \frac{1}{\kappa} s_1 \exp\left[\frac{y_d^+(1+1/s_1)}{y_\varepsilon^+ - y_d^+}\right], \qquad s_1 = 1 + \kappa \max(-y_\nu^+, 0)$$

The last has to be determined iteratively, but converges quickly.

### 4.2.4 Other Variables

Omega

Where wall functions are used with the $k$-$\omega$ model, $\varepsilon$ is deduced as above and the near-wall value of $\omega$ then determined by

$$\omega_P = \frac{\varepsilon_P}{C_\mu k_P}$$

Reynolds stresses

Where wall functions are used with differential-stress models, cell-averaged production is first expressed in a *local* coordinate system with tangential ($t$) and normal ($n$) velocities (relative to any wall velocity). Then:

$$P_{tt}^{(av)} = 2P_k^{(av)}, \qquad P_{tn}^{(av)} = -\frac{0.248}{\sqrt{C_\mu}} P_k^{(av)}, \qquad P_{ab}^{(av)} = 0 \text{ otherwise}$$

Components of production are subsequently rotated to the global Cartesian system and the equations treated in similar fashion to the $k$ equation (including cell-averaged dissipation).

Note:

- Earlier versions of STREAM set individual stresses at near-wall nodes rather than cell-averaged production. This, however, gave problems when the positive tangential direction could not be identified; for example, at impingement or separation points. This does not affect the production terms as they vanish here anyway.

- Because only cell-averaged $P_{ij}$ is specified in this way (not $D_{ij}$ or $S_{ij}$) all differential-stress models using wall functions in STREAM revert to the standard return-to-isotropy form for the fast pressure strain in the near-wall cell. This is not ideal!

### 4.3 Two-Layer Approach (Two-Equation Eddy-Viscosity Models Only)

A blending function $f_b$ is used to blend the eddy viscosity $\mu_t$ and dissipation rate $\varepsilon$ between any two-equation eddy-viscosity model (high-Re or low-Re) and the one-equation model of Wolfshtein (1969) defined earlier. The blending function is here taken as

$$f_b = \frac{1}{2}\left[1 + \tanh(\frac{y^* - 60}{4.352})\right]$$

To blend the eddy viscosity:

$$\mu_t = f_b \mu_t^{(2-eqn)} + (1 - f_b)\mu_t^{(1-eqn)}$$

To blend the dissipation rate within its discretised equation ( $a_P \varepsilon_P - \sum a_F \varepsilon_F = b_p$ ), write

$$\varepsilon_p = f_b(\frac{b_P + \sum a_F \varepsilon_F}{a_P}) + (1 - f_b)\varepsilon_P^{(1-eqn)}$$

This rearranges as:

$$\frac{a_P}{f_b}\varepsilon_p - \sum a_F \varepsilon_F = b_P + \frac{(1 - f_b)}{f_b} a_P \varepsilon_P^{(1-eqn)}$$

so that any under-relaxation step for $\varepsilon$ is preceded by a modification of coefficients:

$$a_P \to a'_p = \frac{a_p}{f_b}, \qquad b_P \to b'_P + (1 - f_b) a'_P \varepsilon_P^{(1-eqn)}$$

## References


Apsley, D.D., 2007, CFD calculation of turbulent flow with arbitrary wall roughness, Flow, Turbulence and Combustion, 78, 153-175.

Apsley, D.D. and Leschziner, M.A., 1998, A new low-Reynolds-number nonlinear two-equation turbulence model for complex flows, Int. J. Heat Fluid Flow, 19, 209-222.

Chien, K.Y., 1982, Predictions of channel and boundary-layer flows with a low-Reynolds-number turbulence model, AIAA J., 20, 33-38.

Craft, T.J. and Launder, B.E., 1992, New wall-reflection model applied to the turbulent impinging jet, AIAA J., 30, 2970-2972.

Craft, T.J., Launder, B.E. and Suga, K., 1996, Development and application of a cubic eddy-viscosity model of turbulence, Int. J. Heat Fluid Flow, 17, 108-115.

Gatski, T.B. and Speziale, C.G., 1993, On explicit algebraic stress models for complex turbulent flows, J. Fluid Mech., 254, 59-78.

Gibson, M.M. and Launder, B.E., 1978, Ground effects on pressure fluctuations in the atmospheric boundary layer, J. Fluid Mech., 86, 491-511.

Jakirlić, S. and Hanjalić, K., 1995, A second-moment closure for non-equilibrium and separating high- and low-Re-number flows, Proc. 10th Symp. Turbulent Shear Flows, Pennsylvania State University.

Lam, C.K.G. and Bremhorst, K.A., 1981, Modified form of the k-e model for predicting wall turbulence, Journal of Fluids Engineering, 103, 456-460.

Launder, B.E., Reece, G.J. and Rodi, W., 1975, Progress in the development of a Reynolds-stress turbulence closure, J. Fluid Mech., 68, 537-566.

Launder, B.E. and Sharma, B.I., 1974, Application of the energy-dissipation model of turbulence to the calculation of flow near a spinning disc, Letters in Heat and Mass Transfer, 1, 131-138.

Launder, B.E. and Spalding, D.B., 1974, The numerical computation of turbulent flows, Computer Meth. Appl. Mech. Eng., 3, 269-289.

Lien, F-S. and Leschziner, M.A., 1993, Second-moment modelling of recirculating flow with a non-orthogonal collocated finite-volume algorithm, 8th Symposium on Turbulent Shear Flows, Munich.

Lien, F.-S., Chen, W.L. and Leschziner, M.A., 1996, Low-Reynolds-number eddy-viscosity modelling based on non-linear stress-strain/vorticity relations, 3rd Symp. Engineering Turbulence Modelling and Measurements, Crete.

Menter, F.R., 1994, Two-equation eddy-viscosity turbulence models for engineering applications, AIAA J., 32, 1598-1605.

Norris, L.H. and Reynolds, W.C., 1975, Turbulent channel flow with a moving wavy boundary, Stanford Univ. Dept. Mech. Eng. Rept. FM-10.

Rubinstein, R. and Barton, J.M., 1990, Nonlinear Reynolds stress models and the renormalisation group, Phys. Fluids A, 2, 1472-1476.

Shih, T.-H., Liou, W.W., Shammir, A., Yang, Z. and Zhu, J., 1995, A new $k$-ε eddy-viscosity model for high Reynolds number turbulent flows, Computers Fluids, 24, 227-238.

Shih, T.-H., Zhu, J. and Lumley, J.L., 1995, A new Reynolds stress algebraic equation model, Comput. Methods Appl. Mech. Engrg., 125, 287-302.

Speziale, C.G., 1987, On nonlinear K-l and K-ε models of turbulence, J. Fluid Mech., 178, 459-475.

Speziale, C.G., Sarkar, S. and Gatski, T.B., 1991, Modelling the pressure-strain correlation of turbulence: an invariant dynamical systems approach, J. Fluid Mech., 227, 245-272.

Wilcox, D.C., 1988a, Reassessment of the scale-determining equation for advanced turbulence models, AIAA J., 26, 1299-1310.

Wilcox, D.C., 1988b, Multi-scale model for turbulent flows, AIAA Journal, 26, 1311-1320.
Wilcox, D.C., 1994, Simulation of transition with a two-equation turbulence model, AIAA J., 32, 247-255.
Wolfshtein, M.W., 1969, The velocity and temperature distribution in one-dimensional flow with turbulence augmentation and pressure gradient, Int. J. Heat Mass Transfer, 12, 301.
Yakhot, V., Orszag, S.A., Thangam, S., Gatski, T.B. and Speziale, C.G., 1992, Development of turbulence models for shear flows by a double expansion technique, Phys. Fluids A, 7, 1510.