
The analysis is applicable to a flat-plate boundary layer or fully-developed pipe or channel flow. First consider smooth walls.

2.1 Shear Stress and Friction Velocity

The shear stress (= rate of transport of momentum per unit area in the positive y direction) is

$$\tau = \mu \frac{\partial U}{\partial y} - \rho \overline{uv} \qquad (1)$$

The viscous part varies from being the sole transporter of momentum at the wall to a negligible fraction of the total stress in the outer part of a turbulent boundary layer.

For $y < 0.1\delta$, τ is approximately constant (why?) and equal to its value at the wall:

$$\tau \approx \tau_w$$

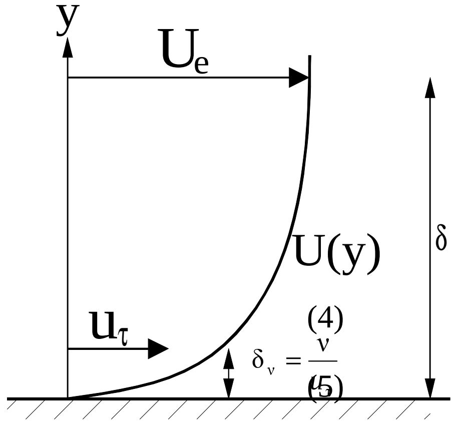
This is the *constant-stress* layer. As τ_w has dimensions of [*density*] $\times$ [*velocity*]^2, it is possible to define an important velocity scale – the *friction velocity*, u_τ – by

$$\tau_w = \rho u_\tau^2 \qquad (2)$$

or

$$u_\tau \equiv \sqrt{\tau_w / \rho} \qquad (3)$$

2.2 Length and Velocity Scales



Wall Units

Very close to the wall the most important scaling parameters are:
- kinematic viscosity ν;
- wall shear stress τ_w.

The characteristic velocity and length scales are:

friction velocity: $$u_\tau \equiv \sqrt{\tau_w/\rho} \qquad (4)$$

viscous length scale: $$\delta_\nu \equiv \frac{\nu}{u_\tau} \qquad (5)$$

From these we can form non-dimensional velocity and height in *wall units*:

$$U^+ \equiv \frac{U}{u_\tau}, \qquad y^+ \equiv \frac{y}{\delta_\nu} = \frac{u_\tau y}{\nu} \qquad (6)$$

y^+ is a sort of local Reynolds number. Its value is a measure of the relative importance of viscous and turbulent transport at different distances from the wall.

Boundary-Layer Units

At large y^+ the direct effect of viscosity on momentum transport is small and heights can be specified as a fraction of the boundary-layer depth δ:

$$\eta = \frac{y}{\delta} \qquad (7)$$

The quantity

$$\mathrm{Re}_\tau \equiv \frac{u_\tau \delta}{\nu} = \delta^+ \qquad (8)$$

is called the *friction Reynolds number* and is a global parameter of the boundary layer.

Since fully-developed boundary-layer flow is completely specified by U, y, ρ, ν, δ and u_τ, dimensional analysis (6 variables, 3 independent dimensions) yields a functional relationship between 6 – 3 = 3 dimensionless groups, conveniently taken as

$$\frac{U}{u_\tau} = f\left(\frac{y}{\delta_\nu}, \frac{y}{\delta}\right)$$

i.e.

$$U^+ = f(y^+, \eta) \qquad (9)$$

Almost all boundary-layer analysis is based upon the smooth overlap of the limiting cases – *inner layer* ($\eta \to 0$) and *outer layer* ($y^+ \gg 1$).

2.3 Inner Layer (Prandtl, 1925)

Dimensional parameters U, y, τ_w, ρ, ν – but *not* δ.

Dimensional analysis (5 parameters, 3 independent dimensions) $\Rightarrow$ 2 independent dimensionless groups, conveniently taken as $U^+ = U / u_\tau$ and $y^+ = u_\tau y / \nu$.

Then we have the *law of the wall*:

$$U^+ = f_w(y^+) \qquad (10)$$

f_w is expected to be a universal function; i.e. independent of the external flow.

According to Pope (2000), the inner layer corresponds roughly to $y / \delta < 0.1$, or the region over which the shear stress is approximately constant.

2.4 Outer Layer (Von Kármán, 1930)

Dimensional parameters U, y, τ_w, ρ, δ – but *not* ν.

Dimensional analysis (5 parameters, 3 independent dimensions) $\Rightarrow$ 2 independent dimensionless groups, conveniently taken as

$$\frac{U_e - U}{u_\tau}, \qquad \eta = \frac{y}{\delta}$$

Then one has the *velocity-defect law*:

$$\frac{U_e - U}{u_\tau} = f_o(\eta) \tag{11}$$

Unlike f_w which is expected to be universal, $f_o(\eta)$ will vary with the particular flow.

2.5 Overlap Layer – the Log Law

As noted by C.B. Millikan (1937) the inner and outer layers can only overlap smoothly if the overlap-region velocity profile is logarithmic.

Outer layer: $U_e^+ - U^+ = f_o(\eta)$

Inner layer: $U^+ = f_w(y^+)$

Introducing $\delta^+ = \delta u_\tau / \nu$, so that $y^+ = \eta\delta^+$, and adding:

$$U_e^+(\delta^+) = f_o(\eta) + f_w(\eta\delta^+)$$

For a function f_w of the *product* $\eta\delta^+$ to be the sum of separate functions of η and δ^+, f_w must be logarithmic. This can be proved formally by differentiating successively with respect to each variable, as follows.

Differentiate wrt δ^+:

$$U_e^{+\prime}(\delta^+) = 0 + \eta f_w'(\eta\delta^+)$$

Differentiate wrt η:

$$\begin{aligned} 0 &= f_w'(\eta\delta^+) + \eta\delta^+ f_w''(\eta\delta^+) \\ &= f_w'(y^+) + y^+ f_w''(y^+) \\ &= \frac{\mathrm{d}}{\mathrm{d}y^+}\left(y^+ \frac{\mathrm{d}f_w}{\mathrm{d}y^+}\right) \end{aligned}$$

Hence,

$$y^+ \frac{\mathrm{d}f_w}{\mathrm{d}y^+} = \text{constant}$$

This constant is conventionally written as $1/\kappa$, where κ (≈ 0.41), is *von Kármán's constant*.

$$\frac{\mathrm{d}f_w}{\mathrm{d}y^+} = \frac{1}{\kappa y^+}$$

which integrates to give

$$f_w = \frac{1}{\kappa}\ln y^+ + B, \qquad B \text{ another constant.}$$

Hence we have the *log-law velocity profile*:

$$U^+ = \frac{1}{\kappa}\ln y^+ + B \qquad (12)$$

or, equivalently,

$$U^+ = \frac{1}{\kappa}\ln Ey^+ \qquad (13)$$

Notes.

(1) There is some variation between sources, but typical values of the constants are $\kappa = 0.41$ ($1/\kappa = 2.44$) and $B = 5.0$ ($E = 7.76$).

(2) Except in strong adverse pressure gradients (e.g. in a diffuser) the logarithmic velocity profile is a good approximation across much of the shear layer. This observation turns out to extremely useful in deriving friction formulae – see Section 3.

(3) In the log law region,

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y} \qquad \text{or} \qquad \frac{y}{u_\tau}\frac{\partial U}{\partial y} = y^+\frac{\partial U^+}{\partial y^+} = constant$$

This is often used as an alternative starting point for the derivation of the log law.

2.6 Viscous Sublayer

Very close to the wall, turbulent fluctuations are damped out and the wall shear stress is almost entirely viscous:

$$\mu\frac{\partial U}{\partial y} = \tau_w, \quad \text{constant}$$

which yields a linear velocity profile:

$$U = \frac{\tau_w}{\mu} y$$

Setting $\tau_w = \rho u_\tau^2$ and rearranging,

$$U^+ = y^+ \qquad (14)$$

Experiment shows that the linear viscous sublayer corresponds roughly to $y^+ < 5$.

2.7 Limits of the Various Regions

Pope (2000) gives the following rough delimiting y^+ and y/δ values.

Inner layer (roughly $y/\delta < 0.1$) – velocity scales on u_τ and y^+, but not on δ.

Outer layer (roughly $y^+ > 50$) – the direct effect of viscosity is negligible.

Overlap region - exists at sufficiently high Reynolds number.
In the overlap region the mean-velocity profile must be logarithmic. In fact the log law is a good approximation beyond the overlap region. Pope suggests:

Viscous sublayer: $y^+ < 5$ – linear velocity profile
Buffer layer: $5 < y^+ < 30$
Log law region: $y^+ > 30$, $y/\delta < 0.3$ – logarithmic velocity profile

2.8 Velocity-Defect Layer: Coles' Law of the Wake

In the outer layer the velocity profile deviates slightly from the log law, particularly in non-equilibrium boundary layers with a pressure gradient. Coles (1956) noted that the *deviation* or excess velocity above the log law had a wake-like shape relative to the free stream; i.e.

$$U = U_{\text{log law}} + \Delta U \; f(\frac{y}{\delta})$$

where f is some S-shaped function with $f(0) = 0$, $f(1) = 1$; popular forms are

$$f(\eta) = \sin^2 \frac{\pi}{2}\eta$$

$$f(\eta) = 3\eta^2 - 2\eta^3$$

Then we have the Coles Law of the Wake:

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln y^+ + B + \frac{2\Pi}{\kappa} f(y/\delta) \qquad (15)$$

where the deviation from the log law is quantified by the *Coles wake parameter* Π.

Typical values are:

pipe flow or channel flow:	$\Pi = 0$
zero-pressure-gradient flat-plate boundary layer:	$\Pi = 0.45$

In general, Π is a function of pressure gradient.

2.9 Effect of Roughness

The seminal experimental work was done by Prandtl's PhD student Johann Nikuradse, who measured the friction factor in pipes artificially roughened with densely-packed sand grains of size k_s. The *relative roughness* k_s/D varied from 1/30 to 1/1000.

The influence of wall roughness is characterised by $k_s^+ = u_\tau k_s/\nu$.

Hydraulically Smooth: ($k_s^+ < 5$; i.e. less than the viscous sublayer depth)
In this regime roughness has no effect on the friction factor or mean-velocity profile.

Fully Rough: ($k_s^+ > 70$)
Transfer of momentum to the wall is predominantly by pressure drag on roughness elements, not viscous stresses, and wall friction becomes essentially independent of Reynolds number for sufficiently large Re. Dimensional analysis implies

$$U^+ = \frac{1}{\kappa}\ln\frac{y}{k_s} + B_k$$

From experimental data, $B_k \approx 8.5$.

Transitional Roughness ($5 < k_s^+ < 70$)
Both roughness and viscous effects operate.

(These k_s^+ limits are those of Schlichting. White gives 4 and 60 instead, whilst Cebeci and Bradshaw's transition formula below uses 2.25 and 90.)

An all-encompassing mean-velocity profile may be written

$$U^+ = \frac{1}{\kappa}\ln y^+ + \tilde{B}(k_s^+)$$

where

$$\tilde{B} \quad \rightarrow \quad \begin{cases} B & (k_s^+ \rightarrow 0;\ \text{hydraulically smooth}) \\ B_k - \frac{1}{\kappa}\ln k_s^+ & (k_s^+ \rightarrow \infty;\ \text{fully rough}) \end{cases}$$

Suitable interpolation formulae are:
Cebeci and Bradshaw (1977):

$$\tilde{B} = (1-\alpha)B + \alpha(B_k - \frac{1}{\kappa}\ln k_s^+)\ , \qquad \alpha = \begin{cases} 0\,, & k_s^+ < 2.25 \\ \sin\left[\frac{\pi}{2}\frac{\ln(k_s^+/2.25)}{\ln(90/2.25)}\right], & 2.25 < k_s^+ < 90 \\ 1\,, & k_s^+ > 90 \end{cases}$$

Apsley (2007):

$$\tilde{B} = B_k - \frac{1}{\kappa}\ln(k_s^+ + C)\,, \qquad C = e^{\kappa(B_k - B)}$$

(Both authors used slightly different values of B and B_k from those used in these Notes).

In practice, we are often more interested in the resulting *friction law* (see Section 3). For pipe flow this is the *Colebrook-White formula*. The effect of surface roughness depends on its form as well as its size. The work of Colebrook (1939) and Moody (1944) helped to define "equivalent sand roughness" for many commercial pipe materials.

Geophysical Flows

Perhaps the ultimate in rough-wall boundary layers is the atmospheric boundary layer. In this case the mean velocity profile is typically written (with the meteorological convention of z for a vertical coordinate):

$$U = \frac{u_\tau}{\kappa}\ln(\frac{z}{z_0}) \qquad (16)$$

z_0 is called the *roughness length* and comparison with the above formulae, fitting all constants inside the natural logarithm and taking $B_k = 8.5$, gives $z_0 \approx k_s/30$. For typical rural conditions z_0 has a value of about 0.1 m.

Examples

Question 1.
Consider airflow at 10 m s^{-1} over a flat plate. If the friction Reynolds number is 1200, calculate (a) the friction velocity; (b) the wall shear stress; (c) the depth of the boundary layer. Assume a Coles wake parameter $\Pi = 0.45$.

Question 2.
Wind velocities over open fields were measured as 5.89 m s^{-1} and 8.83 m s^{-1} at heights of 2 m and 10 m respectively. Use this data to estimate: (a) the roughness length z_0; (b) the friction velocity u_τ; (c) the velocity at height 25 m; (d) the average velocity over a depth of 25 m.

Question 3. (From White, 1994)
J. Laufer's (1954) pipe-flow experiments gave the following data at $Re_D \approx 5\times10^5$

r/R	0.0	0.102	0.206	0.412	0.617	0.784	0.846	0.907	0.963
U/U_0	1.0	0.997	0.988	0.959	0.908	0.847	0.818	0.771	0.690

where U_0 is the centreline velocity. Find the best-fit power-law profile of the form

$$\frac{U}{U_0} = \left(\frac{y}{R}\right)^{1/n}$$

where $y = R - r$ is the distance from the wall.

Answers

(1) (a) $u_\tau = 0.41$ m s^{-1}
(b) $\tau_w = 0.20$ N m^{-2}
(c) $\delta = 44$ mm

(2) (a) $z_0 = 0.080$ m
(b) $u_\tau = 0.75$ m s^{-1}
(c) $U(z = 25\text{ m}) = 10.5$ m s^{-1}
(d) $U_{av} = 8.7$ m s^{-1}

(3) $n = 9$