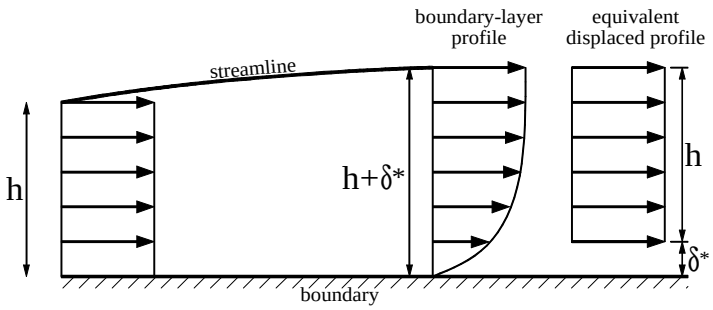

The integral analysis of boundary layers was originally proposed by Theodore von Kármán and K. Pohlhausen in separate papers in 1921.

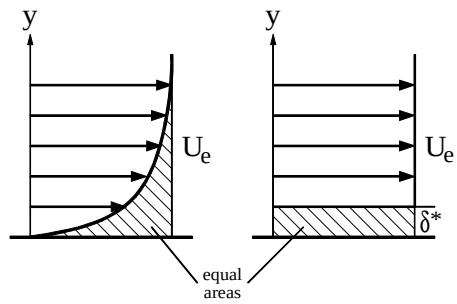
4.1 Characteristic Integral Lengths



Displacement Thickness

The flow near the surface is retarded, so that the streamlines must be displaced outwards to satisfy continuity. To reduce the total mass flow rate of a frictionless fluid by the same amount, the surface would have to be displaced outward by a distance δ*, called the *displacement thickness*:

$$\rho_e U_e \delta^* = \int_0^\infty (\rho_e U_e - \rho U)\ \mathrm{d}y \quad = \textit{mass flux deficit}$$



Momentum Thickness

The loss of momentum flux for the mass flux $\rho U\,\mathrm{d}y$ between adjacent streamlines is $(U_e - U)\times\rho U\ \mathrm{d}y$. Hence the total loss of momentum flux is equivalent to the removal of momentum through a distance θ, called the *momentum thickness*:

$$\rho_e U_e^2 \theta = \int_0^\infty (U_e - U)\,\rho U\ \ \mathrm{d}y \quad = \textit{momentum flux deficit}$$

For incompressible flow (with uniform free-stream density) these take particularly simple forms:

Displacement thickness: $$\delta^* = \int_0^\infty (1 - \frac{U}{U_e})\ \mathrm{d}y \qquad (1)$$

Momentum thickness: $$\theta = \int_0^\infty \frac{U}{U_e}(1 - \frac{U}{U_e})\ \mathrm{d}y \qquad (2)$$

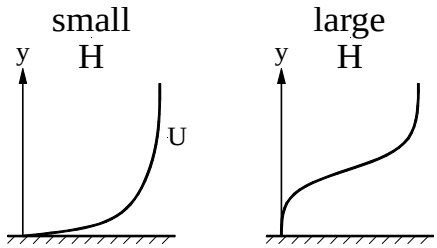
Shape Factor

The ratio of displacement thickness to momentum thickness is called the *shape factor H*:

Shape factor:	$H \equiv \dfrac{\delta^*}{\theta}$	(3)

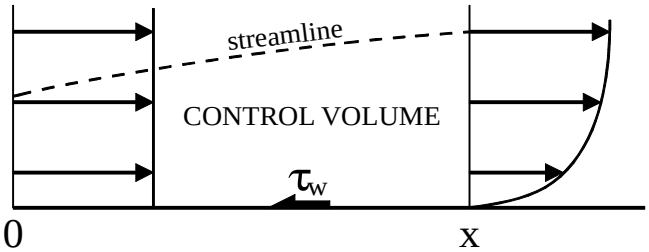
Since $1-\dfrac{U}{U_e} > \dfrac{U}{U_e}(1-\dfrac{U}{U_e})$ the shape factor is always greater than 1.

A large shape factor is an indicator of a boundary layer near separation.



4.2 The Momentum-Integral Relation

4.2.1 Zero-Pressure-Gradient Boundary Layer



For a *zero-pressure-gradient* boundary layer (ρ_e and U_e constant) the *rate of loss* of momentum equals the total drag on the surface up to that point; i.e., per unit width:

$$\underbrace{\rho_e U_e^2 \theta}_{\text{momentum flux deficit}} = \underbrace{D(x)}_{\text{total drag force}} = \int_0^x \tau_w \, \mathrm{d}x$$

Differentiating:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho_e U_e^2 \theta) = \tau_w$$

Dividing by $\rho_e U_e^2$ we can also express it in a simpler non-dimensional form:

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{c_f}{2}$$

Integrating, gives the *total* drag coefficient (= average skin-friction coefficient) over length L:

$$c_D = \frac{2\theta(L)}{L}$$

The drag coefficient can then be deduced entirely from the downstream velocity profile.

4.2.2 General Case

The following derivation applies to a steady, compressible or incompressible, 2-dimensional boundary layer with arbitrary free-stream pressure gradient.

For complete generality we will allow for a wall transpiration velocity V_w as well as a wall shear stress τ_w.

Start with the integral momentum flux deficit:

$$\rho_e U_e^2 \theta = \int_0^\infty (U_e - U)\, \rho U \quad \mathrm{d}y$$

Differentiate with respect to x, taking the differential operator through the integral sign and using the product rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho_e U_e^2 \theta) = \int_0^\infty \frac{\partial}{\partial x}\{\rho U (U_e - U)\}\, \mathrm{d}y$$

$$= \int_0^\infty \left\{ \underline{\frac{\partial(\rho U)}{\partial x}}(U_e - U) + \rho U \frac{\mathrm{d}U_e}{\mathrm{d}x} - \underline{\rho U \frac{\partial U}{\partial x}} \right\} \mathrm{d}y$$

In the first underlined term substitute from the continuity equation:

$$\frac{\partial(\rho U)}{\partial x} = -\frac{\partial(\rho V)}{\partial y}$$

In the second underlined term substitute from the boundary-layer momentum equation:

$$\rho U \frac{\partial U}{\partial x} = -\rho V \frac{\partial U}{\partial y} + \rho_e U_e \frac{\mathrm{d}U_e}{\mathrm{d}x} + \frac{\partial \tau}{\partial y}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho_e U_e^2 \theta) = \int_0^\infty \left\{ -\frac{\partial(\rho V)}{\partial y}(U_e - U) + \rho U \frac{\mathrm{d}U_e}{\mathrm{d}x} + \rho V \frac{\partial U}{\partial y} - \rho_e U_e \frac{\mathrm{d}U_e}{\mathrm{d}x} - \frac{\partial \tau}{\partial y} \right\} \mathrm{d}y$$

$$= \int_0^\infty \left\{ -\frac{\partial(\rho V)}{\partial y}(U_e - U) + \rho V \frac{\partial U}{\partial y} - (\rho_e U_e - \rho U)\frac{\mathrm{d}U_e}{\mathrm{d}x} - \frac{\partial \tau}{\partial y} \right\} \mathrm{d}y$$

Collecting y derivatives together, and simplifying using the product rule and the fact that U_e is independent of y,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho_e U_e^2 \theta) = -\int_0^\infty \frac{\partial}{\partial y}\{\rho V (U_e - U) + \tau\}\, \mathrm{d}y \quad - \frac{\mathrm{d}U_e}{\mathrm{d}x}\int_0^\infty (\rho_e U_e - \rho U) \quad \mathrm{d}y$$

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\rho_e U_e^2 \theta) = -\Big[\rho V (U_e - U) + \tau\Big]_0^\infty - \frac{\mathrm{d}U_e}{\mathrm{d}x}(\rho_e U_e \delta^*)$$

Since $U_e - U$ and τ both vanish at $y = \infty$, whilst $U = 0$, $V = V_w$ and $\tau = \tau_w$ at $y = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho_e U_e^2 \theta) = \rho_w V_w U_e + \tau_w - \frac{\mathrm{d}U_e}{\mathrm{d}x}(\rho_e U_e \delta^*)$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho_e U_e^2 \theta) + \rho_e U_e \frac{\mathrm{d}U_e}{\mathrm{d}x}\delta^* = \tau_w + \rho_w V_w U_e$$

Kármán's integral relation

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}(\rho_e U_e^2\theta)}_{\substack{\text{rate of loss}\\ \text{of momentum}}} + \underbrace{\rho_e U_e \frac{\mathrm{d}U_e}{\mathrm{d}x}\delta^*}_{\substack{\text{effect of free-stream}\\ \text{pressure gradient}}} = \underbrace{\tau_w}_{\substack{\text{wall}\\ \text{stress}}} + \underbrace{\rho_w V_w U_e}_{\substack{\text{wall}\\ \text{transpiration}}} \tag{4}$$

For constant-density flow the most compact form comes from expanding the first differential and dividing by ρU_e^2:

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} + (2+H)\frac{\theta}{U_e}\frac{\mathrm{d}U_e}{\mathrm{d}x} = \frac{c_f}{2} + \frac{V_w}{U_e} \tag{5}$$

where

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_e^2} \qquad \textit{skin-friction coefficient}$$

$$H = \frac{\delta^*}{\theta} \qquad \textit{shape factor}$$

Notes.

(1) The results also hold along a curved-wall boundary layer, provided that the radius of curvature is much greater than the thickness of the boundary layer.

(2) The integral relations hold whether the boundary layer is laminar, turbulent or transitional.

(3) The integral relation is exact only for the correct mean velocity profile. However, the power of the scheme is that it provides an excellent estimate of drag with only a half-decent approximation for $U(y)$.

(4) For a flat-plate boundary layer with zero pressure gradient and no wall transpiration, Kármán's integral relation takes the particularly simple form:

Zero pressure-gradient boundary layer:

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{c_f}{2} \tag{6}$$

In this case the total-drag coefficient $c_D(L) = \dfrac{D(L)}{\frac{1}{2}\rho U_e^2 L}$ where $D(L) = \displaystyle\int_0^L \tau_w \, \mathrm{d}x$ is given by

$$c_D(L) = 2\frac{\theta(L)}{L} \tag{7}$$

The total drag is entirely determined by the downstream momentum thickness.

(5) The shape factor H is only relevant when there is a free-stream pressure gradient ($\mathrm{d}U_e/\mathrm{d}x \neq 0$). A large shape factor indicates a boundary layer approaching separation.

(6) An alternative proof of Kármán's integral relation – starting from the differential boundary-layer equations – is given in the Examples Section.

4.3 Application to Laminar Flow

A fully-developed profile may be approximated by:

$$U = \begin{cases} U_e f(\frac{y}{\delta}), & y \le \delta \\ U_e & y > \delta \end{cases} \tag{8}$$

where $f(0) = 0$, $f'(0) \neq 0$ or ∞, $f(1) = 1$, $f'(1) = 0$. (Why is the second condition important?)

The integral boundary-layer depths are proportional to δ:

$$\delta^* = A\delta, \qquad \theta = B\delta \tag{9}$$

and the skin-friction coefficient

$$c_f = \frac{\mu \left.\frac{\mathrm{d}U}{\mathrm{d}y}\right|_{y=0}}{\frac{1}{2}\rho U_e^2} = \frac{2\mu f'(0)}{\rho U_e \delta} = \frac{C}{\mathrm{Re}_\delta} \tag{10}$$

where A, B and C are constants determined by the particular profile adopted:

$$A = \int_0^1 (1 - f)\ \mathrm{d}\eta, \qquad B = \int_0^1 f(1 - f)\ \mathrm{d}\eta, \qquad C = 2f'(0) \tag{11}$$

Substituting the expressions for θ and c_f into the integral relation:

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{c_f}{2} \quad \Rightarrow \quad B\frac{\mathrm{d}\delta}{\mathrm{d}x} = \frac{C\nu}{2U_e\delta} \quad \Rightarrow \quad \frac{\mathrm{d}\delta^2}{\mathrm{d}x} = \frac{C}{B}\left(\frac{\nu}{U_e}\right)$$

Hence:

$$\delta = \left(\frac{C}{B}\right)^{1/2}\left(\frac{\nu x}{U_e}\right)^{1/2} \qquad \text{or} \qquad \frac{\delta}{x} = \left(\frac{C}{B}\right)^{1/2} \mathrm{Re}_x^{-1/2} \tag{12}$$

(x is measured from some virtual origin; in practice, this can be taken as the leading edge).

Since $\theta \propto x^{1/2}$ we have $\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{2}\frac{\theta}{x}$ and hence

$$c_f = 2\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\theta}{x}$$

Summary of Results For a Zero-Pressure-Gradient Laminar Boundary Layer

$$\delta^* = A\delta, \qquad \theta = B\delta, \qquad H = \frac{A}{B}$$

$$c_f = \frac{\theta}{x}, \qquad c_D(L) = 2c_f(L)$$

where

$$A = \int_0^1 (1 - f)\ \mathrm{d}\eta, \qquad B = \int_0^1 f(1 - f)\ \mathrm{d}\eta, \qquad C = 2f'(0)$$

$$\delta = \left(\frac{C}{B}\right)^{1/2}\sqrt{\frac{\nu x}{U_e}}$$

Thus, assuming only that the profile is *self-similar*, but without any knowledge about its precise form, we can confirm that

$$\delta \propto x^{1/2} \quad \left(\text{or} \quad \frac{\delta}{x} \propto \mathrm{Re}_x^{-1/2}\right) \qquad \text{and} \qquad c_D(L) \propto \mathrm{Re}_L^{-1/2}$$

The simplest profile satisfying

$$U = 0 \qquad \text{on } y = 0$$

$$U = U_e, \ \frac{\mathrm{d}U}{\mathrm{d}y} = 0 \quad \text{on } y = \delta$$

is (the channel-flow profile)

$$U = U_e \frac{y}{\delta}\left(2 - \frac{y}{\delta}\right) \qquad \text{or} \qquad f(\eta) = 2\eta - \eta^2$$

This has $A = \frac{1}{3}$, $B = \frac{2}{15}$, $C = 4$ and gives the following approximations.

	Approximation	Blasius solution
δ^*	$1.83\sqrt{\frac{\nu x}{U_e}}$	$1.72\sqrt{\frac{\nu x}{U_e}}$
θ	$0.730\sqrt{\frac{\nu x}{U_e}}$	$0.664\sqrt{\frac{\nu x}{U_e}}$
H	2.5	2.59

In both cases,

$$c_f = \frac{\theta}{x}$$

$$c_D(L) = 2c_f(L)$$

Thus, the drag is correct to within 10% and the shape factor to within 3.5%, even for a simple approximation. Much better approximations are given in the Examples section.

4.4 Application to Turbulent Flow

The momentum integral relation for a zero-pressure-gradient boundary layer is

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{c_f}{2}$$

To make this tractable we adopt the power-law approximations that we have met before:

Velocity profile: $$\frac{U}{U_e} = (\frac{y}{\delta})^{1/7} \tag{13}$$

Skin-friction coefficient: $$c_f = 0.0205\,\mathrm{Re}_\delta^{-1/6} \quad (\text{where } \mathrm{Re}_\delta = \frac{U_e\delta}{\nu}) \tag{14}$$

From the first of these we can compute displacement and momentum thickness:

$$\frac{U}{U_e} = \eta^{1/7}, \qquad \eta = y/\delta$$

Displacement thickness:

$$\delta^* \equiv \int_0^\delta (1-\frac{U}{U_e})\,\mathrm{d}y \quad = \delta\int_0^1 (1-\eta^{1/7})\;\;\mathrm{d}\eta \quad = \frac{1}{8}\delta \tag{15}$$

Momentum thickness:

$$\theta \equiv \int_0^\delta \frac{U}{U_e}(1-\frac{U}{U_e})\,\mathrm{d}y \quad = \delta\int_0^1 \eta^{1/7}(1-\eta^{1/7})\;\;\mathrm{d}\eta \quad = \frac{7}{72}\delta \tag{16}$$

Shape factor:

$$H \equiv \frac{\delta^*}{\theta} \quad = \frac{9}{7} \tag{17}$$

Substituting for skin-friction coefficient c_f and momentum thickness θ in the momentum integral relation,

$$\frac{7}{72}\frac{\mathrm{d}\delta}{\mathrm{d}x} = 0.01025\,(\frac{\nu}{U_e\delta})^{1/6}$$

$$\Rightarrow \qquad \frac{7}{72}\delta^{1/6}\;\mathrm{d}\delta = 0.01025\,(\frac{\nu}{U_e})^{1/6}\;\mathrm{d}x$$

$$\Rightarrow \qquad \frac{7}{72}\times\frac{6}{7}\delta^{7/6} = 0.01025\,(\frac{\nu}{U_e})^{1/6}\,x$$

$$\Rightarrow \qquad (\frac{\delta}{x})^{7/6} = 0.123\,(\frac{\nu}{U_e x})^{1/6}$$

Hence

$$\frac{\delta}{x} = 0.166\,\mathrm{Re}_x^{-1/7} \qquad \text{or} \qquad \mathrm{Re}_\delta = 0.166\,\mathrm{Re}_x^{6/7} \tag{18}$$

Thus, the boundary layer grows as

$$\delta \propto x^{6/7}$$

Since $\theta \propto x^{6/7}$ we have $\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{6}{7}\frac{\theta}{x} = \frac{1}{12}\frac{\delta}{x}$, and hence:

$$c_f = 2\frac{\mathrm{d}\theta}{\mathrm{d}x} = 2\times\frac{1}{12}\frac{\delta}{x} = 0.0277\,\mathrm{Re}_x^{-1/7} \tag{19}$$

(Alternatively, use $c_f = 0.0205\,\mathrm{Re}_\delta^{-1/6}$ and substitute for Re_δ.)

$$c_D(L) = \frac{2\theta}{L} = \frac{7}{6}c_f(L) = 0.032\,\mathrm{Re}_L^{-1/7} \tag{20}$$

i.e. the total drag coefficient is only 7/6 times the local skin friction coefficient at the end of the plate.

Summary of Results for a Zero-Pressure-Gradient Turbulent Boundary Layer

Boundary-layer depth:

$$\delta^* = \frac{1}{8}\delta, \qquad \theta = \frac{7}{72}\delta, \qquad H = \frac{9}{7}$$

where

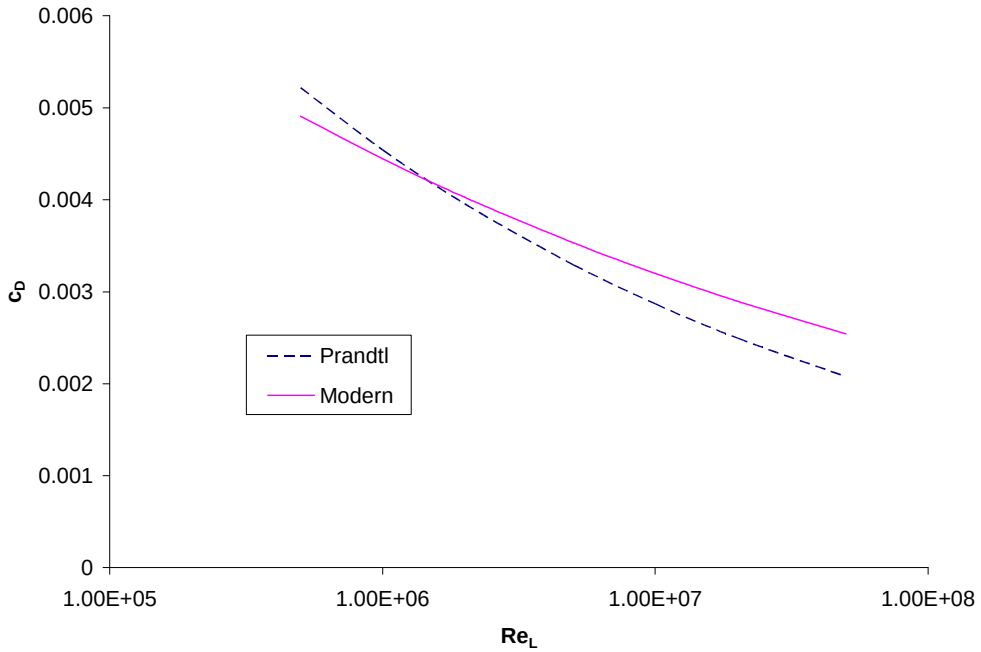
$$\frac{\delta}{x} = 0.166\,\mathrm{Re}_x^{-1/7} \qquad \text{or} \qquad \delta \propto x^{6/7}$$

Frictional drag:

$$c_f = 0.0277\,\mathrm{Re}_x^{-1/7}$$

$$c_D(L) = \frac{7}{6}c_f(L) = 0.032\,\mathrm{Re}_L^{-1/7}$$

Note. An earlier and alternative correlation by Prandtl (see Q3 in the Examples) and based on pipe-flow data gave a skin friction coefficient proportional to $x^{-1/5}$ and resulted in a drag coefficient $c_D(L) = 0.072\,\mathrm{Re}_L^{-1/5}$. This is still widely used today.



Examples

Question 1.
Show that, for a power-law velocity profile of the form

$$\frac{U}{U_e} = \left(\frac{y}{\delta}\right)^{1/n} \qquad (y < \delta)$$

the displacement thickness, momentum thickness and shape factor are given by

$$\delta^* = \frac{1}{n+1}\delta, \qquad \theta = \frac{n}{(n+1)(n+2)}\delta, \qquad H = \frac{n+2}{n}$$

Question 2.
Laminar boundary layer profiles may be assumed to be approximately of the form

$$U = U_e f(\eta), \qquad \eta = \frac{y}{\delta}$$

where $f(0) = 0, \quad f(1) = 1, \quad f'(1) = 0$. Use an integral analysis with the following profiles to find expressions for δ^*, θ, H, c_f and c_D and compare with Blasius' solution.

(a) $\frac{3}{2}\eta - \frac{1}{2}\eta^3$

(b) $2\eta - 2\eta^3 + \eta^4$

(c) $\sin(\frac{\pi}{2}\eta)$

Question 3. (White)
An alternative (but less accurate) analysis of turbulent flat-plate flow was given by Prandtl in 1927, using a wall shear-stress formula from *pipe flow*:

$$\tau_w = 0.0225\rho U_e^2 \left(\frac{\nu}{U_e\delta}\right)^{1/4}$$

Show that an integral analysis with this formula and a 1/7 power law profile for velocity leads to the following relations for turbulent flat-plate flow:

$$\frac{\delta}{x} = \frac{0.37}{\mathrm{Re}_x^{1/5}}, \qquad c_f = \frac{0.058}{\mathrm{Re}_x^{1/5}}, \qquad c_D(L) = \frac{0.072}{\mathrm{Re}_L^{1/5}}$$

Question 4.
Show that, in general, if the skin-friction coefficient and momentum thickness for a zero-pressure-gradient flat-plate boundary layer are related by

$$c_f = A\mathrm{Re}_\theta^{-1/n}$$

then

$$\frac{\theta}{x} = \left[(\frac{n+1}{n})\frac{A}{2}\right]^{\frac{n}{n+1}} \mathrm{Re}_x^{-\frac{1}{n+1}}, \qquad c_f = \frac{2n}{n+1}\frac{\theta}{x}, \qquad c_D(L) = (\frac{n+1}{n})c_f(L)$$

Use this to confirm the results in Section 4.4 and Question 3 above.

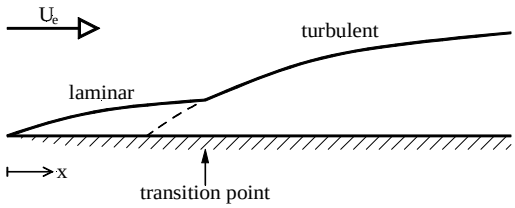
Question 5.
Why would a simple power law be a bad approximation to a laminar boundary-layer velocity profile?

Question 6.
For purely laminar or purely turbulent flat-plate boundary layers the momentum thickness θ at distance x from the leading edge is given by:

$$\frac{\theta}{x} = \begin{cases} 0.664\,\mathrm{Re}_x^{-1/2}, & \text{(laminar)} \\ 0.0158\,\mathrm{Re}_x^{-1/7}, & \text{(turbulent)} \end{cases}$$

where $\mathrm{Re}_x = U_e x/\nu$ and U_e = free-stream velocity, ν = kinematic viscosity.

In practice, a turbulent boundary layer is preceded by an initial laminar region. Since $\mathrm{d}\theta/\mathrm{d}x = \frac{1}{2}c_f$ and the skin-friction coefficient c_f is finite, the momentum thickness must be continuous at the transition point.



Irrespective of any intermediate development, the total drag coefficient for one side of a plate of length L depends only upon the momentum thickness at the end:

$$c_D(L) = 2\frac{\theta(L)}{L}$$

(a) Develop a procedure, *illustrated by a flowchart showing calculation steps, or by a computer program*, to find the momentum thickness at the end of, and total drag coefficient for, a flat plate of *arbitrary* length L. ($\mathrm{Re}_{x,tr}$ is assumed to be given.)

Using this procedure, find:

(b) the transition length;
(c) the momentum thickness at the downstream end;
(d) the drag coefficient c_D;
(e) the total drag on one side of a smooth rectangular plate of size 4 m × 2 m in an air flow of 8 m s^{-1} parallel to the long side.

Take $\mathrm{Re}_{x,tr} = 5\times10^5$ and, for air, ρ = 1.2 kg m^{-3}, ν = 1.5×10^{-5} m^2 s^{-1}.

Question 7.
A wind tunnel has a test section 0.5 m square and 6 m long. To preserve a constant free-stream velocity it is sometimes desirable to slant the walls outward.

(a) Explain the purpose of this wall adjustment.
(b) If only the roof is to be adjusted, estimate the angle at which it should be slanted in order to preserve a free-stream velocity of 40 m s^{-1} over a working section from x = 2 m to x = 4 m, stating any assumptions made.

Question 8.
A square-section duct of side 30 cm and length 12 m carries a throughflow of air at flow rate 3 m^3 s^{-1}. Assuming that turbulent boundary layers grow along the side walls from the front edge, estimate:

(a) the pressure drop along the duct;
(b) the drag on the side walls.

Question 9.
Starting from the boundary-layer equations in differential form:

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} = \rho_e U_e \frac{\mathrm{d}U_e}{\mathrm{d}x} + \frac{\partial \tau}{\partial y} \qquad \text{(momentum)}$$

$$\frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0 \qquad \text{(continuity)}$$

derive Kármán's integral relation.

Question 10.
A boundary-layer mean-velocity profile is approximated by

$$U = \begin{cases} \dfrac{u_\tau}{\kappa} \ln E \dfrac{u_\tau y}{\nu} & (y < \delta) \\ U_e & (y > \delta) \end{cases}$$

where U is continuous at $y = \delta$. Find δ^* and θ in terms of u_τ/U_e and δ.

Question 11.
Assume that the following mean-velocity profile holds across a turbulent boundary layer.

$$\frac{U}{u_\tau} = \frac{1}{\kappa} \ln\left(\frac{u_\tau y}{\nu}\right) + B + \frac{2\Pi}{\kappa} f\left(\frac{y}{\delta}\right) \qquad \text{where} \qquad f(\eta) = 3\eta^2 - 2\eta^3$$

(a) Find an expression for U/U_e as a function of η and the quantity γ, where

$$\gamma = \frac{u_\tau}{\kappa U_e}, \qquad U_e \equiv U(\delta)$$

(b) Hence, or otherwise, show that the displacement thickness δ^* is given by:

$$\frac{\delta^*}{\delta} = \gamma(1 + \Pi)$$

(c) Find a similar expression for θ/δ where θ is the momentum thickness.

Question 12.
In a 2-dimensional, incompressible boundary layer the *energy thickness* δ_E may be defined by the integral energy flux deficit:

$$U_e^3 \delta_E = \int_0^\infty U(U_e^2 - U^2) \; \mathrm{d}y$$

where U_e is the free-stream velocity and U is the wall-parallel mean-velocity component in the boundary layer.

Using the boundary-layer equations with shear stress τ, derive the following relation for incompressible boundary-layer flow over a plane surface with wall transpiration velocity V_w:

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_e^3 \delta_E) = 2\int_0^\infty \frac{\tau}{\rho} \frac{\partial U}{\partial y} \, \mathrm{d}y + V_w U_e^2$$

Give an interpretation of all the terms.

Answers

(2)

Profile	$\delta^* \sqrt{\frac{U_e}{\nu x}}$	$\theta \sqrt{\frac{U_e}{\nu x}}$	H
(a) $\frac{3}{2}\eta - \frac{1}{2}\eta^3$	$\frac{3}{4}\left(\frac{70}{13}\right)^{1/2} = 1.74$	$\frac{1}{2}\left(\frac{117}{70}\right)^{1/2} = 0.646$	$\frac{35}{13} = 2.69$
(b) $2\eta - 2\eta^3 + \eta^4$	$\frac{9}{5}\left(\frac{35}{37}\right)^{1/2} = 1.75$	$\frac{2}{3}\left(\frac{37}{35}\right)^{1/2} = 0.685$	$\frac{189}{74} = 2.55$
(c) $\sin(\frac{\pi}{2}\eta)$	$\frac{\pi - 2}{(2-\pi/2)^{1/2}} = 1.74$	$(2-\pi/2)^{1/2} = 0.655$	$\frac{\pi - 2}{2-\pi/2} = 2.66$
(d) Blasius	1.72	0.664	2.59

In all cases,

$$c_f = \frac{\theta}{x}, \qquad c_D(L) = 2c_f(L)$$

(6) (b) x_{tr} = 0.94 m
(c) $\theta_L = 6.8\times10^{-3}$ m
(d) c_D = 0.0034
(e) D = 1.05 N

(7) (b) 0.41 degrees

(8) (a) 560 Pa
(b) 56 N

(10) $\delta^* = \frac{u_\tau}{\kappa U_e}\delta$

$$\theta = \frac{u_\tau}{\kappa U_e}(1 - 2\frac{u_\tau}{\kappa U_e})\delta$$

(11) (a) $\frac{U}{U_e} = 1 + \gamma\{\ln\eta + 2\Pi(3\eta^2 - 2\eta^3 - 1)\}$

(c) $\frac{\theta}{\delta} = \gamma(1+\Pi) - \gamma^2\left(2 + \frac{19}{6}\Pi + \frac{52}{35}\Pi^2\right)$