7. HEAT TRANSFER SPRING 2009

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7.1 Definitions

Heat transfer is energy in transit due to a temperature difference.
The *heat flux* $\mathbf{q}_h$ is the rate of heat transfer per unit area.

7.2 Modes of Heat Transfer

7.2.1 Conduction

Heat transfer due to molecular activity in the absence of a bulk motion.

Fourier's Law

$$\mathbf{q}_h = -k_h \nabla T \tag{1}$$

k_h is the *conductivity of heat*.

At 20 ºC, k_h(water) = 0.604 W m^{-1} K^{-1}; k_h(air) = 0.0257 W m^{-1} K^{-1}.

7.2.2 Advection

Heat transfer due to bulk motion of fluid.

$$\mathbf{q}_h = \rho c_p (T - T_e)\mathbf{u} \tag{2}$$

c_p is the *specific heat capacity* (at constant pressure). T_e is a reference (usually the external or free-stream) temperature.

At 20 ºC, c_p(water) = 4182 J kg^{-1} K^{-1}; c_p(air) = 1007 J kg^{-1} K^{-1}.

7.2.3 Radiation

Heat transfer by emission of electromagnetic radiation.

Stefan-Boltzmann Law

$$\mathbf{q}_h = \varepsilon\sigma T_s^4 \tag{3}$$

σ (= 5.670×10^{-8} W m^{-2} K^{-4}) is the *Stefan-Boltzmann constant*.
ε is the *emissivity* (ε = 1 for a perfect black body)

7.2.4 Free and Forced Convection

For fluids, conduction and advection are usually combined as *convection*, which may be either:

forced convection	– flow driven by external means;
free (or *natural*) *convection*	– flow driven by buoyancy.

At low speeds, both are present and we refer to *mixed convection*.

Regardless of the convective heat-transfer mechanism it is common to write

Newton's law of cooling

$$q_h = h(T_s - T_e) \tag{4}$$

h is the *convective heat-transfer coefficient*; T_s is the surface temperature; T_e is the free-stream temperature (external flows) or bulk temperature (internal flows).

7.3 The Prandtl Number

The *Prandtl number*

$$\mathrm{Pr} = \frac{\nu}{\kappa_h} = \frac{\mu c_p}{k_h} \tag{5}$$

is the ratio of momentum diffusivity (ν) to heat diffusivity ($\kappa_h = k_h/\rho c_p$). It is a measure of the relative effectiveness of momentum and energy transport by diffusion in the velocity and thermal boundary layers. For air at normal temperatures, Pr = 0.71.

For a *laminar* boundary layer:

gases:	$\mathrm{Pr} \approx 1$	$\delta \approx \delta_T$
liquid metals:	$\mathrm{Pr} \ll 1$	$\delta \gg \delta_T$
oils:	$\mathrm{Pr} \gg 1$	$\delta \ll \delta_T$

where δ and δ_T are, repectively, the thicknesses of momentum and thermal boundary layers. In general, $\delta/\delta_T = f(\mathrm{Pr})$.

Note that many fluid properties, notably viscosity, vary substantially with temperature. In boundary-layer calculations it is common to evaluate them at the *film temperature* T_f, the average of the free-stream and surface temperatures:

$$T_f = \tfrac{1}{2}(T_s + T_e) \tag{6}$$

7.4 Dimensionless Numbers in Free and Forced Convection

7.4.1 Free Convection

Thermal expansion coefficient $\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}$. The buoyancy force per unit mass is $\beta g \Delta T$.

Grashof number: $\mathrm{Gr} = \frac{\beta g \Delta T\, L^3}{\nu^2}$ (7)

Ratio of buoyancy to viscous forces. Its role in free convection is much the same as that of the Reynolds number in forced convection.

Rayleigh number: $\mathrm{Ra} = \frac{\beta g \Delta T\, L^3}{\nu \kappa_h} = \mathrm{Gr.Pr}$ (8)

7.4.2 Forced Convection

Nusselt number: $\mathrm{Nu} = \frac{q_h L}{k_h \Delta T}$ (9)

Ratio of actual to conductive heat flux

Stanton number: $\mathrm{St} = \frac{q_h}{\rho U c_p \Delta T} = \frac{\mathrm{Nu}}{\mathrm{Re.Pr}}$ (10)

Ratio of actual to advective heat flux.

In all cases it is necessary to specify what length, velocity and temperature scales are being used. For calculations, engineers require c_f and Nu or St. By dimensional analysis,

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Re}, \mathrm{Pr})$$

7.5 The Energy Equation

The energy equation for fluid flow can be written in any of the convenient forms:

Internal energy: $\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p\nabla \bullet \mathbf{u} - \nabla \bullet \mathbf{q}_h + \Phi_h$ (11a)

Enthalpy ($h = e + \frac{p}{\rho}$): $\rho \frac{\mathrm{D}h}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} - \nabla \bullet \mathbf{q}_h + \Phi_h$ (11b)

Total enthalpy ($h_0 = e + \frac{p}{\rho} + \frac{1}{2}u^2$): $\rho \frac{\mathrm{D}h_0}{\mathrm{D}t} = \frac{\partial p}{\partial t} - \nabla \bullet \mathbf{q}_h + \frac{\partial}{\partial x_j}(u_i \tau_{ij})$ (11c)

The rate of transfer of mechanical to heat energy (i.e. the rate of working of viscous forces) is given by

$$\Phi_h = \frac{\partial u_i}{\partial x_j}\tau_{ij} = 2\mu(S_{ij} - \tfrac{1}{3}S_{kk}\delta_{ij})^2 \quad (12)$$

The LHS of any of the energy equations can also be written in *conservative form*; e.g.

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} \quad \rightarrow \quad \frac{\partial(\rho e)}{\partial t} + \nabla \bullet (\rho e \mathbf{u})$$

For ideal gases,

$$e = c_v T\,, \qquad h = c_p T \tag{13}$$

where c_v and c_p are the specific heat capacities at constant volume and constant pressure, respectively.

The *Eckert number*

$$\mathrm{Ec} = \frac{U^2}{c_p \Delta T} \tag{14}$$

is the ratio of kinetic energy to enthalpy change across a layer and is usually assumed small when heat transfer is of interest. At low speeds a change in energy or enthalpy is proportional to a change in temperature:

$$\Delta(enthalpy) = c_P \Delta T \qquad \text{(per unit mass)} \tag{15}$$

and hence the energy equation can be written as a temperature equation.

The *incompressible boundary-layer equations* for momentum and energy may be written:

$$\begin{aligned} \rho(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}) &= -\frac{\mathrm{d}P_e}{\mathrm{d}x} + \frac{\partial}{\partial y}(\mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'}) \\ \rho c_p (\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}) &= \frac{\partial}{\partial y}(k_h \frac{\partial \bar{T}}{\partial y} - \rho c_p \overline{v'T'}) \end{aligned} \tag{16}$$

The mean vertical heat flux is:

$$q_h = -k_h \frac{\partial \bar{T}}{\partial y} + \rho c_p \overline{v'T'} \tag{17}$$

The effect of turbulence is an additional *net* heat flux due to vertical migration of fluid elements carrying their own temperature or energy content.

7.6 Laminar Boundary Layers With Isothermal Walls

Substituting the Blasius solution into the temperature equation gives

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} + \frac{\mathrm{Pr}}{2} f \frac{\mathrm{d}\theta}{\mathrm{d}\eta} = 0 \tag{18}$$

where the boundary-layer coordinate η and dimensionless stream function $f(\eta)$ are:

$$\eta = y\sqrt{\frac{U_e}{\nu x}} \qquad \psi = U_e \sqrt{\frac{\nu x}{U_e}} f(\eta) \tag{19}$$

and θ is the dimensionless temperature defined by

$$\theta = \frac{T - T_e}{T_s - T_e}, \qquad \theta(0) = 1, \qquad \theta(\infty) = 0 \tag{20}$$

The temperature equation in boundary-layer coordinates is readily solved in terms of the Blasius function f (see Examples). The solution, which is clearly a function of Pr, can be obtained numerically.

To a good approximation,

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right|_{\eta=0} \approx -0.332\,\mathrm{Pr}^{1/3} \qquad (0.1 \le \mathrm{Pr} \le 10000)$$

and hence

$$\mathrm{Nu}_x \equiv \frac{q_w x}{k_h \Delta T} = -\mathrm{Re}_x^{1/2} \left.\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right|_{\eta=0} \approx 0.332\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3} \tag{21}$$

Comparing with the skin-friction result $c_f = 0.664\,\mathrm{Re}_x^{-1/2}$,

$$\mathrm{Nu}_x \approx \tfrac{1}{2} c_f\,\mathrm{Re}_x\,.\mathrm{Pr}^{1/3}$$

or

$$\boxed{\mathrm{St} \approx \frac{1}{\mathrm{Pr}^{2/3}} \frac{c_f}{2}} \tag{22}$$

This is sometimes called the *Chilton-Colburn formula*.

Since $q_w \propto x^{-1/2}$, integrating the heat flux over a finite length L and averaging gives

$$\overline{\mathrm{Nu}} = 2\mathrm{Nu}_x(x = L) \approx 0.664\,\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3} \tag{23}$$

(Again, compare the momentum result: $c_D(L) = 2c_f(L) = 1.328\,\mathrm{Re}_L^{-1/2}$.)

7.7 Temperature Profile in a Turbulent Boundary Layer

The effect of turbulence is an additional vertical heat flux $\rho c_p \overline{v'T'}$ due to the wall-normal fluctuations of particles carrying their temperature with them.

In turbulent forced convection one would expect the same eddying motions to be responsible for diffusing both heat and momentum so that

$$\delta \approx \delta_T$$

A Reynolds analogy leads to an *eddy-diffusivity* model:

momentum: $\quad -\overline{u'v'} = \nu_t \dfrac{\partial \bar{u}}{\partial y}$

heat: $\quad -\overline{v'T'} = \kappa_t \dfrac{\partial \bar{T}}{\partial y}$ (24)

where

$$\frac{\nu_t}{\kappa_t} = \mathrm{Pr}_t \qquad (\text{sometimes } \sigma_t) \tag{25}$$

is the *turbulent Prandtl number*. Pr_t is a function of the molecular Prandtl number Pr, but usually (except in liquid metals) $\mathrm{Pr}_t \approx 0.9 - 1.0$.

For momentum we have already defined a velocity scale u_* ($\equiv u_\tau$) from the wall shear stress:

$$\tau_w = \rho u_*^2 \tag{26}$$

In like manner, for heat transfer we can define a temperature scale T_* from the wall heat flux:

$$q_w = \rho c_p u_* T_* \tag{27}$$

(Note that T_* can have either sign.)

In wall units,

$$U^+ = \frac{\bar{u}}{u_*}, \qquad T^+ = \frac{T_w - \bar{T}}{T_*} \tag{28}$$

In the overlap layer dimensional analysis gives

$$\frac{y}{T_*}\frac{\partial \bar{T}}{\partial y} = y^+ \frac{\partial T^+}{\partial y^+} = constant$$

from which one may recover a logarithmic profile. The gradient-transfer hypothesis gives

$$q_h = -\rho c_p (\kappa_h + \kappa_t)\frac{\partial \bar{T}}{\partial y}$$

and, writing $q = q_w = \rho c_p u_* T_*$,

$$u_* T_* = -\left(\frac{\nu}{\mathrm{Pr}} + \frac{\nu_t}{\mathrm{Pr}_t}\right)\frac{\partial \bar{T}}{\partial y}$$

whence we have the *temperature law of the wall*

$$\frac{\partial T^+}{\partial y^+} = \frac{1}{\mathrm{Pr}^{-1} + \mathrm{Pr}_t^{-1}(\nu_t/\nu)} \qquad \text{or} \qquad T^+ = \int_0^{y^+} \frac{\mathrm{d}y^+}{\mathrm{Pr}^{-1} + \mathrm{Pr}_t^{-1}(\nu_t/\nu)} \tag{29}$$

In the near-wall limit ($\nu_t / \nu \to 0$),

$$T^+ = \mathrm{Pr}\, y^+ \tag{30}$$

In the overlap region ($\nu_t = \kappa u_* y \quad \Rightarrow \quad \nu_t/\nu = \kappa y^+ >> 1$),

$$\boxed{T^+ = \frac{\mathrm{Pr}}{\kappa}\ln y^+ + A(\mathrm{Pr}) \qquad (31)}$$

The function $A(\mathrm{Pr})$ is derived by integration of (29). An accepted curve fit for $\mathrm{Pr} \geq 0.7$ is

$$A(\mathrm{Pr}) = 13\,\mathrm{Pr}^{2/3} - 7 \tag{32}$$

but more complicated expressions have been given (see Kader, 1981).

7.8 Heat-Transfer Coefficients

We obtained skin-friction coefficients (defining *momentum* transfer) by assuming the logarithmic profile to apply across the shear layer. An exactly analogous technique can be used for heat transfer.

Note that the results for flat plate and pipe flows below are strictly for constant-temperature (isothermal) walls; varying wall temperatures may be treated by an integral analysis – see Section 7.8.

7.8.1 External Flow – Flat-Plate Boundary Layer

Applying temperature and velocity laws at the boundary layer depth:

$$\frac{T_w - T_e}{T_*} = \frac{\mathrm{Pr}_t}{\kappa} \ln \frac{\delta_T u_*}{\nu} + A \qquad (A = 13\,\mathrm{Pr}^{2/3} - 7)$$

$$\frac{U_e}{u_*} = \frac{1}{\kappa} \ln \frac{\delta u_*}{\nu} + B + \frac{2\Pi}{\kappa} \qquad (B = 5.0,\ \Pi = 0.45)$$

Subtracting, assuming $\mathrm{Pr}_t = 1$ and $\delta/\delta_T = 1$,

$$\frac{T_w - T_e}{T_*} - \frac{U_e}{u_*} = A - B - \frac{2\Pi}{\kappa}$$

Substituting for T_* in terms of the heat flux gives

$$\frac{\rho c_P U_e (T_w - T_e)}{q_w} \frac{u_*}{U_e} = A - B - \frac{2\Pi}{\kappa} + \frac{U_e}{u_*}$$

which can be rearranged to give

$$\mathrm{St} = \frac{c_f/2}{1 + (A - B - 2\Pi/\kappa)\sqrt{c_f/2}}$$

where $\mathrm{St} = \dfrac{q_w}{\rho c_P U_e (T_w - T_e)}$ is the Stanton number. Substituting for $A(\mathrm{Pr})$ this gives

$$\mathrm{St} \approx \frac{c_f/2}{1 + (13\,\mathrm{Pr}^{2/3} - 14)\sqrt{c_f/2}} \tag{33}$$

However, for typical parameter values this is well-approximated by the laminar expression - equation (22) - and it is usually the simpler form which is used in calculations.

7.8.2 Internal Flow – Pipe Flow

The Stanton number for pipe flow is defined in terms of the *bulk* temperature T_{av}, not the centreline temperature:

$$\mathrm{St} = \frac{q_w}{\rho c_p U_{av} (T_w - T_{av})} \tag{34}$$

where

$$T_{av} U_{av} \pi R^2 = \int_0^R \bar{u}\bar{T}(2\pi r)\ \mathrm{d}r$$

$$= 2\pi \int_0^R \bar{u}\bar{T}(R - y)\ \mathrm{d}y$$

where $y = R - r$ is the distance to the nearest wall.

Assuming $\mathrm{Pr}_t = 1$, much algebra (see the Examples section) produces

$$\mathrm{St} = \frac{c_f/2}{1 + (13\,\mathrm{Pr}^{2/3} - 12)\sqrt{c_f/2} + 7.4 c_f/2} \tag{35}$$

7.9 Temperature Integral Results

Writing the boundary-layer temperature equation in conservative form,

$$\frac{\partial}{\partial x}(\bar{u}\bar{T}) + \frac{\partial}{\partial y}(\bar{v}\bar{T}) = -\frac{1}{\rho c_p}\frac{\partial q_h}{\partial y}$$

and subtracting T_e times the continuity equation

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

gives

$$\frac{\partial}{\partial x}[\bar{u}(\bar{T} - T_e)] + \frac{\partial}{\partial y}[\bar{v}(\bar{T} - T_e)] = -\frac{1}{\rho c_p}\frac{\partial q_h}{\partial y}$$

Integrating from $y = 0$ to ∞, assuming impermeable walls, yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \bar{u}(\bar{T} - T_e)\ \ \mathrm{d}y = \frac{q_w}{\rho c_p} \tag{36}$$

The *enthalpy thickness* δ_h is defined by

$$\delta_h = \int_0^\infty \frac{\bar{u}}{U_e}\frac{\bar{T} - T_e}{T_s - T_e}\,\mathrm{d}y \tag{37}$$

Hence, from the above,

$$\frac{\mathrm{d}}{\mathrm{d}x}[U_e(T_s - T_e)\delta_h] = \frac{q_w}{\rho c_p} \tag{38}$$

Differentiating the product, and dividing by $U_e(T_s - T_e)$ this becomes

$$c_h = \frac{\mathrm{d}\delta_h}{\mathrm{d}x} + \delta_h\left(\frac{1}{T_s - T_e}\frac{\mathrm{d}T_s}{\mathrm{d}x} + \frac{1}{U_e}\frac{\mathrm{d}U_e}{\mathrm{d}x}\right) \tag{39}$$

where the Stanton number

$$c_h = \frac{q_w}{\rho c_p U_e(T_s - T_e)} \tag{40}$$

The integral relation (39) may be used to derive total heat transfer in cases where surface temperature and/or free-stream velocity are changing. If these are constants, however, the integral relation reduces simply to

$$c_h = \frac{\mathrm{d}\delta_h}{\mathrm{d}x} \tag{41}$$

with obvious similarities to the corresponding momentum integral theory.

7.10 Engineering Heat-Transfer Calculations

For practical engineering calculations are made with largely empirical correlations for Nusselt or Stanton numbers in terms of Re and Pr.

For circular cylinders, Churchill and Bernstein (1977) propose

$$\overline{Nu}_D = 0.3 + \frac{0.62\,\mathrm{Re}_D^{1/2}\,\mathrm{Pr}^{1/3}}{[1+(0.4/\mathrm{Pr})^{2/3}]^{1/4}}\left[1+\left(\frac{\mathrm{Re}_D}{282000}\right)^{5/8}\right]^{4/5} \tag{42}$$

(where all fluid properties are evaluated at the film temperature).

For correlations for many other geometries, including tube bundles, see, e.g. Incropera and De Witt, 1990.

7.11 References

Churchill, S.W. and Bernstein, M., 1977, *J. Heat Transfer*, 99, 300.

Incropera, F.P. and De Witt, D.P., 1990, Introduction to Heat Transfer, 2nd Ed., Wiley.

Kader, B.A., 1981, Temperature and concentration profiles in fully turbulent boundary layers, *Internat. J. Heat and Mass Transfer*, 24, 1541-1544.

Examples

Question 1.

(a) Show that, for self-similar laminar boundary-layer profiles of the form (defined by a streamfunction):

$$\psi = U_e \delta f(\eta) \qquad \text{where} \qquad \eta = \frac{y}{\delta(x)}, \quad \delta = \sqrt{A\frac{\nu x}{U_e}}$$

then the normalised temperature profile $\theta = \dfrac{T - T_e}{T_s - T_e}$ satisfies

$$\frac{d^2\theta}{d\eta^2} + \frac{A}{2}\Pr f \frac{d\theta}{d\eta} = 0$$

(This gives Blasius (A = 1) and Falkner-Skan (A = 2/(m+1)) boundary-layer solutions as special cases).

(b) Show that the solution in terms of f is

$$\theta = \frac{\displaystyle\int_\eta^\infty \exp\left(-\frac{A}{2}\Pr\int_0^{\eta'} f(s)\ ds\right) d\eta'}{\displaystyle\int_0^\infty \exp\left(-\frac{A}{2}\Pr\int_0^{\eta'} f(s)\ ds\right) d\eta'}$$

Question 2.

Calculate the Eckert number $\mathrm{Ec} = \dfrac{U^2}{c_P \Delta T}$ for air flow at 20 m s^{-1} past a wall with temperature maintained at 50 ºC above ambient.

Question 3.

By assuming logarithmic mean velocity and temperature profiles hold right to the centre of a pipe, show that the Stanton number

$$\mathrm{St} = \frac{q_w}{\rho c_p U_{av}(T_w - T_{av})}$$

in fully-developed pipe flow is given by

$$\mathrm{St} = \frac{c_f/2}{1 + (13\Pr^{2/3} - 12)\sqrt{c_f/2} + 7.4(c_f/2)}$$

Question 4.

Air at 20 ºC and 1 atm. flows at 60 m s^{-1} past a smooth flat plate 2 m long and 0.8 m wide. The plate surface temperature is 50 ºC. Assuming turbulent flow throughout, estimate the total rate of heat loss from one side of the plate.

Question 5.
Calculate the rate of heat transfer per unit length for a 15-mm-diameter cylinder maintained at a constant 60 ºC in an airstream of temperature 15 ºC and velocity 10 m s^{-1}.

Answers

(2) 0.0079

(4) 7.16 kW

(5) 195 W m^{-1}