3. FRICTION LAWS SPRING 2009

3.1 Drag Coefficients

The *local* contributions to forces on a body can be quantified by the ratio of a *stress* (force per unit area) to the *dynamic pressure*:

τ_w (*shear stress*) → *skin-friction coefficient* $c_f = \dfrac{\tau_w}{\frac{1}{2}\rho U_0^2}$ (1)

p (*pressure*/*normal stress*) → *pressure coefficient* $c_p = \dfrac{p}{\frac{1}{2}\rho U_0^2}$ (2)

The reference velocity U_0 is taken as:
– the free-stream velocity (U_e) for external boundary layers;
– the average velocity (U_{av}) for internal flows (e.g. pipes or channels).

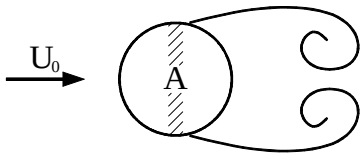
The *total* streamwise force F on a body in a flow can be non-dimensionalised by dividing by (*dynamic pressure* × *area*) as:

drag coefficient $c_D = \dfrac{F}{\frac{1}{2}\rho U_0^2 A}$ (3)

In general, this drag force has both pressure ("form drag") and viscous parts, with the former dominating for separated flows (bluff bodies) and the latter for streamlined bodies.

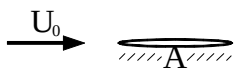
For *bluff* bodies (separated flow):

- force is predominantly *pressure drag*, $-\oint p \;\; \mathrm{d}A_x$
- A is the *projected* area
- $c_D = O(1)$

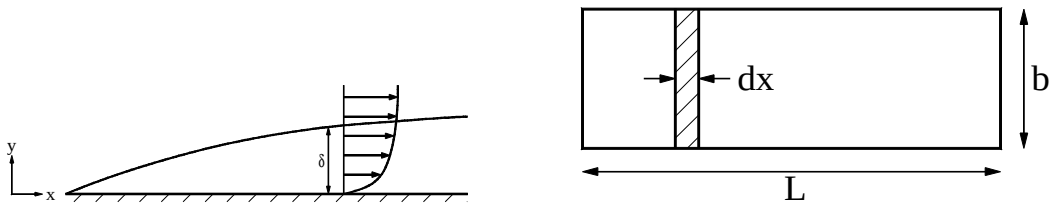


For *streamlined* bodies (no flow separation):

- force is predominantly *viscous drag* $\oint \tau_{wx} \;\; \mathrm{d}A$
- A is the *plan* area
- $c_D \ll 1$



This course will focus almost exclusively on attached boundary layers and the viscous contribution to drag (but see the Examples for section 1 for estimation of the pressure drag on an object).



For boundary layers, the frictional drag on a plate of span b and length L is

$$F = \int_0^L \tau_w \quad b\,\mathrm{d}x$$

$$c_D(L) = \frac{F}{\frac{1}{2}\rho U_0^2 (bL)} \quad = \frac{1}{L}\int_0^L c_f \quad \mathrm{d}x \quad = \text{average skin friction coefficient} \tag{4}$$

Remember from Section 2 that boundary-layer velocity profiles are typically given in terms of the *friction velocity* u_τ defined by

$$\tau_w = \rho u_\tau^2 \tag{5}$$

Substituting (5) in (1), the skin-friction coefficient and friction velocity are related by

$$c_f = 2\left(\frac{u_\tau}{U_0}\right)^2 \qquad \text{or} \qquad \frac{u_\tau}{U_0} = \sqrt{\frac{c_f}{2}} \tag{6}$$

Many valuable friction laws can be derived from the observation that the logarithmic velocity profile (with Coles wake extension in the case of an external flow):

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln\frac{u_\tau y}{\nu} + B + \frac{2\Pi}{\kappa} f\left(\frac{y}{\delta}\right)$$

is often a good approximation right across the shear layer. We use this in two ways:

- for an *external* flow (flat-plate boundary layer):
  - apply the formula *at the edge of the boundary layer*
  - relate the *free-stream* velocity to wall friction;
- for an *internal* flow (pipe or channel):
  - *integrate* across the duct
  - relate the *average* velocity to wall friction.

3.2 Flat-Plate Boundary Layer

Reynolds numbers:

$$\text{streamwise:}\quad \mathrm{Re}_x \equiv \frac{U_e x}{\nu}; \qquad \text{cross-stream:}\quad \mathrm{Re}_\delta \equiv \frac{U_e \delta}{\nu} \tag{7}$$

Applying the velocity law at the edge of the boundary layer:

$$\frac{U_e}{u_\tau} = \frac{1}{\kappa}\ln\frac{u_\tau \delta}{\nu} + B + \frac{2\Pi}{\kappa} \tag{8}$$

Noting that

$$\frac{u_\tau}{U_e} = \sqrt{\frac{c_f}{2}}$$

and

$$\frac{u_\tau \delta}{\nu} = \frac{U_e \delta}{\nu}\frac{u_\tau}{U_e} \quad = \mathrm{Re}_\delta \sqrt{\frac{c_f}{2}}$$

equation (8) can be rewritten as

$$\sqrt{\frac{2}{c_f}} = \frac{1}{\kappa}\ln(\mathrm{Re}_\delta \sqrt{\frac{c_f}{2}}) + B + \frac{2\Pi}{\kappa}$$

With $\kappa = 0.41$, $B = 5.0$ and $\Pi = 0.45$ this gives

$$\sqrt{\frac{2}{c_f}} = 2.44\ln(\mathrm{Re}_\delta \sqrt{\frac{c_f}{2}}) + 7.20 \tag{9}$$

This gives c_f *implicitly* in terms of Re_δ. A more convenient form is found by solving it for a few representative values of Re_δ and fitting a power-law approximation (see the Examples).

Rearranging (9) into a convenient form for iteration:

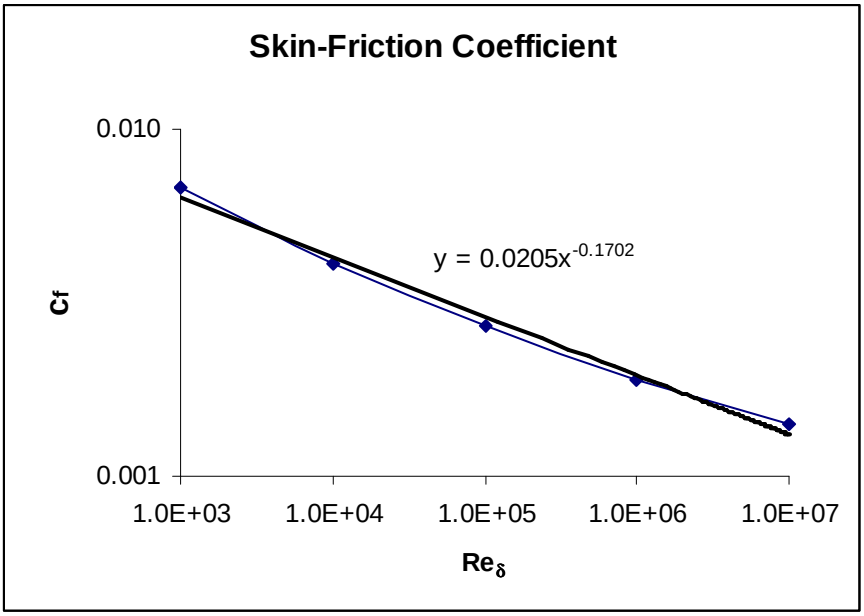
$$c_f = \frac{2}{[2.44\ln(\mathrm{Re}_\delta \sqrt{c_f/2}) + 7.20]^2}$$

then solving iteratively gives

Re_δ	10^3	10^4	10^5	10^6	10^7
c_f	0.00682	0.00409	0.00270	0.00190	0.00141

A suitable power-law fit (see the figure below) is:

$$c_f = 0.0205\,\mathrm{Re}_\delta^{-1/6} \tag{10}$$



This is only useful if $\delta(x)$ and hence Re_δ is known. We shall derive this by an integral analysis in Section 4. For now we quote the result:

$$\mathrm{Re}_\delta = 0.166\,\mathrm{Re}_x^{6/7}$$

Then

$$c_f = 0.0277\,\mathrm{Re}_x^{-1/7} \tag{11a}$$

$$c_D(L) = \frac{7}{6} c_f(L) \quad = 0.032\,\mathrm{Re}_L^{-1/7} \tag{11b}$$

These friction laws may be used, for example, to give a simple estimate of the viscous drag on thin aerofoils or other streamlined shapes (see Questions 3 and 4 in the Examples).

The above relations are for a smooth-walled boundary layer. Schlichting gives the following approximate formulae for the fully-rough regime:

$$c_f = [2.87 + 1.58\log_{10}(\frac{x}{k_s})]^{-2.5} \tag{12a}$$

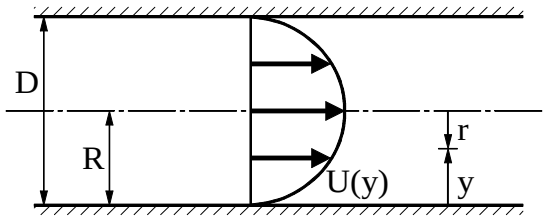
$$c_D(L) = [1.89 + 1.62\log_{10}(\frac{L}{k_s})]^{-2.5} \tag{12b}$$

3.3 Pipe Flow

The Reynolds number is usually defined in terms of diameter D and average velocity U_{av}:

$$\mathrm{Re} = \frac{U_{av} D}{\nu} \tag{13}$$

3.3.1 Smooth-Walled Pipes

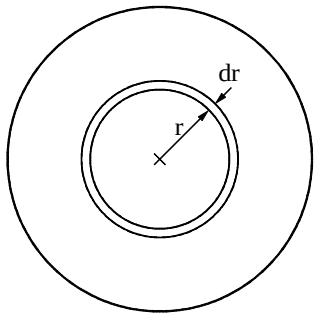


To a good approximation the logarithmic velocity profile holds right across (but see the note below) a pipe of radius R:

$$\frac{U}{u_\tau} = \frac{1}{\kappa} \ln E y^+, \qquad (\kappa = 0.41,\ E = 7.76)$$

where $y = R - r$ is the distance from the pipe wall.

If U_{av} is the average velocity and Q the flow rate, then



$$U_{av} \pi R^2 = Q = \int_0^R U\ 2\pi r \quad \mathrm{d}r$$

$$\Rightarrow \qquad U_{av} \pi R^2 = 2\pi \int_0^R \frac{u_\tau}{\kappa} \ln(E \frac{u_\tau y}{\nu})\ (R - y)\ \mathrm{d}y$$

$$\Rightarrow \qquad \frac{U_{av}}{u_\tau} = \frac{2}{\kappa} \int_0^R \ln(E \frac{u_\tau y}{\nu})\ (1 - \frac{y}{R})\ \frac{\mathrm{d}y}{R}$$

Change variables to the boundary-layer variable $\eta = y/R$:

$$\frac{U_{av}}{u_\tau} = \frac{2}{\kappa} \int_0^1 \ln(E \frac{u_\tau R}{\nu} \eta)\ (1 - \eta)\ \mathrm{d}\eta$$

This integrates to give

$$\frac{U_{av}}{u_\tau} = \frac{1}{\kappa} \left\{ \ln(E \frac{u_\tau R}{\nu}) - \frac{3}{2} \right\} \tag{14}$$

Noting that

$$\frac{u_\tau}{U_{av}} = \sqrt{\frac{c_f}{2}} \qquad \text{and} \qquad \frac{u_\tau R}{\nu} = \frac{1}{2} \frac{U_{av} D}{\nu} \frac{u_\tau}{U_{av}} = \frac{1}{2} \mathrm{Re} \sqrt{\frac{c_f}{2}}$$

then

$$\frac{1}{\sqrt{c_f}} = \frac{1}{\kappa\sqrt{2}} \ln(\frac{e^{-3/2} E}{2\sqrt{2}} \mathrm{Re} \sqrt{c_f}) = 1.72 \ln(\frac{\mathrm{Re}\sqrt{c_f}}{1.63})$$

The logarithm is traditionally converted to base 10 (for “engineering” reasons?):

$$\frac{1}{\sqrt{c_f}} = 4.0 \log_{10}(\frac{\mathrm{Re}\sqrt{c_f}}{1.63})$$

Prandtl deduced this in 1935, and then adjusted the constants to give a better fit to actual pipe flow data. The result is

$$\frac{1}{\sqrt{c_f}} = 4.0 \log_{10}\left(\frac{\mathrm{Re}\sqrt{c_f}}{1.26}\right) \tag{15}$$

Important note. Applying the log law right across the pipe would imply a *negative* velocity very close to the wall (when $y^+ < 1/E$). Fortunately, the actual contribution to the flow rate from this region is negligible. Fortunately also, the integral of the logarithm converges (since $\ln x$ integrates to $x \ln x - x$, which tends to 0 as $x \to 0$).

3.3.2 Rough-Walled Pipes

For fully-rough pipes $(k_s u_\tau / \nu > 70)$ the mean velocity profile may be written

$$\frac{U}{u_\tau} = \frac{1}{\kappa} \ln\left(\frac{y}{k_s}\right) + 8.5$$

An exactly comparable analysis to that above (see Question 7 in the examples) yields

$$\frac{1}{\sqrt{c_f}} = 4.0 \log_{10}\left(\frac{3.7D}{k_s}\right) \tag{16}$$

In practice, for commercial pipes,
(a) the roughness distribution is very different to that of uniformly-distributed sand;
(b) roughness and viscous effects are both significant.

The smooth-wall and fully-rough limits were accommodated by the *Colebrook-White formula*, which can be written in terms of the skin-friction coefficient as:

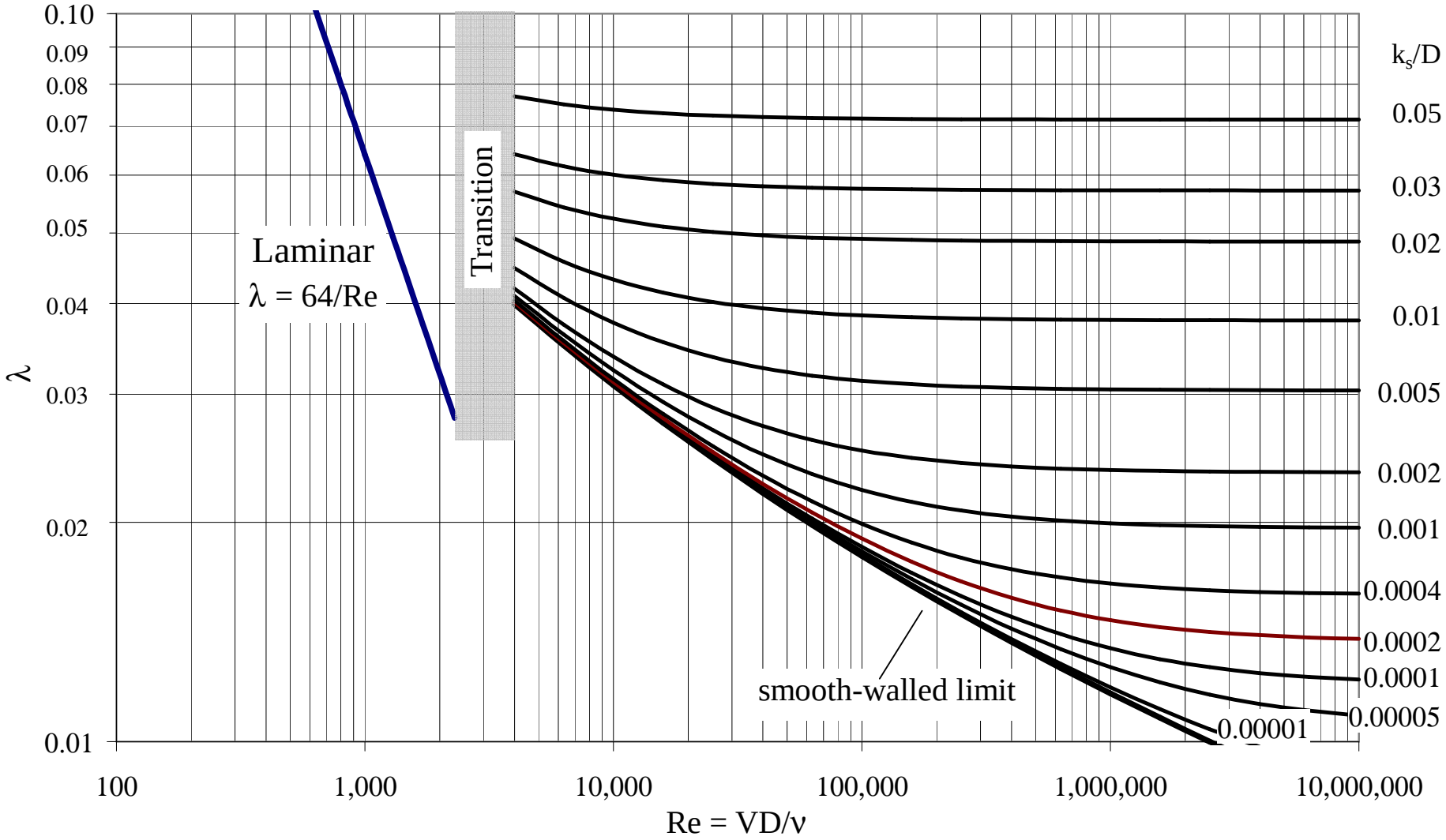
$$\frac{1}{\sqrt{c_f}} = -4.0 \log_{10}\left(\frac{k_s}{3.7D} + \frac{1.26}{\mathrm{Re}\sqrt{c_f}}\right) \tag{17}$$

In the context of pipe flow it is common to use a *friction factor* rather than skin-friction coefficient. Unfortunately, different authors define this as c_f or $4c_f$ (calling them the Fanning friction factor and Darcy friction factor, respectively). Adopting the definition $\lambda = 4c_f$ (which gives the most convenient frictional head-loss formula – see below) yields the more common version of the Colebrook-White formula:

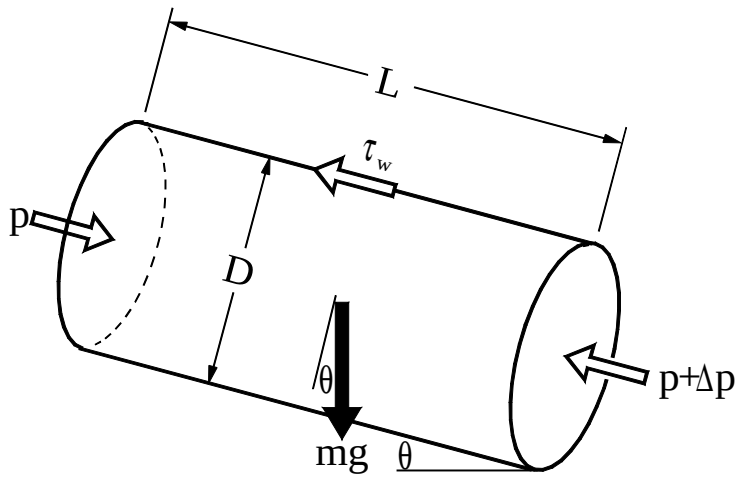
$$\boxed{\frac{1}{\sqrt{\lambda}} = -2.0 \log_{10}\left(\frac{k_s}{3.7D} + \frac{2.51}{\mathrm{Re}\sqrt{\lambda}}\right), \qquad \lambda = 4c_f \qquad (18)}$$

Colebrook (1939) and Moody (1944) helped compile *equivalent sand roughness* k_s for commercial pipe materials. Typical values are of order 0.01 to 0.1 mm.

Moody plotted solutions of (17) for friction factor in terms of Re for a set of typical values of k_s in his so-called *Moody chart*.


0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
λ
Laminar
λ = 64/Re
Transition
smooth-walled limit
k_s/D
0.05
0.03
0.02
0.01
0.005
0.002
0.001
0.0004
0.0002
0.0001
0.00005
0.00001
100
1,000
10,000
100,000
1,000,000
10,000,000
Re = VD/ν

3.4 Frictional Losses



Balancing pressure, weight and friction forces along the pipe axis in fully-developed flow,

$$-\Delta p \times \frac{\pi D^2}{4} + mg\sin\theta - \tau_w \times \pi DL = 0$$

Substituting

$$m = \rho\frac{\pi D^2}{4}L, \qquad \sin\theta = -\Delta z / L$$

where z is the height of the pipe centreline, this gives

$$-\Delta(p + \rho gz) \times \frac{\pi D^2}{4} = \tau_w \times \pi DL$$

or, dividing by the cross-sectional area,

$$-\Delta(p + \rho gz) = 4\tau_w \frac{L}{D}$$

$p + \rho gz$ is called the *piezometric pressure*, p^*. It represents the combined effect of pressure and weight.

If τ_w is written in terms of the skin-friction coefficient c_f and the *dynamic pressure*:

$$\tau_w = c_f(\tfrac{1}{2}\rho U_{av}^2)$$

then

$$-\Delta p^* = 4c_f \frac{L}{D}(\tfrac{1}{2}\rho U_{av}^2) \tag{19}$$

If the flow is predominantly gravity-driven, then it may be more convenient to work with head loss h_f and *dynamic head* $U_{av}^2/2g$ rather than pressure loss and dynamic pressure:

$$h_f = \frac{-\Delta p^*}{\rho g}$$

Then:

$$h_f = 4c_f \frac{L}{D}\left(\frac{U_{av}^2}{2g}\right) \tag{20}$$

Either of (19) or (20) is known as the *Darcy-Weisbach equation* for frictional losses in pipes.

With the common hydraulic definition $\lambda = 4c_f$ for *friction factor*, and writing V for average velocity, the loss equation is more commonly written as

$$-\Delta p^* = \lambda \frac{L}{D}\left(\frac{1}{2}\rho V^2\right)$$

or

$$h_f = \lambda \frac{L}{D}\left(\frac{V^2}{2g}\right) \tag{21}$$

This relationship between the frictional pressure loss and dynamic pressure, or between head loss and dynamic head, is used in conjunction with the Colebrook-White formula for the friction factor to relate head losses h_f, diameter D and discharge Q in pipe-flow calculations; (see the Examples).

Examples

Question 1.
(a) Find a relationship between the skin friction coefficient c_f and Reynolds number Re_δ for a flat-plate boundary layer.

(b) Solve this implicit equation for c_f when $\mathrm{Re}_\delta = 10^3$, 10^4, 10^5, 10^6 and 10^7.

(c) Find a suitable power-law approximation of the form $c_f = A\mathrm{Re}_\delta^{-1/n}$ over this Reynolds-number range.

Question 2.
Find a relation similar to equation (9) for a fully-rough boundary layer.

Question 3.
A boat has a streamlined shape with length 35 m and wetted area 280 m^2. Estimate its maximum velocity if the engines have a power output of 400 kW. (Make some reasonable modelling assumptions - which should be stated – and look up any fluid properties which you require.)

Question 4.
Find the terminal fall velocity if a thin square plate of mass 0.3 kg and side 0.2 m is dropped in water of density 1000 kg m^{-3} and kinematic viscosity 1.0×10^{-6} m^2 s^{-1}; assume that the plate falls vertically with two opposite sides vertical and that the boundary layer is turbulent from the leading edge.

Question 5.
Consider the power-law approximation to the velocity profile in fully-developed pipe flow:

$$\frac{U}{U_0} = \left(\frac{y}{R}\right)^\alpha$$

where U_0 is the centreline velocity, R is the radius of the pipe and $y = R - r$ is the distance from the nearest wall. Show that the bulk velocity is given by

$$\frac{U_{av}}{U_0} = \frac{2}{(1+\alpha)(2+\alpha)}$$

and evaluate this when $\alpha = 1/7$. How does this compare with laminar flow?

Question 6.
A hydraulically-smooth pipe of diameter 200 mm carries water. The flow is fully-developed and the *centreline* velocity is 2.5 m s^{-1}. Calculate the friction velocity and the pressure drop along a 100 m length.

(Note that it is rare to know the centreline velocity; it is more common to know the average velocity, which is directly related to the total volume flow rate).

Question 7.
Assuming a fully-rough pipe-flow profile

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln(\frac{y}{k_s}) + 8.5$$

where y is the distance from the pipe wall, and noting that the skin-friction coefficient is defined in terms of the *average* velocity, derive the friction law for fully rough pipes:

$$\frac{1}{\sqrt{c_f}} = 4.0\log_{10}(\frac{3.7D}{k_s})$$

Question 8.
Write a computer program to solve the Colebrook-White equation

$$\frac{1}{\sqrt{\lambda}} = -2.0\log_{10}(\frac{k_s}{3.7D} + \frac{2.51}{\mathrm{Re}\sqrt{\lambda}})$$

for λ (= $4c_f$), given values of relative roughness k_s/D and pipe Reynolds number $\mathrm{Re} = U_{av}D/\nu$. Check some typical solutions against those of a Moody chart.

Question 9.
Develop a friction law (equation relating c_f and Re_h) for a 2-d channel flow of height h in a manner similar to that for pipe flow, by assuming that the log law holds right to the centre of the channel. (For channel flow, the conventional definition is $\mathrm{Re}_h = U_{av}h/\nu$.)

Question 10.
Using the implicit friction formula derived in Question 9 above, evaluate c_f for $\mathrm{Re}_h = 10^4$, 10^5, 10^6 and 10^7 and find a power-law curve fit.

Question 11.
Show that, for fully-developed pipe flow,

$$\frac{U_{\max}}{U_{av}} = 1 + 2.59\sqrt{c_f}$$

How does this compare with laminar pipe flow?

Question 12.

A smooth straight pipe of length 40 m and internal diameter 100 mm carries water and connects points *A* and *B*. At point *A* the absolute pressure and elevation are 180 kPa and 2 m respectively. At point *B* they are 120 kPa and 7 m respectively.

(a) Which way is the flow going?

(b) If the friction factor λ (= 4 × skin-friction coefficient) is given by

$$\frac{1}{\sqrt{\lambda}} = 2.0\log_{10}\left(\frac{\mathrm{Re}\sqrt{\lambda}}{2.51}\right)$$

where Re is the Reynolds number based on bulk velocity and diameter, find the volumetric flow rate.

(c) Confirm that the flow is indeed turbulent.

Question 13.

The known outflow from a branch of a distribution system is 30 L s^{-1}. The pipe is of diameter 150 mm, length 500 m and has roughness parameter estimated as 0.06 mm. Find the head loss in the pipe.

Question 14.

(a) By balancing forces, show that, for uniform flow in a *wide* channel, the bed shear stress τ_w is related to the depth of water *h* by

$$\tau_w = \rho g h S$$

where ρ is density, *g* is the gravitational acceleration and *S* is the slope (assumed small enough for the small-angle approximation $\sin\theta \approx \tan\theta$ and any difference between vertical depth and that normal to the bed to be negligible).

(b) In a channel with a rough bed, the mean-velocity $U(y)$ is given by

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln\frac{y}{k_s} + B_k$$

where u_τ is the friction velocity, κ (= 0.41) is Von Kármán's constant, k_s is the equivalent sand roughness and B_k (= 8.5) is another constant. Find the depth-averaged velocity U_{av} as a function of u_τ and h/k_s.

A particular channel has slope 1 in 200 and bed roughness k_s = 1 mm, and carries water (density 1000 kg m^{-3}).

(c) If the depth of flow is 0.6 m find the volume flow rate per metre width of channel.

(d) If the volume flow rate is 0.5 m^3 s^{-1} per metre width of channel find the depth of flow.

Question 15.

A generalised form of the log-law mean-velocity profile which satisfies both smooth- and rough-wall limits can be written in wall units as

$$U^+ = \frac{1}{\kappa}\ln\left(\frac{y^+}{1+ck_s^+}\right) + B \qquad (15.1)$$

where κ, B and c are constants and

$$U^+ = \frac{U}{u_\tau}, \qquad y^+ = \frac{u_\tau y}{\nu}, \qquad k_s^+ = \frac{u_\tau k_s}{\nu}$$

y is the distance from the wall, u_τ is the friction velocity and k_s is the Nikuradse roughness.

(a) Assuming that (15.1) holds from the wall to the centreline of a pipe of diameter D, integrate to find an implicit relationship between the skin-friction coefficient c_f, pipe Reynolds number Re and relative roughness k_s/D.

(b) By comparing your results with the Colebrook-White formula:

$$\frac{1}{\sqrt{c_f}} = -4.0\log_{10}\left(\frac{k_s}{3.7D} + \frac{1.26}{\mathrm{Re}\sqrt{c_f}}\right)$$

deduce values for κ, B and c.

(c) Show that equation (15.1) can also be written in the form

$$U^+ = \frac{1}{\kappa}\ln y^+ + \tilde{B}(k_s^+) \qquad (15.2)$$

and deduce the functional form of $\tilde{B}(k_s^+)$.

(d) In the fully-rough limit ($k_s^+ >> 1$) equation (15.2) can be written as

$$U^+ = \frac{1}{\kappa}\ln\frac{y}{k_s} + B_k$$

Use your answers to parts (b) and (c) to deduce a value for B_k.

Answers

(1) (a) $\sqrt{\frac{2}{c_f}} = \frac{1}{\kappa}\ln(\mathrm{Re}_\delta \sqrt{\frac{c_f}{2}}) + B + \frac{2\Pi}{\kappa}$ or $c_f = \frac{2}{[2.439\ln(\mathrm{Re}_\delta \sqrt{c_f/2}) + 7.195]^2}$

(b)

Re_δ	10^3	10^4	10^5	10^6	10^7
c_f	0.006827	0.004092	0.002700	0.001904	0.001410

(c) A suitable power-law fit is $c_f = 0.0205\,\mathrm{Re}_\delta^{-1/5.88}$

(2) $\sqrt{2/c_f} = 2.44\ln(\delta/k_s) + 10.7$

(3) 11.5 m s^{-1}

(4) 4.0 m s^{-1}

(5) When $\alpha = 1/7$, $U_{av}/U_0 = 49/60$ (compared with ½ for laminar flow)

(6) $u_\tau = 0.092$ m s^{-1}; $\Delta p = 16.8$ kPa

(9) $1/\sqrt{c_f} = 1.72\ln(1.01\,\mathrm{Re}_h \sqrt{c_f})$

(10) $c_f = 0.040\,\mathrm{Re}_h^{-1/5.3}$

Re_h	c_f
10^4	0.00738
10^5	0.00436
10^6	0.00285
10^7	0.00199

(12) (a) From A to B

(b) 14.6 L s^{-1}

(c) $\mathrm{Re}_D = 1.86\times10^5$ (» 2300; hence, fully turbulent)

(13) 8.8 m

(14) (b) $\frac{U_{av}}{u_\tau} = \frac{1}{\kappa}(\ln\frac{h}{k_s} - 1) + B_k$

(c) 2.23 m^2 s^{-1}

(d) 0.238 m

(15) (a) $\frac{1}{\sqrt{c_f}} = \frac{-1}{\kappa\sqrt{2}}\ln\left(\frac{2\sqrt{2}e^{3/2-\kappa B}}{\mathrm{Re}\sqrt{c_f}} + 2e^{3/2-\kappa B}c\frac{k_s}{D}\right)$

(b) $\kappa = 0.41$, $B = 5.7$, $c = 0.30$

(c) $\tilde{B}(k_s^+) = B - \frac{1}{\kappa}\ln(1 + ck_s^+)$

(d) 8.6