Example 1.
The minute hand makes one complete revolution in one hour (i.e. 60 minutes of 60 seconds each). Hence,

$$\omega = \frac{\text{angle turned}}{\text{time}} = \frac{2\pi \text{ rad}}{60 \times 60 \text{ s}} = 1.745 \times 10^{-3} \text{ rad s}^{-1}$$

Answer: 1.75×10^{-3} rad s^{-1}.

Example 2.
Given:

$$\omega_0 = 0$$

$$\alpha = 0.1 \text{ rad s}^{-2}$$

$$\omega = 50\frac{\text{rev}}{\text{min}} = 50 \times \frac{2\pi \text{ rad}}{60 \text{ s}} = 5.236 \text{ rad s}^{-1}$$

Need θ, first in radians, then revolutions.

The analogous problem for linear motion would be ... given u, a, v, find s, for which we would use

$$v^2 - u^2 = 2as$$

The corresponding equation for rotation is

$$\omega^2 - \omega_0^2 = 2\alpha\theta$$

$$\Rightarrow \quad \theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{5.236^2 - 0}{2 \times 0.1} = 137.1 \text{ rad}$$

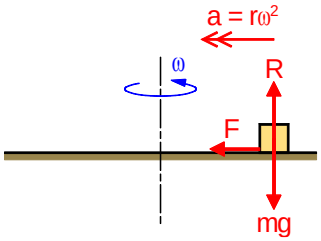
Converting to revolutions:

$$\frac{137.1}{2\pi} = 21.82 \text{ rev}$$

Answer: 21.8 revolutions.

Example 3.
(a) Angular velocity:

$$\omega = 78\frac{\text{rev}}{\text{min}} \quad = 78 \times \frac{2\pi \text{ rad}}{60 \text{ s}} \quad = 8.168 \text{ rad s}^{-1}$$



Without friction, the particle would continue moving in a straight line, increasing the distance from the centre. Friction provides the centripetal force needed to move in a circle. At constant speed the acceleration is $v^2/r = r\omega^2$ toward the centre. Hence,

$$F = mr\omega^2$$

The minimum coefficient of friction corresponds to the point where the particle is just about to slip; hence:

$$F = \mu R \quad = \mu mg$$

In the limiting case, then:

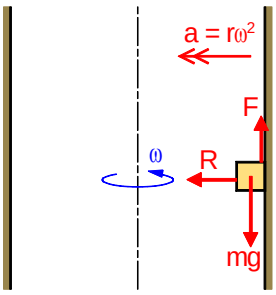
$$\mu mg = mr\omega^2$$

$$\Rightarrow \qquad \mu = \frac{r\omega^2}{g} \quad = \frac{0.1 \times 8.168^2}{9.81} \quad = 0.6801$$

Answer: 0.680.

(b) Angular velocity:

$$\omega = 200\frac{\text{rev}}{\text{min}} \quad = 200 \times \frac{2\pi \text{ rad}}{60 \text{ s}} \quad = 20.94 \text{ rad s}^{-1}$$



Without friction, the particle would drop down the side under its own weight. Here, friction balances weight:

$$F = mg$$

In this instance it is the normal reaction from the wall which provides the centripetal force. With centripetal acceleration $r\omega^2$ then:

$$R = mr\omega^2$$

The minimum coefficient of friction corresponds to the point where the particle is just about to slip; hence:

$$F = \mu R \quad = \mu mr\omega^2$$

In the limiting case, then:

$$\mu mr\omega^2 = mg$$

$$\Rightarrow \qquad \mu = \frac{g}{r\omega^2} \quad = \frac{9.81}{0.3 \times 20.94^2} \quad = 0.07458$$

Answer: 0.0746.

Note. The expression for μ in (a) is $\mu = \frac{r\omega^2}{g}$ and that in part (b) is $\mu = \frac{g}{r\omega^2}$. The fact that these are different (in fact, reciprocals) should serve as a reminder that it is vital to understand the problem, rather than just learn formulae. In part (a) the centripetal force is friction; in part (b) the centripetal force is the normal reaction.

Example 4.

(a) The particle moves in a horizontal plane, so forces in the vertical balance:

$$T \sin\theta = mg$$

$$\Rightarrow \qquad T = \frac{mg}{\sin\theta} \quad = \frac{3 \times 9.81}{\sin 15°} \quad = 113.7 \text{ N}$$

Answer: 114 N.

(b) By Hooke's Law:

$$T = kx$$

$$\Rightarrow \qquad x = \frac{T}{k} \quad = \frac{113.7}{90} \quad = 1.263 \text{ m}$$

Answer: 1.26 m.

(c) The particle moves in a circle of radius

$$r = (L + x)\cos\theta \quad = (1.5 + 1.263)\cos 15° \quad = 2.669 \text{ m}$$

The horizontal component of the tension provides the centripetal acceleration:

$$T\cos\theta = \frac{mv^2}{r}$$

$$\Rightarrow \qquad v = \sqrt{\frac{Tr\cos\theta}{m}} \quad = \sqrt{\frac{113.7 \times 2.669 \times \cos 15°}{3}} \quad = 9.885 \text{ m s}^{-1}$$

Answer: 9.88 m s^{-1}.

Example 5.
(a)
(i) Measure potential energy relative to the point of release ($v = 0$ and $\theta = 0$):

kinetic energy + potential energy = constant

$$\frac{1}{2}mv^2 - mgr\sin\theta = 0 - 0$$

$$v^2 = 2gr\sin\theta$$

Hence,

$$v = \sqrt{2gr\sin\theta}$$

(ii) Centripetal acceleration:

$$|a_r| = \frac{v^2}{r} \quad = 2g\sin\theta$$

Tangential acceleration is entirely due to the tangential component of the weight:

$$mg\cos\theta = ma_\theta$$

Hence,

$$a_\theta = g\cos\theta$$

(iii) Resolving inward:

$$R - mg\sin\theta = m|a_r|$$

$$R - mg\sin\theta = 2mg\sin\theta$$

Hence,

$$R = 3mg\sin\theta$$

(b) Before collision, with $\theta = 90°$, and with positive velocity left-to-right, the velocities of the mass m and mass $\frac{1}{2}m$ boxes are, respectively, $u_B = \sqrt{2gr}$ and $u_C = 0$. After collision, let the corresponding velocities by v_B and v_C.

Momentum:

$$mv_B + \frac{1}{2}mv_C = m\sqrt{2gr} + 0$$

i.e.

$$v_B + \frac{1}{2}v_C = \sqrt{2gr}$$

Restitution:

$$v_C - v_B = e\sqrt{2gr}$$

Adding these last two equations:

$$\frac{3}{2}v_C = (1+e)\sqrt{2gr}$$

Hence,

$$v_C = \frac{2}{3}(1+e)\sqrt{2gr}$$

Rearranging the restitution equation, $v_B = v_C - e\sqrt{2gr}$. Hence,

$$v_B = \frac{1}{3}(2-e)\sqrt{2gr}$$

Example 6.
Initial angular velocity:

$$\omega = 0.05\frac{\text{rev}}{\text{min}} \quad = 0.05 \times \frac{2\pi\ \text{rad}}{60\ \text{s}} \quad = 5.236 \times 10^{-3}\ \text{rad s}^{-1}$$

Equation of (angular) motion:

$$\text{torque} = \text{rate of change of angular momentum}$$
$$= \frac{\text{change of angular momentum}}{\text{time}}$$

Now,

$$\text{angular momentum} = \text{moment of momentum} \quad = Mv \times R$$

(since all the contributing particles making up mass M move with speed v at radius R) and

$$v = R\omega$$

Hence,

$$\text{torque} = \frac{0 - (Mv)R}{t}$$
$$= -\frac{MR^2\omega}{t}$$
$$= -\frac{1.9 \times 10^6 \times 38^2 \times 5.236 \times 10^{-3}}{30} \quad = -4.788 \times 10^5\ \text{N m}$$

The minus sign is unimportant here: it just signifies slowing down; i.e. there is a *braking* torque of 4.788×10^5 N m.

Now,

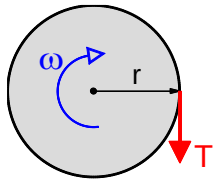
torque = force × radius

$\Rightarrow$ $4.788 \times 10^5 = \text{braking force} \times 38$

$\Rightarrow$ braking force = 12600 N

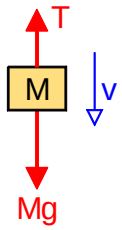
Answer: torque = 4.79×10^5 N m; force = 12600 N.

Example 7.
Let T be the tension in the rope. This will couple the linear motion of the bucket with the rotational motion of the cylinder.



Linear (downward) motion of the bucket:

$$Mg - T = M\frac{\mathrm{d}v}{\mathrm{d}t} \qquad \text{(A)}$$



Rotational motion of the cylinder:

$$Tr = I\frac{\mathrm{d}\omega}{\mathrm{d}t} \qquad \text{(B)}$$

We are interested in the linear acceleration $\mathrm{d}v/\mathrm{d}t$, so write ω in terms of v. Since the rate of fall of the bucket (v) equals the rate at which cable runs off the cylinder, which equals the speed ($r\omega$) of a particle on its circumference:

$$v = r\omega$$

$$\Rightarrow \qquad \omega = \frac{v}{r}$$

Hence, (B) can be rewritten

$$Tr = \frac{I}{r}\frac{\mathrm{d}v}{\mathrm{d}t} \qquad \text{(B)}'$$

Eliminating T from (A) and (B)′ by forming (A) × r + (B)′ gives

$$Mgr = \left(Mr + \frac{I}{r}\right)\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\Rightarrow \qquad g = \left(1 + \frac{I}{Mr^2}\right)\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\Rightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{g}{1 + \frac{I}{Mr^2}}$$

This is the acceleration of the bucket. It is less than the free-fall acceleration g because of the rotational inertia of the cylinder.

Example 8.
Initial angular velocity:

$$\omega_0 = 300\,\frac{\text{rev}}{\text{min}} \quad = 300 \times \frac{2\pi\ \text{rad}}{60\ \text{s}} \quad = 31.42\ \text{rad s}^{-1}$$

Impulse form of the equation of motion:

$$\text{angular impulse} = \text{change of angular momentum}$$

$$= 0 - I\omega_0 \quad = -1000 \times 31.42 \quad = -3.142 \times 10^4\ \text{N m s}$$

The minus sign is unimportant here: it just signifies that it is slowing down; i.e. there is a *braking* angular impulse of 3.142×10^4 N m s.

Now,

$$\text{angular impulse} = \text{torque} \times \text{time}$$

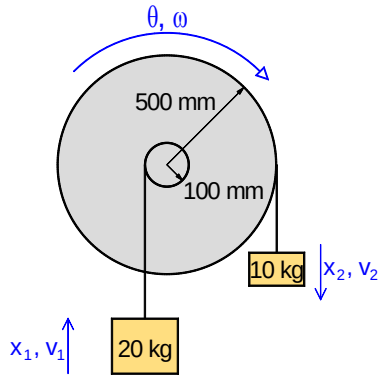
$$\Rightarrow \quad 3.142 \times 10^4 = \text{torque} \times 600$$

$$\Rightarrow \quad \text{torque} = 52.37\ \text{N m}$$

Answer: angular impulse = 3.14×10^4 N m s; torque = 52.4 N m.

Example 9.

Use subscript 1 to designate the 20 kg mass and shaft radius; use subscript 2 to designate the 10 kg mass and flywheel radius.



Suppose the displacement of the 20 kg mass *measured positive upwards* is x_1. (Actually, we have not determined yet whether this mass will move up or down, although the question implies up; strictly, x_1 could be positive or negative, but we shall see later that it must indeed be positive.) Similarly, let the displacement of the 10 kg *measured positive downwards* be x_2. Let the corresponding velocities in the same directions be v_1 and v_2 and let the angular velocity of the flywheel and shaft in the same positive sense be ω.

Because of the unequal radii, masses 1 and 2 do not move the same distance. Considering how much cable runs on or off the shaft and flywheel when they turn by angle θ:

$$x_2 = R_2\theta \quad \text{(whence } v_2 = R_2\omega\text{)}$$

$$x_1 = R_1\theta \quad \text{(whence } v_1 = R_1\omega\text{)}$$

Hence,

$$\frac{x_2}{x_1} = \frac{R_2}{R_1} \quad = 5$$

$$\Rightarrow \quad x_2 = 5x_1$$

Thus, if $x_1 = 1$ m, then $x_2 = 5$ m.

Answer: the 10 kg mass drops 5 m.

(b) Energy principle (for the whole system):

change in (KE + PE) = 0

$$\Rightarrow \quad \frac{1}{2}I\omega^2 + \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \qquad + m_1gx_1 - m_2gx_2 = 0$$

Since it is x_1 and ω that are involved in the question, write all displacements in terms of x_1 using $x_2 = 5x_1$ and all velocities in terms of ω using $v_2 = R_2\omega$, $v_1 = R_1\omega$:

$$\frac{1}{2}I\omega^2 + \frac{1}{2}m_1R_1^2\omega^2 + \frac{1}{2}m_2R_2^2\omega^2 + m_1gx_1 - m_2g(5x_1) = 0$$

$$\Rightarrow \quad \frac{1}{2}(I + m_1R_1^2 + m_2R_2^2)\omega^2 + (m_1 - 5m_2)gx_1 = 0$$

$$\Rightarrow \quad \frac{1}{2}(500 + 0.2 + 2.5)\omega^2 - 30 \times 9.81x_1 = 0$$

(Note that the kinetic energy is dominated by that of the flywheel.)

$$\Rightarrow \quad 251.35\omega^2 = 294.3x_1$$

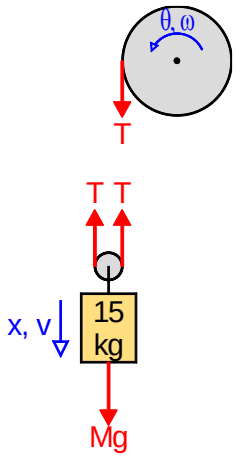
(Since ω^2 must be positive, this confirms x_1 positive; i.e. the 20 kg mass moves upwards.)

When $x_1 = 1$ m, this gives

$$\omega = 1.082 \text{ rad s}^{-1}$$

Answer: 1.08 rad s^{-1}.

Example 10.



(a)

When the mass drops a distance x ...
... each side of cable lengthens by x ...
... so the amount of cable run off the wheel is $2x$

Hence,

$$r\theta = 2x$$

and differentiating to get velocities:

$$r\omega = 2v$$

(Note that the answer is *not* the naïve one, $v = r\omega$, because the speed of a particle on the circumference is twice the speed of the mass in this instance.)

(b)

Linear (downward) motion of the mass:

$$Mg - 2T = M\frac{\mathrm{d}v}{\mathrm{d}t} \qquad \text{(A)}$$

Rotational motion of the wheel:

$$Tr = I\frac{\mathrm{d}\omega}{\mathrm{d}t} \qquad \text{(B)}$$

We are interested in the linear acceleration $\mathrm{d}v/\mathrm{d}t$, so write ω in terms of v; i.e., from part (a),

$$\omega = \frac{2v}{r}$$

Hence, (B) can be rewritten

$$Tr = \frac{2I}{r}\frac{\mathrm{d}v}{\mathrm{d}t} \qquad \text{(B)}'$$

Eliminating T from (A) and (B)′ by forming (A) $\times\, r + 2 \times$ (B)′ gives

$$Mgr = \left(Mr + \frac{4I}{r}\right)\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\Rightarrow \qquad g = \left(1 + \frac{4I}{Mr^2}\right)\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\Rightarrow \qquad \frac{g}{1 + \frac{4I}{Mr^2}} = \frac{\mathrm{d}v}{\mathrm{d}t}$$

Substituting values,

$$a \equiv \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{g}{1 + \frac{4I}{Mr^2}} = \frac{9.81}{1 + \frac{4 \times 0.9}{15 \times 0.3^2}} = 2.675 \text{ m s}^{-2}$$

Answer: 2.68 m s^{-2}.

(c) Given (or derived):

$$u = 0$$

$$a = 2.675 \text{ m s}^{-2}$$

$$x = 2 \text{ m}$$

We require v.

Use:

$$v^2 - u^2 = 2ax$$

$$\Rightarrow \quad v = \sqrt{u^2 + 2ax} \quad = \sqrt{0 + 2 \times 2.675 \times 2} \quad = 3.271 \text{ m s}^{-1}$$

Answer: 3.27 m s^{-1}.

(d)
From

$$r\omega = 2v$$

differentiate to relate the angular and linear accelerations:

$$r\alpha = 2a$$

Hence,

$$\alpha = \frac{2}{r}a \quad = \frac{2}{0.3} \times 2.675 \quad = 17.83 \text{ rad s}^{-2}$$

Given:

$$\omega_0 = 0$$

$$\alpha = 17.83 \text{ rad s}^{-2}$$

$$\omega = 300 \frac{\text{rev}}{\text{min}} \quad = 300 \times \frac{2\pi \text{ rad}}{60 \text{ s}} \quad = 31.42 \text{ rad s}^{-1}$$

Need θ, first in radians, then revolutions.

The analogous problem for linear motion would be: given u, a, v, find s, for which we would use

$$v^2 - u^2 = 2as$$

The corresponding equation for rotation is

$$\omega^2 - \omega_0^2 = 2\alpha\theta$$

$$\Rightarrow \quad \theta = \frac{\omega^2 - \omega_0^2}{2\alpha} \quad = \frac{31.42^2 - 0}{2 \times 17.83} \quad = 27.68 \text{ rad}$$

Converting to revolutions:

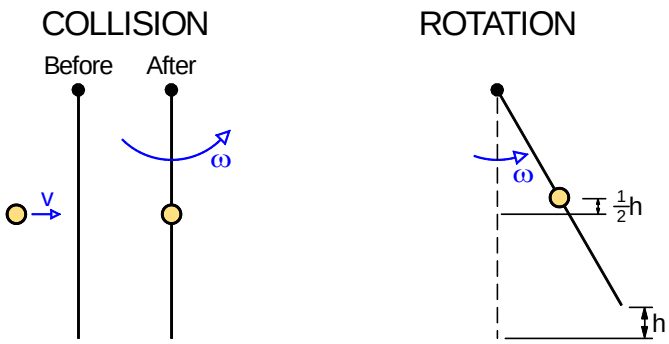
$$\frac{27.68}{2\pi} = 4.405 \text{ rev}$$

Answer: 4.41 revolutions.

(This part could also be done by finding the final linear velocity using $v = \frac{1}{2}r\omega$, using constant-linear-acceleration formulae with a from part (c), then converting back from distance of travel to angle via $\theta = 2x/r$.)

Example 11.
There are two stages:

- *collision* – conservation of angular momentum gives the angular velocity immediately after collision;
- *rotation* – conservation of energy as the initial rotational kinetic energy is converted to gravitational potential energy as the centre of mass rises.



Let M (= 6 kg) and m (= 1 kg) be the mass of plate and clay, respectively. The plate has side $a = 0.2$ m.

<u>Collision</u>

The moment of inertia of plate + attached clay is

$$I = I_{\text{plate}} + I_{\text{clay}}$$
$$= \frac{1}{3}Ma^2 + m(\frac{1}{2}a)^2$$
$$= \frac{1}{3} \times 6 \times 0.2^2 + 1 \times (0.1)^2 \quad = 0.09 \text{ kg m}^2$$

Conservation of angular momentum:

angular momentum before = angular momentum after

$$\Rightarrow \quad mv \times \frac{1}{2}a = I\omega$$

$$\Rightarrow \quad 1 \times 10 \times 0.1 = 0.09\omega$$

$$\Rightarrow \quad \omega = 11.11 \text{ rad s}^{-1}$$

<u>Rotation</u>

Energy principle:

gain in PE = loss of KE

If the bottom of the plate rises h (which is what we want to find), the centre of mass of the plate and clay rises $\frac{1}{2}h$. Hence,

$$(M+m)g(\frac{1}{2}h) = \frac{1}{2}I\omega^2$$

$$\Rightarrow \quad 7 \times 9.81 \times \frac{1}{2} \times h = \frac{1}{2} \times 0.09 \times 11.11^2$$

$$\Rightarrow \quad h = 0.1618 \text{ m}$$

Answer: 0.162 m.

Example 12.
The object is made up by (adding or subtracting) cylinders of different densities. First rewrite the moment of inertia of a generic cylinder ($\frac{1}{2}MR^2$) about its longitudinal axis in terms of density, height and radius:

$$I_{\text{cylinder}} = \frac{1}{2}MR^2$$
$$= \frac{1}{2}(\rho \times \pi R^2 h)R^2$$

Hence,

$$I_{\text{cylinder}} = \frac{1}{2}\pi\rho R^4 h$$

(a)

$$I_{\text{aluminium}} = I_{\text{flywheel}} + I_{\text{shaft}}$$
$$= \frac{1}{2}\pi\rho_{\text{aluminium}}(0.2)^4 \times 0.06 + \frac{1}{2}\pi\rho_{\text{aluminium}}(0.05)^4 \times 0.3$$

where $\rho_{\text{aluminium}} = 2650 \text{ kg m}^{-3}$. Hence,

$$I_{\text{aluminium}} = 0.4074 \text{ kg m}^2$$

Answer: 0.407 kg m^2.

(b) The steel rim can be constructed by subtracting a steel cylinder of radius 0.2 m and height 0.06 m from a steel cylinder of radius R_o and height 0.06 m, where R_o is to be found. Since the overall moment of inertia is to double, we must have

$$I_{\text{steel}} = I_{\text{aluminium}} = 0.4074 \text{ kg m}^2$$

Then,

$$I_{\text{steel}} = \frac{1}{2}\pi\rho_{\text{steel}}R_o^4 \times 0.06 - \frac{1}{2}\pi\rho_{\text{steel}}0.2^4 \times 0.06$$

where $\rho_{\text{steel}} = 7850 \text{ kg m}^{-3}$. Hence,

$$0.4074 = 739.8R_o^4 - 1.1838$$

$$\Rightarrow \quad R_o = 0.2154 \text{ m}$$

Answer: 0.215 m (= 215 mm).

(c) The total moment of inertia (steel + aluminium parts) is

$$0.4074 \times 2 = 0.8148 \text{ kg m}^2$$

The initial angular velocity is

$$\omega = 100\frac{\text{rev}}{\text{min}} = 100 \times \frac{2\pi \text{ rad}}{60 \text{ s}} = 10.47 \text{ rad s}^{-1}$$

From the angular impulse equation:

torque × time = change of angular momentum

$\Rightarrow \quad (F \times R_o) \times t = 0 - (I\omega)$

$\Rightarrow \quad F \times 0.2154 \times 30 = -0.8148 \times 10.47$

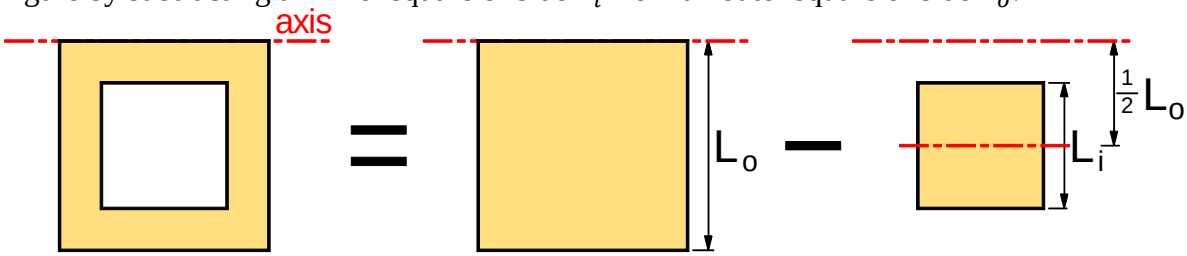
$\Rightarrow \quad F = -1.320$ N

(The minus simply indicates a braking force; we are only interested in the magnitude.)

Answer: 1.32 N.

Example 13.
Find moment of inertia I, then deduce radius of gyration from the defining equation $I = Mk^2$.

Let L_o (= 0.5 m) be the outer side length and L_i (= 0.3 m) be the inner side length. Form the figure by subtracting an inner square of side L_i from an outer square of side L_o.



Let the area mass density of the material be ρ. Then the masses of outer and inner squares are, in kg-m-s units:

$$M_o = \rho L_o^2 \quad = 0.25\rho$$
$$M_i = \rho L_i^2 \quad = 0.09\rho$$

<u>Method 1</u>
For the outer square, the axis is along one side, so that:

$$I_o = \frac{1}{3} M_o L_0^2 \quad = \frac{1}{3}(\rho L_0^2) L_o^2 \quad = 0.02083\rho$$

For the inner square, the axis is offset, so find the moment of inertia about its centre-of-mass axis and use the parallel-axis theorem to shift it by $\frac{1}{2}L_o$:

$$I_i = \frac{1}{12} M_i L_i^2 + M_i(\frac{1}{2}L_o)^2 \quad = \frac{1}{12}(\rho L_i^2) L_i^2 + (\rho L_i^2)(\frac{1}{2}L_o)^2 \quad = 0.0063\rho$$

Then

$$I = I_o - I_i \quad = 0.01453\rho$$

But

$$M = M_o - M_i \quad = 0.16\rho$$

Hence:

$$k = \sqrt{\frac{I}{M}} \quad = \sqrt{\frac{0.01453\rho}{0.16\rho}} \quad = 0.3014 \text{ m}$$

<u>Method 2</u>
Find the moment of inertia about the overall centre of mass, then use the parallel-axis theorem for the whole figure. There is some slightly fortuitous algebra here.

For the centre-of-mass axis:

$$
\begin{aligned}
I_G &= \frac{1}{12}M_o L_o^2 - \frac{1}{12}M_i L_i^2 \\
&= \frac{1}{12}\rho(L_o^4 - L_i^4) \\
&= \frac{1}{12}\rho(L_o^2 - L_i^2)(L_o^2 + L_i^2) \\
&= \frac{1}{12}M(L_o^2 + L_i^2)
\end{aligned}
$$

where we have used the difference of two squares between second and third lines and the fact that the mass of the whole is $M = \rho(L_o^2 - L_i^2)$.

Translating a distance $\frac{1}{2}L_o$ by the parallel-axis theorem:

$$I = I_G + M(\frac{1}{2}L_o)^2 \quad = \frac{1}{12}M(L_o^2 + L_i^2) + \frac{1}{4}ML_o^2 \quad = \frac{1}{12}M(4L_o^2 + L_i^2)$$

Then,

$$k = \sqrt{\frac{I}{M}} \quad = \sqrt{\frac{1}{12}(4L_o^2 + L_i^2)} \quad = \sqrt{\frac{1}{12}(4 \times 0.5^2 + 0.3^2)} \quad = 0.3014 \text{ m}$$

Answer: 0.301 m.

Note. The unknown density ρ cancels out. k is a purely geometric quantity, independent of the material and its density, as long as that is uniform.

Example 14.
(a) Break down into:

2 × (rod about axis through centre) + 2 × (all mass at distance $L/2$ from axis)

$$I = 2 \times \frac{1}{12}ML^2 + 2 \times M(\frac{1}{2}L)^2$$
$$= \frac{2}{3}ML^2$$

(b) Use the result from part (a) and parallel-axis rule for an assembly of mass $4M$ moved $L/2$:

$$I = \frac{2}{3}ML^2 + 4M(\frac{1}{2}L)^2$$
$$= \frac{5}{3}ML^2$$

(c) Use the result from part (a) and the perpendicular-axis rule:

$$I = 2 \times \frac{2}{3}ML^2$$
$$= \frac{4}{3}ML^2$$

(d) Use the result from part (c) and parallel-axis rule for assembly of mass $4M$ moved $L/\sqrt{2}$:

$$I = \frac{4}{3}ML^2 + 4M(\frac{L}{\sqrt{2}})^2$$
$$= \frac{10}{3}ML^2$$

(Could also use the result from part (b) and the perpendicular-axis rule).

(e) Use the result from part (c) and the perpendicular-axis rule in reverse:

$$2I = \frac{4}{3}ML^2$$

Hence,

$$I = \frac{2}{3}ML^2$$

Example 15.
The point of contact is instantaneously at rest. Since the velocity of this point is the forward velocity of the centre (v) combined with the backward velocity relative to the centre ($r\omega$),

$$v - r\omega = 0$$

$$\Rightarrow \quad r\omega = v$$

The moment of inertia is

$$I = \underbrace{Mr^2}_{\text{rim}} + n \times \underbrace{(\frac{1}{3}mr^2)}_{\text{spoke}} \quad = (M + \frac{1}{3}nm)r^2$$

The total kinetic energy is

KE = translational KE (total mass $M + nm$ at speed v) + rotational KE

Hence,

$$\begin{aligned}
\text{KE} &= \frac{1}{2}(M + nm)v^2 + \frac{1}{2}I\omega^2 \\
&= \frac{1}{2}(M + nm)v^2 + \frac{1}{2}(M + \frac{1}{3}nm)r^2\omega^2 \\
&= \frac{1}{2}(M + nm)v^2 + \frac{1}{2}(M + \frac{1}{3}nm)v^2 \\
&= Mv^2 + \frac{2}{3}nmv^2 \\
&= (M + \frac{2}{3}nm)v^2
\end{aligned}$$

Answer: $(M + \frac{2}{3}nm)v^2$.