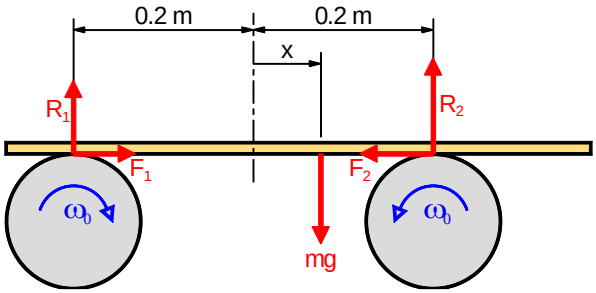
Example 1.
Assume that ω_0 is sufficiently large that the bar never acquires the speed of a wheel and is always slipping relative to the wheels. Then friction forces are maximal and given, in magnitude, by $F = \mu R$. The system oscillates because when, for example, the bar moves to the right, the normal reaction and hence the friction force at that wheel is greater, causing a net restoring force toward the centre. We shall see that the *net* friction force from the two wheels is proportional to the displacement and hence gives rise to SHM.



Let R_1 and R_2 be the normal reactions on the bar at left and right wheels respectively and F_1 and F_2 the corresponding friction forces. These act in directions such that they try to make the bar move in the same direction as the wheels. kg-m-s units are assumed where necessary.

Considering moments about the right-hand and left-hand points of contact:

$$R_1 \times 0.4 = mg(0.2 - x) \quad \Rightarrow \quad R_1 = mg(0.5 - 2.5x)$$
$$R_2 \times 0.4 = mg(0.2 + x) \quad \Rightarrow \quad R_2 = mg(0.5 + 2.5x)$$

With $\mu = 0.2$ the corresponding friction forces are:

$$F_1 = \mu R_1 \quad = mg(0.1 - 0.5x)$$
$$F_2 = \mu R_2 \quad = mg(0.1 + 0.5x)$$

Equation of motion ("force = mass × acceleration") to the right:

$$F_1 - F_2 = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

$$\Rightarrow \quad -1.0 \times mgx = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

$$\Rightarrow \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -9.81x$$

Comparison with the standard SHM equation ($\mathrm{d}^2x/\mathrm{d}t^2 = -\omega^2x$) shows that this is SHM in x with natural circular frequency

$$\omega = \sqrt{9.81} \quad = 3.132 \text{ rad s}^{-1}$$

$$\Rightarrow \quad \text{period} = \frac{2\pi}{\omega} \quad = \frac{2\pi}{3.132} \quad = 2.006 \text{ s}$$

Answer: 2.01 s.

Example 2.
(a) Let:

$$12\cos 3t + 5\sin 3t = A\sin(\omega t + \phi)$$
$$= A(\sin\omega t\cos\phi + \cos\omega t\sin\phi)$$

Since this has to be true for all t, the circular frequency is $\omega = 3$. Then, equating coefficients,

cos$3t$: $A\sin\phi = 12$

sin$3t$: $A\cos\phi = 5$

Squaring and adding to eliminate ϕ:

$$A^2(\sin^2\phi + \cos^2\phi) = 12^2 + 5^2$$

$\Rightarrow$ $A^2 = 169$

By assumption the amplitude is positive. Hence,

$$A = 13$$

Alternatively, dividing to eliminate A:

$$\tan\phi = \frac{12}{5}$$

Hence, $\phi = 67.4°$ (1.18 radians) or 247.4° (4.32 radians). As $\sin\phi$ is positive, the former is correct here.

Answer: $13\sin(3t + 67.4°)$ (or, with the phase angle in radians, $13\sin(3t + 1.18)$).

(b) Let:

$$4\cos 2t - 3\sin 2t = A\sin(\omega t + \phi)$$
$$= A(\sin\omega t\cos\phi + \cos\omega t\sin\phi)$$

Since this has to be true for all t, the circular frequency is $\omega = 2$. Then, equating coefficients,

cos $2t$: $A\sin\phi = 4$

sin $2t$: $A\cos\phi = -3$

Squaring and adding to eliminate ϕ:

$$A^2(\sin^2\phi + \cos^2\phi) = 4^2 + (-3)^2$$

$\Rightarrow$ $A^2 = 25$

By assumption the amplitude is positive. Hence,

$$A = 5$$

Alternatively, dividing to eliminate A:

$$\tan\phi = \frac{-4}{3}$$

Hence, $\phi = -53.1°$ (–0.927 radians) or 126.9° (2.21 radians). As $\sin\phi$ is positive (or $\cos\phi$ is negative), the latter is correct here.

Answer: $5\sin(2t + 126.9°)$ (or, with the phase angle in radians, $5\sin(2t + 2.21)$).

Example 3.

(a)

Equation of motion ("force = mass × acceleration") to the right:

$$-100x - 60x = 10\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

$$\Rightarrow \qquad -160x = 10\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \qquad (*)$$

Inspection of the LHS shows that the same total elastic force would arise from a single spring of stiffness 160 N m^{-1}.

Answer: 160 N m^{-1}.

(b) From (*),

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -16x$$

Comparison with the standard SHM equation ($\mathrm{d}^2x/\mathrm{d}t^2 = -\omega^2x$) shows that this is SHM in x with natural circular frequency

$$\omega = \sqrt{16} \quad = 4 \text{ rad s}^{-1}$$

Answer: 4 rad s^{-1}.

*** (d) and (c) done in reverse order below... ***

(d) Period:

$$T = \frac{2\pi}{\omega} \quad = \frac{2\pi}{4} \quad = 1.571 \text{ s}$$

Answer: 1.57 s.

(c) Natural frequency:

$$f = \frac{1}{T} \quad = \frac{1}{1.571} \quad = 0.6365 \text{ Hz}$$

Answer: 0.637 Hz.

(e) General solution:

$$x = A\cos 4t + B\sin 4t$$

$$\Rightarrow \qquad \dot{x} = -4A\sin 4t + 4B\cos 4t$$

Released from rest; i.e. $\dot{x} = 0$ when $t = 0$:

$$0 = 4B \qquad \Rightarrow \qquad B = 0$$

Released from displacement 0.1 m; i.e. $x = 0$ when $t = 0$:

$$0.1 = A$$

Hence the general solution and corresponding velocity are:

$$x = 0.1 \cos 4t$$

$$\dot{x} = -0.4 \sin 4t$$

Since the sine function cycles between –1 and +1, $\dot{x}$ cycles between –0.4 and +0.4. The maximum speed is therefore 0.4 m s^{-1}.

Answer: 0.4 m s^{-1}.

Example 4.

(a) Period is $T = 7$ s, so

$$\omega = \frac{2\pi}{T} \quad = \frac{2\pi}{7} \quad = 0.8976 \text{ rad s}^{-1}$$

The amplitude is half the range (8 – 2 m), or

$$A = 3 \text{ m}$$

(i) $v_{\max} = \omega A \quad = 0.8976 \times 3 \quad = 2.693 \text{ m s}^{-1}$

(ii) $a_{\max} = \omega^2 A \quad = 0.8976^2 \times 3 \quad = 2.417 \text{ m s}^{-2}$

Answer: 2.69 m s^{-1}; 2.42 m s^{-2}.

(b) Equation of motion for the box:

$$R - mg = ma_y$$

or

$$R = m(g + a_y)$$

where a_y is the vertically-upward component of acceleration.

(i) At the top, $a_y = -2.417$ m s^{-2}, so

$$R = 15(9.81 - 2.417) \quad = 110.9 \text{ N}$$

(ii) At the bottom, $a_y = +2.417$ m s^{-2}, so

$$R = 15(9.81 + 2.417) \quad = 183.4 \text{ N}$$

(iii) In the middle, $a_y = 0$, so

$$R = 15 \times 9.81 \quad = 147.2 \text{ N}$$

Answer: 111 N; 183 N; 147 N.

Example 5.
(a) Let x be the extension. Assume kg-m-s units where necessary.

Equation of motion ("force = mass × acceleration") downwards:

$$-kx - kx + mg = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

$$\Rightarrow \qquad -400x + 16g = 16\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \qquad (*)$$

Inspection of the LHS shows that the same total elastic force would be obtained with a single spring of stiffness 400 N m^{-1}.

Answer: 400 N m^{-1}.

(b)
Method 1
Balance weight against total elastic force at equilibrium:

$$16g = 400x_{\text{eqm}}$$

$$\Rightarrow \qquad x_{\text{eqm}} = \frac{16g}{400} \quad = \frac{16 \times 9.81}{400} \quad = 0.3924 \text{ m}$$

Method 2
Divide (*) by 16 and rearrange:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -25x + g$$

$$\Rightarrow \qquad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -25\left(x - \frac{g}{25}\right)$$

The system is in equilibrium (zero acceleration $\mathrm{d}^2x/\mathrm{d}t^2$) when

$$x_{\text{eqm}} = \frac{g}{25} \quad = 0.3924 \text{ m}$$

Answer: 0.392 m.

(c) From the Method 2 working above,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -25\left(x - \frac{g}{25}\right)$$

Setting

$$X = x - \frac{g}{25}$$

(which is the displacement from equilibrium) and noting that $\mathrm{d}^2X/\mathrm{d}t^2 = \mathrm{d}^2x/\mathrm{d}t^2$,

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2} = -25X$$

Comparison with the standard SHM equation ($\mathrm{d}^2X/\mathrm{d}t^2 = -\omega^2X$) shows that this is SHM in X with

$$\omega = \sqrt{25} \quad = 5 \text{ rad s}^{-1}$$

The period is then

$$T = \frac{2\pi}{\omega} \quad = \frac{2\pi}{5} \quad = 1.257 \text{ s}$$

Answer: 1.26 s.

Example 6.
(a) Balancing forces:

$$kx = mg$$

Hence,

$$x = \frac{mg}{k} \quad = \frac{7.6 \times 9.81}{1800} \quad = 0.04142 \text{ m}$$

Answer: 0.0414 m.

(b) Equation of motion, in terms of string extension measured downward:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx + mg \quad = -k\left(x - \frac{mg}{k}\right)$$

Change variables to the displacement from equilibrium, X:

$$X = x - \frac{mg}{k}$$

Since $\mathrm{d}^2X/\mathrm{d}t^2 = \mathrm{d}^2x/\mathrm{d}t^2$,

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2} = -\frac{k}{m}X$$

This is SHM with natural circular frequency

$$\omega = \sqrt{\frac{k}{m}} \quad = \sqrt{\frac{1800}{7.6}} \quad = 15.39 \text{ rad s}^{-1}$$

and period

$$T = \frac{2\pi}{\omega} \quad = \frac{2 \times \pi}{15.39} \quad = 0.4083 \text{ s}$$

If the amplitude is $A = 0.12$ m then the maximum speed is

$$v_{\max} = A\omega \quad = 0.12 \times 15.39 \quad = 1.847 \text{ m s}^{-1}$$

Answer: (i) 0.408 s; (ii) 1.85 m s^{-1}.

(c) If the spring is replaced by a string then it ceases to be taut for part of the motion and so its oscillation will *not* be SHM. Instead, an oscillation is made up partly of the harmonic motion associated with SHM and partly of motion under gravity whilst the string is not taut.

Let t_1 be the time for the particle to rise from maximum string extension ($X = 1.2$ m) to string just going loose ($X = -0.04142$ m) and t_2 be the subsequent time to maximum rise under constant acceleration $-g$. Then the total period of oscillation is $2(t_1 + t_2)$.

In the first part of the motion,

$$X = A\cos\omega t \quad = 0.12\cos 15.39t$$

and t_1 is the first time such that $X = -0.04142$. i.e.

$$0.12\cos 15.39t_1 = -0.04142$$

or (noting that the $\cos^{-1}$ function needs to give a result in radians):

$$t_1 = \frac{1}{15.39}\cos^{-1}\left(-\frac{0.04142}{0.12}\right) \quad = 0.1250 \text{ s}$$

Since the velocity in the first part of the motion is

$$v = \frac{\mathrm{d}X}{\mathrm{d}t} = -A\omega\sin\omega t$$

we have a velocity at the point where the string becomes loose of

$$u = -A\omega\sin\omega t_1 \quad = -0.12 \times 15.39\sin(15.39 \times 0.1250) \quad = -1.733 \text{ m s}^{-1}$$

Then from "$v = u + at$", with velocity 0 and $a = g$, we have

$$t_2 = \frac{-u}{g} \quad = \frac{1.733}{9.81} \quad = 0.1767 \text{ s}$$

The full period of oscillation is then

$$T = 2(t_1 + t_2) \quad = 2(0.1250 + 0.1767) \quad = 0.6034 \text{ s}$$

Answer: 0.603 s.

Example 7.
Note that the degree of freedom is the downward movement of the *mass*, m. Let this be x. Then the whole of cable + two springs has stretched by $2x$.

There are two conditions:
(1) The total stretch of both springs is $2x$:

$$e_1 + e_2 = 2x \qquad (*)$$

where e_1 and e_2 are the extensions of left and right springs.

(2) There is the same force in each spring (equal to the tension T in the cable). Thus,

$$ke_1 = T = 2ke_2 \qquad (**)$$

Hence, from (**),

$$e_1 = 2e_2$$

and hence, from (*),

$$3e_2 = 2x$$

$$\Rightarrow \qquad e_2 = \frac{2}{3}x$$

whence the tension T is, from (**):

$$T = \frac{4}{3}kx$$

Since the looped cable's total pull on the mass is $2T$, the equation of motion is

$$mg - 2T = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

$$\Rightarrow \qquad m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = mg - \frac{8}{3}kx$$

$$\Rightarrow \qquad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{8}{3}\frac{k}{m}\left(x - \frac{3mg}{8k}\right)$$

Changing variables to $X = x - 3mg/8k$, and noting that $\mathrm{d}^2X/\mathrm{d}t^2 = \mathrm{d}^2x/\mathrm{d}t^2$:

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2} = -\frac{8}{3}\frac{k}{m}X$$

Comparison with the standard SHM equation ($\mathrm{d}^2X/\mathrm{d}t^2 = -\omega^2X$) gives circular frequency

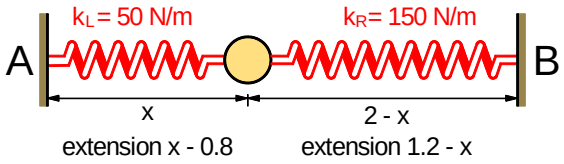
$$\omega = \sqrt{\frac{8}{3}\frac{k}{m}}$$

and period

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3m}{8k}} \quad = \pi\sqrt{\frac{3m}{2k}}$$

Answer: $\pi\sqrt{\frac{3m}{2k}}$.

Example 8.
Assume kg-m-s units where necessary.



(a) If x is the distance of the particle from A then:

- its distance from B is $2 - x$;
- the extension of the spring attached to A is $x - 0.8$
- the extension of the spring attached to B is $(2 - x) - 0.8 = 1.2 - x$

Equation of motion ("force = mass × acceleration"), being careful of the directions of the elastic forces:

$$-50(x - 0.8) + 150(1.2 - x) = 0.4\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

$$\Rightarrow \quad -200x + 220 = 0.4\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

$$\Rightarrow \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -500x + 550$$

$$\Rightarrow \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -500(x - 1.1)$$

(Any of the last 4 equations would answer part (a) of the question.)

(b) The equilibrium point is where net force, and hence acceleration, is zero. From the above this occurs at

$$x_{\text{eqm}} = 1.1 \text{ m}$$

(You can also get this by balancing the elastic forces.)

Answer: 1.1 m away from A.

(c) Change variables to

$$X = x - 1.1$$

(i.e. X is the displacement from equilibrium.) Then, as $\mathrm{d}^2X/\mathrm{d}t^2 = \mathrm{d}^2x/\mathrm{d}t^2$,

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2} = -500X$$

Comparison with the standard SHM equation ($\mathrm{d}^2X/\mathrm{d}t^2 = -\omega^2X$) shows that this is SHM in X with

$$\omega = \sqrt{500} \quad = 22.36 \text{ rad s}^{-1}$$

(i) The period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{22.36} = 0.2810 \text{ s}$$

(ii) Initially, $x = 1.0$ m and so $X = -0.1$ m. The particle starts from rest, so this is the furthest it will be from equilibrium. Hence, the amplitude of the motion is $A = 0.1$ m.

The maximum speed and acceleration are:

$$v_{\max} = \omega A = 22.36 \times 0.1 = 2.236 \text{ m s}^{-1}$$
$$a_{\max} = \omega^2 A = 500 \times 0.1 = 50 \text{ m s}^{-2}$$

Answer: period = 0.281 s; maximum speed = 2.24 m s^{-1}; maximum acceleration = 50 m s^{-2}.

Example 9.
(a) From the given formula for a circular disk, and using the parallel axis theorem:

$$I(\text{disk}) = \frac{1}{2}M_{\text{disk}}R^2 + M_{\text{disk}}R^2$$
$$= \frac{3}{2}M_{\text{disk}}R^2$$
$$= \frac{3}{2} \times 3 \times 0.4^2 \quad = 0.72 \text{ kg m}^2$$

As the particle is (by definition) just a point mass, at distance $2R$ from the given axis:

$$I(\text{particle}) = M_{\text{particle}}(2R)^2$$
$$= 2 \times 0.8^2 \quad = 1.28 \text{ kg m}^2$$

The moment of inertia of the whole is

$$I = 0.72 + 1.28 \quad = 2.0 \text{ kg m}^2$$

Answer: 2.0 kg m^2.

(b) Rotational equation of motion:

$$T = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

where T is torque The gravitational torques can be summed separately: that of the disk (line of action $R\sin\theta$ from the axis) and that of the particle (line of action $2R\sin\theta$ from the axis). Noting that, in each case, the sense of the torque is opposite to that of the rotation:

$$-(M_{\text{disk}}g)(R\sin\theta) - (M_{\text{particle}}g)(2R\sin\theta) = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

$$\Rightarrow \quad -\left(M_{\text{disk}} + 2M_{\text{particle}}\right)gR\sin\theta = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

$$\Rightarrow \quad -(3 + 2 \times 2) \times 9.81 \times 0.4\sin\theta = 2.0\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

With the small-angle approximation $\sin\theta \approx \theta$, this gives

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -13.734\,\theta$$

This is SHM in θ, with

$$\omega = \sqrt{13.734} \quad = 3.706 \text{ rad s}^{-1}$$
$$\text{period} = \frac{2\pi}{\omega} \quad = \frac{2\pi}{3.706} \quad = 1.695 \text{ s}$$

Answer: 1.70 s.

Example 10.
When the bar is rotated a small angle θ:

each spring is stretched a distance $(\frac{1}{2}L)\theta$ (assuming the small-angle approximation);

$\Rightarrow$ the restoring force in each spring has magnitude $k(\frac{1}{2}L\theta)$

$\Rightarrow$ the restoring torque from each spring has magnitude $k(\frac{1}{2}L\theta) \times \frac{1}{2}L = \frac{1}{4}kL^2\theta$

(again, assuming the small-angle approximation)

Equation of angular motion:

$$T = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

$$\Rightarrow \quad -2 \times \frac{1}{4}kL^2\theta = \frac{1}{12}ML^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

$$\Rightarrow \quad \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -6\frac{k}{M}\theta$$

Comparison with the standard SHM equation ($\mathrm{d}^2\theta/\mathrm{d}t^2 = -\omega^2\theta$) shows that this is SHM in θ with natural circular frequency

$$\omega = \sqrt{\frac{6k}{M}}$$

Answer: $\sqrt{\frac{6k}{M}}$.

Example 11.

(a) The potential energy is that of a mass M, a distance $L\cos\theta + \frac{1}{2}H$ below the fixed line; i.e.

$$\text{PE} = -Mg(L\cos\theta + \frac{1}{2}H) \quad = -MgL\cos\theta + \text{constant } (C)$$

(The $\frac{1}{2}H$ just reflects the distance of the centroid of the sign below the end of a rod.)

Since the sign doesn't rotate, its kinetic energy is $\frac{1}{2}MV^2$, where V is the speed of the centre of mass. But also, since it isn't rotating, the velocity of the centre is the same as the end of a rod; i.e. $V = L\dot{\theta}$. Hence,

$$\text{KE} = \frac{1}{2}MV^2 \quad = \frac{1}{2}ML^2\dot{\theta}^2$$

(b) Maximum speed occurs at the bottom of the swing ($\theta = 0$: the position of minimum potential energy).

Conserving energy:

$$(\text{KE} + \text{PE})_{\text{start}} = (\text{KE} + \text{PE})_{\text{end}}$$

where "start" corresponds to $\theta = \pi/3$ and "end" to $\theta = 0$. Hence,

$$0 - MgL\cos\frac{\pi}{3} + C = \frac{1}{2}MV^2 - MgL\cos 0 + C$$

$$\Rightarrow \qquad MgL(1 - \cos\frac{\pi}{3}) = \frac{1}{2}MV^2$$

$$\Rightarrow \qquad 2gL(1 - \frac{1}{2}) = V^2$$

$$\Rightarrow \qquad V = \sqrt{gL}$$

Answer: $\sqrt{gL}$.

(c)

$$\text{KE} + \text{PE} = \frac{1}{2}ML^2\dot{\theta}^2 - MgL\cos\theta + C$$
$$= \frac{1}{2}ML^2\dot{\theta}^2 - MgL(1 - \frac{1}{2}\theta^2 + \cdots) + C$$

To quadratic order:

$$\text{KE} + \text{PE} = \frac{1}{2}ML^2\dot{\theta}^2 + \frac{1}{2}MgL\theta^2 + \text{constant}$$

(d) Since energy is conserved, $\mathrm{d}/\mathrm{d}t(\text{KE} + \text{PE}) = 0$, so that differentiating wrt t gives:

$$0 = \frac{1}{2}ML^2 \times 2\dot{\theta}\ddot{\theta} + \frac{1}{2}MgL \times 2\theta\dot{\theta}$$

$$\Rightarrow \quad 0 = ML\dot{\theta}(L\ddot{\theta} + g\theta)$$

Dividing by $ML\dot{\theta}$ (by assumption, not everywhere zero):

$$L\ddot{\theta} + g\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{g}{L}\theta$$

Comparison with the standard SHM equation ($\mathrm{d}^2\theta/\mathrm{d}t^2 = -\omega^2\theta$) shows that this is SHM in θ with

$$\omega = \sqrt{\frac{g}{L}}$$

$$\Rightarrow \quad \text{period} = \frac{2\pi}{\omega} \quad = 2\pi\sqrt{\frac{L}{g}}$$

Answer: $2\pi\sqrt{\frac{L}{g}}$.

Example 12.

(a) "mass × acceleration = force" gives, in kg-m-s units:

$$20\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -40\frac{\mathrm{d}x}{\mathrm{d}t} - 180x$$

If desired, rearrange and divide by 20:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = 0$$

(b) If there is no damping (i.e. no $\mathrm{d}x/\mathrm{d}t$ term) then the equation becomes

$$20\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -180x$$

or

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -9x$$

Comparison with the standard SHM equation ($\mathrm{d}^2x/\mathrm{d}t^2 = -\omega^2x$) gives $\omega = 3$ rad s^{-1} and period

$$T = \frac{2\pi}{\omega} \quad = \frac{2\pi}{3} \quad = 2.094 \text{ s}$$

Answer: 2.09 s.

(c) If there is damping then seek a solution $x = A\mathrm{e}^{pt}$ to the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = 0$$

The resulting auxiliary equation is

$$p^2 + 2p + 9 = 0$$

with solution

$$p = \frac{-2 \pm \sqrt{4 - 36}}{2} \quad = -1 \pm \sqrt{(-8)} \quad = -1 \pm 2\sqrt{2}\ \mathrm{i}$$

Hence, the general solution is

$$x = \mathrm{e}^{-t}\{C\cos(2\sqrt{2}t) + D\sin(2\sqrt{2}t)\}$$

A full oscillation is completed over time T (the damped period) such that

$$2\sqrt{2}T = 2\pi$$

$$\Rightarrow \qquad T = \frac{\pi}{\sqrt{2}} \qquad = 2.221 \text{ s}$$

Over this interval the exponential term e^{-t} goes to $e^{-(t+T)} = e^{-t}e^{-T}$, so is reduced by factor

$$e^{-T} = e^{-2.221} = 0.1085$$

Answer: period = 2.22 s; amplitude reduction factor per cycle = 0.109.

Example 13.
Given:

$$m = 40 \text{ kg}$$
$$k = 700 \text{ N m}^{-1}$$
$$c = 60 \text{ N s m}^{-1}$$

Then (by formula):

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{700}{40}} = 4.183 \text{ rad s}^{-1}$$
$$\zeta = \frac{c}{2m\omega} = \frac{60}{2 \times 40 \times 4.183} = 0.1793$$

Answer: 0.179.

Since damping ratio $\zeta < 1$, the system does oscillate and it does so with damped frequency:

$$\omega_d = \omega\sqrt{1 - \zeta^2} = 4.183 \times \sqrt{1 - 0.1793^2} = 4.115 \text{ rad s}^{-1}$$

The period is

$$\frac{2\pi}{\omega_d} = \frac{2\pi}{4.115} = 1.527 \text{ s}$$

Answer: oscillates, with period 1.53 s.

For critical damping we require

$$\zeta = 1$$
$$\Rightarrow \quad \frac{c}{2m\omega} = 1$$
$$\Rightarrow \quad c = 2m\omega = 2 \times 40 \times 4.183 = 334.6 \text{ N s m}^{-1}$$

Answer: 335 N s m^{-1}.

Example 14.

Complementary function

Auxiliary equation:

$$p^2 + 4p + 5 = 0$$

$$\Rightarrow \quad p = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 5}}{2} \quad = \frac{-4 \pm \sqrt{-4}}{2} \quad = -2 \pm i$$

Hence, the complementary function is

$$x = \mathrm{e}^{-2t}(C \cos t + D \sin t)$$

Particular integral

The form of the RHS suggests that one try

$$x = E \cos 2t + F \sin 2t$$

Substituting:

$$(-4E \cos 2t - 4F \sin 2t) + 4(-2E \sin 2t + 2F \cos 2t) + 5(E \cos 2t + F \sin 2t) = 130 \sin 2t$$

Equating coefficients:

cos 2t:

$$-4E + 8F + 5E = 0$$

$$\Rightarrow \quad E = -8F$$

sin 2*t*:

$$-4F - 8E + 5F = 130$$

$$\Rightarrow \quad F - 8E = 130$$

$$\Rightarrow \quad 65F = 130 \quad \text{(using } E = -8F \text{ from above)}$$

$$\Rightarrow \quad F = 2$$

$$E = -16$$

The general solution is the sum of complementary function and particular integral:

$$x = \mathrm{e}^{-2t}(C \cos t + D \sin t) - 16 \cos 2t + 2 \sin 2t$$

Note that the first part decays (exponentially), leaving only the second part at large times.

Example 15.
Note: in this question no harmonic force is applied directly to the mass, but it is in an accelerating reference frame, the acceleration of the seismometer giving rise to an "apparent force".

The seismometer has a harmonic (i.e. sinusoidal) oscillation at 3 Hz; i.e.

$$x_B = A\sin\Omega t$$

where A is to be found. Since one cycle takes 1/3 s,

$$\Omega \times \frac{1}{3} = 2\pi$$

$$\Rightarrow \qquad \Omega = 6\pi$$

The question posed is ... given that the amplitude of response of m is $x_{max} = 0.005$ m, what is the amplitude of the ground shaking, A.

The absolute position of the mass relative to a fixed point is

$$x_B + \text{constant} + x$$

Hence, the absolute acceleration is

$$\frac{\mathrm{d}^2 x_B}{\mathrm{d}t^2} + \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

or, since $x_B = A\sin\Omega t$,

$$\text{absolute acceleration} = -\Omega^2 A\sin\Omega t + \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

where, from above, $\Omega = 6\pi$.

Applying the equation of motion ("force = mass × acceleration"):

$$-150x - 3\frac{\mathrm{d}x}{\mathrm{d}t} = 0.5(-\Omega^2 A\sin\Omega t + \frac{\mathrm{d}^2 x}{\mathrm{d}t^2})$$

$$\Rightarrow \qquad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 300x = \Omega^2 A\sin\Omega t \qquad (*)$$

This is an inhomogeneous second-order equation with constant coefficients; hence it has a solution of the form "complementary function + particular integral". The complementary function will decay exponentially and we are left with a particular integral, which, guided by the RHS, should be of the form

$$x = C\sin\Omega t + D\cos\Omega t$$

where C and D are to be found. Moreover, this can be written in a corresponding "amplitude/phase-angle" form which, with some minor algebra, is

$$x = \sqrt{C^2 + D^2}\sin(\Omega t + \phi)$$

But we know that $x_{\max} = 0.005$ and hence

$$\sqrt{C^2 + D^2} = 0.005 \qquad (**)$$

Differentiating x and substituting in (*):

$$(-\Omega^2 C \sin \Omega t - \Omega^2 D \cos \Omega t) + 6(\Omega C \cos \Omega t - \Omega D \sin \Omega t) + 300(C \sin \Omega t + D \cos \Omega t) = \Omega^2 A \sin \Omega t$$

Equating coefficients of $\cos \Omega t$:

$$-\Omega^2 D + 6\Omega C + 300D = 0$$

$$\Rightarrow \quad 6\Omega C = (\Omega^2 - 300)D$$

$$\Rightarrow \quad C = (\frac{\Omega^2 - 300}{6\Omega})D \qquad \text{where } \Omega = 6\pi$$

$$\Rightarrow \quad C = 0.4890D$$

Equating coefficients of $\sin \Omega t$:

$$-\Omega^2 C - 6\Omega D + 300C = \Omega^2 A$$

$$\Rightarrow \quad (300 - \Omega^2)C - 6\Omega D = \Omega^2 A$$

$$\Rightarrow \quad \{ (300 - \Omega^2) \times 0.4890 - 6\Omega \} D = \Omega^2 A, \quad \text{where} \quad \Omega = 6\pi$$

$$\Rightarrow \quad D = -2.535A$$

and, from the relationship between C and D above,

$$C = -1.240A$$

But, from (**),

$$\sqrt{C^2 + D^2} = 0.005$$

$$\Rightarrow \quad \sqrt{1.240^2 + 2.535^2} \times A = 0.005$$

$$\Rightarrow \quad A = 0.001772$$

Answer: 0.00177 m (= 1.77 mm).

Note: using resonance, the seismometer has amplified the original shaking (amplitude 1.77 mm) to a recordable response (amplitude 5 mm).