Q1.

Use "$s = ut + \frac{1}{2}at^2$" (with $u = 0$ in this case) for each particle. For particle A note that the coordinate-wise distances are $s\cos 45°$ and $s\sin 45°$. For absolute coordinates remember to add the starting positions.

In metre-second units:

$$x_A = 100 + \left(0 \times t + \frac{1}{2} \times 2 \times t^2\right)\cos 45° \quad = 100 + 0.7071t^2$$

$$y_A = 0 + \left(0 \times t + \frac{1}{2} \times 2 \times t^2\right)\sin 45° \quad = 0.7071t^2$$

$$x_B = 0 + \left(0 \times t + \frac{1}{2} \times 4 \times t^2\right) \quad = 2t^2$$

$$y_B = 100 + 0 \quad = 100$$

Distance (squared) by Pythagoras:

$$\begin{aligned} d^2 &= (x_A - x_B)^2 + (y_A - y_B)^2 \\ &= (100 - 1.2929t^2)^2 + (0.7071t^2 - 100)^2 \\ &= 20000 - 400t^2 + 2.172t^4 \end{aligned}$$

This is quadratic in t^2 and we need to find its minimum.

<u>Easiest Method – Just Complete the Square</u>

$$\begin{aligned} d^2 &= 2.172(t^4 - 184.16t^2) + 20000 \\ &= 2.172(t^2 - 92.08)^2 + 1584 \end{aligned}$$

The minimum d^2 is 1584 (occurring when $t^2 = 92.08$). Hence,

$$d_{\min} = \sqrt{1584} \quad = 39.80 \text{ m}$$

<u>Alternative method – Differentiation</u>

(For convenience) write $D = d^2$ and $T = t^2$. Then d is minimised when D is minimised.

$$D = 20000 - 400T + 2.172T^2$$

$$\frac{\mathrm{d}D}{\mathrm{d}T} = -400 + 4.344T$$

Setting $\mathrm{d}D/\mathrm{d}T = 0$ gives

$$T = \frac{400}{4.344} \quad = 92.08$$

$$D_{\min} = 20000 - 400 \times 92.08 + 2.172 \times 92.08^2 \quad = 1584$$

$$d_{\min} = \sqrt{D_{\min}} \quad = \sqrt{1584} \quad = 39.80 \text{ m})$$

Answer: 39.8 m

Q2.

(a) Let the vertical distance between the tether level and the ring be d and the angle made by the string with the vertical be θ.

$$\text{Length of string} = \sqrt{1.5^2 + d^2} \qquad \text{(m)}$$

$$\text{Extension of string,} \quad e = \sqrt{1.5^2 + d^2} - 1.5 \qquad \text{(m)}$$

$$\text{Tension in string,} \quad T = \frac{\lambda e}{L} \quad = 8\left(\sqrt{1.5^2 + d^2} - 1.5\right) \qquad \text{(N)}$$

$$\text{Weight,} \quad W = mg \quad = 5.886 \text{ N}$$

Initially,

$$d = 2.85 \text{ m}$$

$$T = 8\left(\sqrt{1.5^2 + 2.85^2} - 1.5\right) \quad = 13.77 \text{ N}$$

$$\theta = \tan^{-1}\left(\frac{1.5}{2.85}\right) \quad = 27.76°$$

The net *upward* force is

$$T\cos\theta - W = 13.77\cos(27.76°) - 5.886 \quad = 12.185 - 5.886 \quad = 6.299 \text{ N}$$

This is positive, so the initial motion is upward.

Answer: upward.

(b) The maximum (and minimum) displacements occur when the velocity, and hence kinetic energy, is zero: i.e., when the potential energy (elastic + gravitational) is the same as its starting value: $V - V_0 = 0$.

Potential energy as a function of d:

$$\begin{aligned} V \quad &= \frac{1}{2}\frac{\lambda}{L}e^2 - mgd \\ &= 4\left(\sqrt{1.5^2 + d^2} - 1.5\right)^2 - 5.886d \qquad \text{(J)} \end{aligned}$$

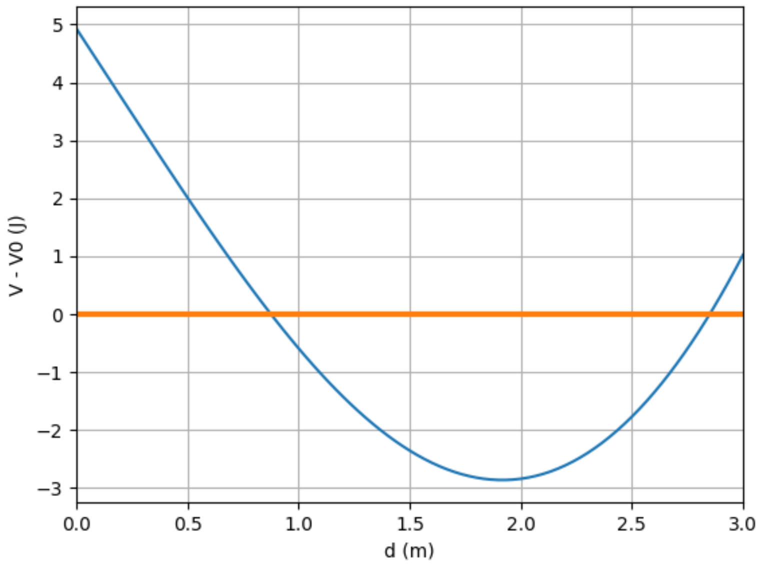
Initially, $d = 2.85$ m and

$$V_0 = -4.933 \text{ J}$$

So, solve for the smaller root (since the larger one is just the starting point) of

$$4\left(\sqrt{1.5^2 + d^2} - 1.5\right)^2 - 5.886d + 4.933 = 0$$

This can be done graphically by successively homing in on a graph:



Alternatively, rearrange as an *implicit* equation for d and solve by single-point iteration:

$$d = \frac{4\left(\sqrt{1.5^2 + d^2} - 1.5\right)^2 + 4.933}{5.886}$$

starting from a small value: $d = 0$, say.

Either approach (or many other numerical ones) gives:

$$d = 0.8763 \text{ m}$$

The distance from the start point is then

$$2.85 - 0.8763 = 1.9737$$

Answer: 1.97 m

(c) From part (a), the normal reaction initially is

$$R = T\sin\theta \quad = 13.77 \times \sin(27.76°) \quad = 6.414 \text{ N}$$

Also from the net (upward) force in part (a),

$$\text{motive force} = 6.299 \text{ N}$$

Motion occurs if and only if

$$\text{motive force} > \mu R$$

Hence, the minimum coefficient of friction to prevent motion is

$$\mu = \frac{\text{motive force}}{R} \quad = \frac{6.299}{6.414} \quad = 0.9821$$

Answer: 0.982

Q3.
(a) The centre of mass coincides (here) with the centroid and lies on the vertical symmetry axis. Find $\bar{y}$, where y is the distance from the base.

Assume length units of m unless otherwise stated. Feel free to use mm instead, but make sure that you are consistent.

Outer rectangle:

$$a_1 = 1.981 \times 0.762 \quad = 1.510$$

$$y_1 = \frac{1}{2} \times 1.981 \quad = 0.9905$$

Rectangular cut-out (subtracted):

$$a_2 = (-)0.9 \times 0.5 \quad = -0.45$$

$$y_2 = 0.6 + \frac{1}{2} \times 0.9 \quad = 1.05$$

Semi-circular cut-out (subtracted):

$$a_3 = (-)\frac{1}{2} \times \pi \times 0.25^2 \quad = -0.09817$$

$$d = \frac{4}{3\pi} \times 0.25 \quad = 0.1061$$

$$y_3 = (0.6 + 0.9 + 0.1) + 0.1061 \quad = 1.706$$

(For the last shape, d is the distance from the base of the semi-circle to its centroid. It is needed here and later.)

Area:

$$A = \sum a_i \quad = 1.510 - 0.45 - 0.09817 \quad = 0.9618$$

Centroid:

$$\bar{y} = \frac{\sum a_i\, y}{A} \quad = \frac{1.510 \times 0.9905 - 0.45 \times 1.05 - 0.09817 \times 1.706}{0.9618} \quad = 0.8897$$

Answer: On the symmetry line, 0.890 m from the lowest side.

(b) Since both mass and moment of inertia are proportional to area, the area mass density in their ratio will cancel. So, **work in area terms (i.e. 2nd moment of area and area rather than moment of inertia and mass), in this part**.

For each shape, use the parallel-axis theorem to switch between C.O.M. axes and the actual axis. *For the semi-circle this means using it twice – firstly in reverse to get to the C.O.M. and then in the forward direction to get to the final axis.*

$$I_{\text{outer}} = \frac{1}{12}a_1 h_1^2 + a_1 y_1^2 \quad = 1.975 \quad (\text{m}^4)$$

$$I_{\text{rectangle}} = \frac{1}{12}a_2 h_2^2 + a_2 y_2^2 \quad = (-)0.5265 \quad (\text{m}^4; \text{to be subtracted})$$

$$I_{\text{semicircle}} = \left(\frac{1}{4}a_3 R^2 - a_3 d^2\right) + a_3 y_3^2 \quad = (-)0.2861 \quad (\text{m}^4; \text{to be subtracted})$$

Overall:

$$I^{(\text{area})} = 1.975 - 0.5265 - 0.2861 \quad = 1.162\ \text{m}^4$$

Radius of gyration:

$$k = \sqrt{\frac{I^{(\text{area})}}{A}} \quad = \sqrt{\frac{1.162}{0.9618}} \quad = 1.099\ \text{m}$$

Answer: 1.10 m

(c) Writing σ as area mass density, although it will simply cancel later:

$$\text{Gain in KE} = \text{loss in PE}$$

$$\frac{1}{2}\left(\sigma I^{(\text{area})}\right)\omega^2 = (\sigma A)g\bar{y}$$

Hence,

$$\omega = \sqrt{\frac{2Ag\bar{y}}{I^{(\text{area})}}} \quad = \sqrt{\frac{2Ag\bar{y}}{Ak^2}} \quad = \frac{\sqrt{2g\bar{y}}}{k}$$

Note: we must use consistent length units. If you chose to have $\bar{y}$ and k in mm then it would be necessary to express g in mm s^{-2}. Here we choose to use m throughout, so that

$$\bar{y} = 0.8897\ \text{m}$$

$$k = 1.099$$

$$g = 9.81\ \text{m s}^{-2}$$

Hence,

$$\omega = \frac{\sqrt{2 \times 9.81 \times 0.8897}}{1.099} \quad = 3.802\ \text{rad s}^{-1}$$

Answer: 3.80 rad s^{-1}

Q4.
(a)

$$\sin\theta = \frac{0.28}{0.5} \quad = 0.56$$

Hence,

$$\theta = 34.06°$$

Answer: 34.1°

(b)

Gain in KE = loss in PE

$$\frac{1}{2}mu^2 = mgh \qquad \text{where} \qquad h = 0.5\cos\theta$$

Hence,

$$u = \sqrt{2gh} \quad = \sqrt{2 \times 9.81 \times (0.5 \times \cos 34.06°)} \quad = 2.851 \text{ m s}^{-1}$$

Answer: 2.85 m s^{-1}

(c) Before collision the normal and tangential components of velocity are:

$$u_{\text{norm}} = u\cos\theta \quad = 2.851 \times \cos 34.06° \quad = 2.362 \text{ m s}^{-1}$$

$$u_{\text{tang}} = u\sin\theta \quad = 2.851 \times \sin 34.06° \quad = 1.597 \text{ m s}^{-1}$$

In the collision only the *normal* component is affected by restitution:

$$v_{\text{norm}} = e \times u_{\text{norm}} \quad = 0.3 \times 2.362 \quad = 0.7086 \text{ m s}^{-1}$$

$$v_{\text{tang}} = u_{\text{tang}} \quad = 1.597 \text{ m s}^{-1}$$

(Obviously, there is also a reversal of direction of the normal component, but we only need its size here.)

Speed:

$$v = \sqrt{v_{\text{norm}}^2 + v_{\text{tang}}^2} \quad = \sqrt{0.7086^2 + 1.597^2} \quad = 1.747 \text{ m s}^{-1}$$

Answer: 1.75 m s^{-1}