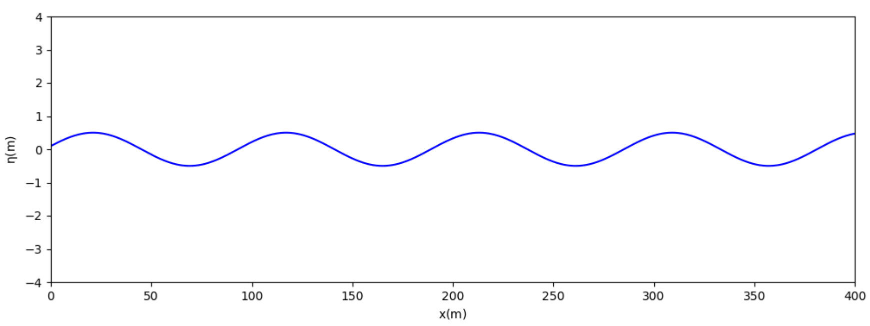
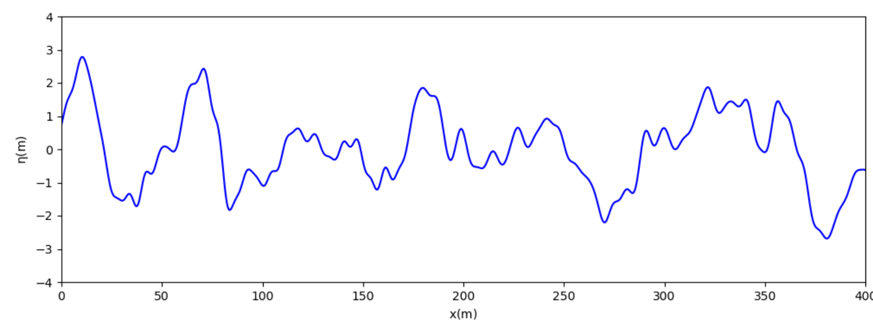
Waves

3. Statistics and Irregular Waves

Statistics and Irregular Waves

- Real wave fields are not regular
- Combination of many periods, heights, directions
- Design and simulation require realistic wave statistics:
  - **probability distribution** of heights
  - **energy spectrum** of frequencies (and directions)





Statistics and Irregular Waves

3. STATISTICS AND IRREGULAR WAVES

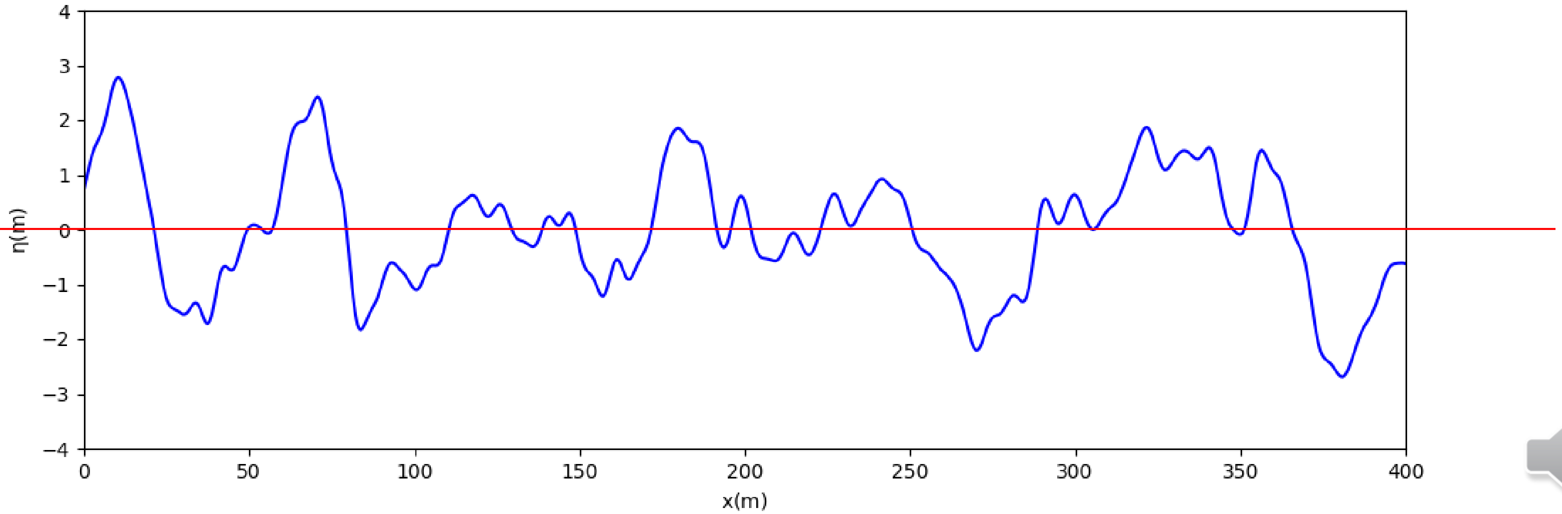
3.1 Measures of height and period
3.2 Probability distribution of wave heights
3.3 Wave spectra
3.4 Reconstructing a wave field
3.5 Prediction of wave climate

Measures of Wave Height

$H_{\max}$	Largest wave height in the sample	$H_{\max} = \max_i(H_i)$
H_{av}	Mean wave height	$H_{\mathrm{av}} = \frac{1}{N}\sum H_i$
H_{rms}	Root-mean-square wave height	$H_{\mathrm{rms}} = \sqrt{\frac{1}{N}\sum H_i^2}$
$H_{1/3}$	Average of the highest $N/3$ waves	$H_{1/3} = \frac{1}{N/3}\sum_{1}^{N/3} H_i$
H_{m0}	Estimate based on the rms surface elevation	$H_{m0} = 4\left(\overline{\eta^2}\right)^{1/2}$
H_s	Significant wave height	Either $H_{1/3}$ or H_{m0}

Measures of Wave Period

T_s	Significant wave period (average of highest $N/3$ waves)
T_p	Peak period (from peak frequency of energy spectrum)
T_e	Energy period (period of a regular wave with same significant wave height and power density; used in wave-energy prediction; derived from energy spectrum)
T_z	Mean zero up-crossing period



Statistics and Irregular Waves

3. STATISTICS AND IRREGULAR WAVES

3.1 Measures of height and period

3.2 Probability distribution of wave heights

3.3 Wave spectra

3.4 Reconstructing a wave field

3.5 Prediction of wave climate

Probability Distribution of Wave Heights

For a narrow-banded frequency spectrum the **Rayleigh probability distribution** is appropriate:

$$P(\text{height} > H) = \exp\left[-\left(H/H_{\text{rms}}\right)^2\right]$$

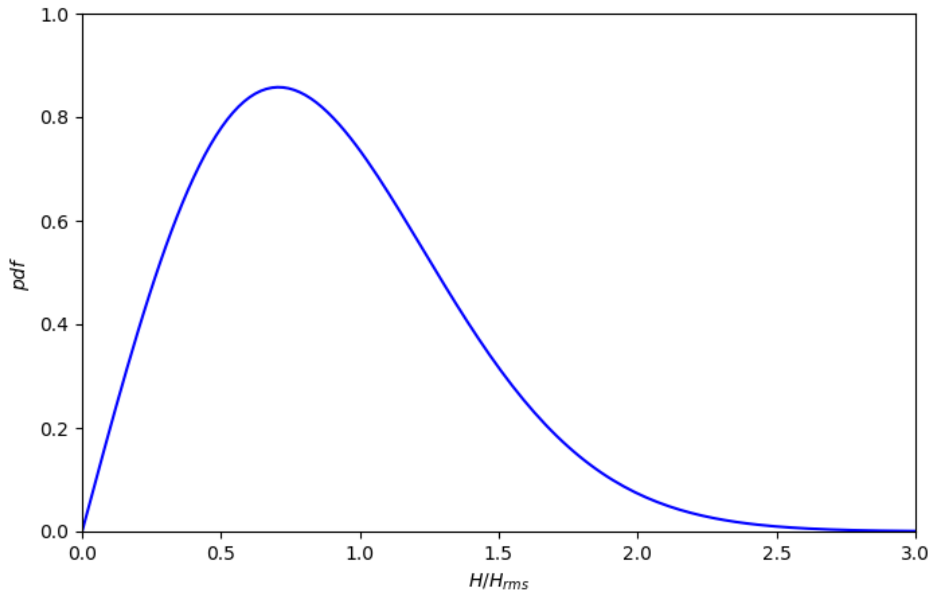
Cumulative distribution function:

$$F(H) = P(\text{height} < H) \quad = 1 - \mathrm{e}^{-(H/H_{\text{rms}})^2}$$

Probability density function:

$$f(H) = \frac{\mathrm{d}F}{\mathrm{d}H} \quad = 2\frac{H}{H_{\text{rms}}^2}\mathrm{e}^{-(H/H_{\text{rms}})^2}$$

Rayleigh Distribution



$$f(H) = 2\frac{H}{H_{\mathrm{rms}}^2}\,\mathrm{e}^{-(H/H_{\mathrm{rms}})^2}$$

Single parameter: H_{rms}

$$H_{\mathrm{av}} = \frac{\sqrt{\pi}}{2} H_{\mathrm{rms}} = 0.886\, H_{\mathrm{rms}}$$

$$H_{\mathrm{rms}}^2 \equiv E(H^2) \quad = \int_0^\infty H^2\, f(H)\,\mathrm{d}H$$

$$H_{1/3} = 1.416\, H_{\mathrm{rms}}$$

$$H_{1/10} = 1.800\, H_{\mathrm{rms}}$$

$$H_{\mathrm{av}} \equiv E(H) \quad = \int_0^\infty H\, f(H)\,\mathrm{d}H$$

$$H_{1/100} = 2.359\, H_{\mathrm{rms}}$$

Example

Near a pier, 400 consecutive wave heights are measured. Assume that the sea state is narrow-banded.

(a) How many waves are expected to exceed $2H_{\mathrm{rms}}$?

(b) If the significant wave height is 2.5 m, what is H_{rms}?

(c) Estimate the wave height exceeded by 80 waves.

(d) Estimate the number of waves with a height between 1.0 m and 3.0 m.

Near a pier, 400 consecutive wave heights are measured. Assume that the sea state is narrow-banded.

(a) How many waves are expected to exceed $2H_{\mathrm{rms}}$?

(b) If the significant wave height is 2.5 m, what is H_{rms}?

(c) Estimate the wave height exceeded by 80 waves.

(d) Estimate the number of waves with a height between 1.0 m and 3.0 m.

(a) $P(\text{height} > H) = \mathrm{e}^{-(H/H_{\mathrm{rms}})^2}$

$P(\text{height} > 2H_{\mathrm{rms}}) = \mathrm{e}^{-4} \quad = 0.01832$

In 400 waves, $n = 400 \times 0.01832 \quad = \mathbf{7.328}$

(b) $H_{1/3} = 1.416\, H_{\mathrm{rms}}$

$2.5 = 1.416\, H_{\mathrm{rms}}$

$\boldsymbol{H_{\mathrm{rms}} = 1.766\ \mathrm{m}}$

(c) $P(\text{height} > H) = \mathrm{e}^{-(H/H_{\mathrm{rms}})^2} \quad = \dfrac{80}{400} \quad = 0.2$

$(H/H_{\mathrm{rms}})^2 = -\ln 0.2$

$H = H_{\mathrm{rms}} \times \sqrt{-\ln 0.2} \quad = \mathbf{2.240\ m}$

(d) $P(1.0 < \text{height} < 3.0) = P(\text{height} > 1.0) - P(\text{height} > 3.0)$

$$= \mathrm{e}^{-(1.0/H_{\mathrm{rms}})^2} - \mathrm{e}^{-(3.0/H_{\mathrm{rms}})^2}$$

$$= 0.6699$$

In 400 waves, $n = 400 \times 0.6699 \quad = \mathbf{268.0}$

Statistics and Irregular Waves

3. STATISTICS AND IRREGULAR WAVES

3.1 Measures of height and period

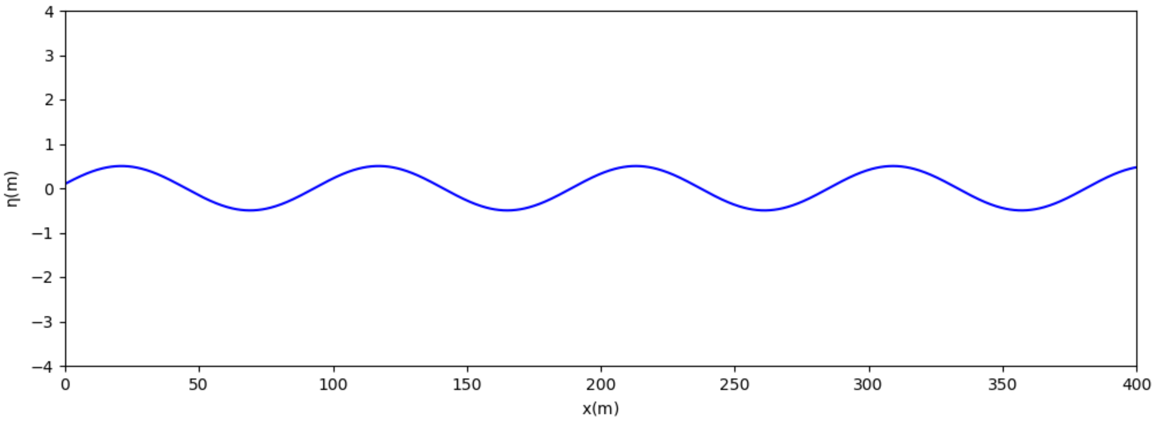
3.2 Probability distribution of wave heights

3.3 Wave spectra

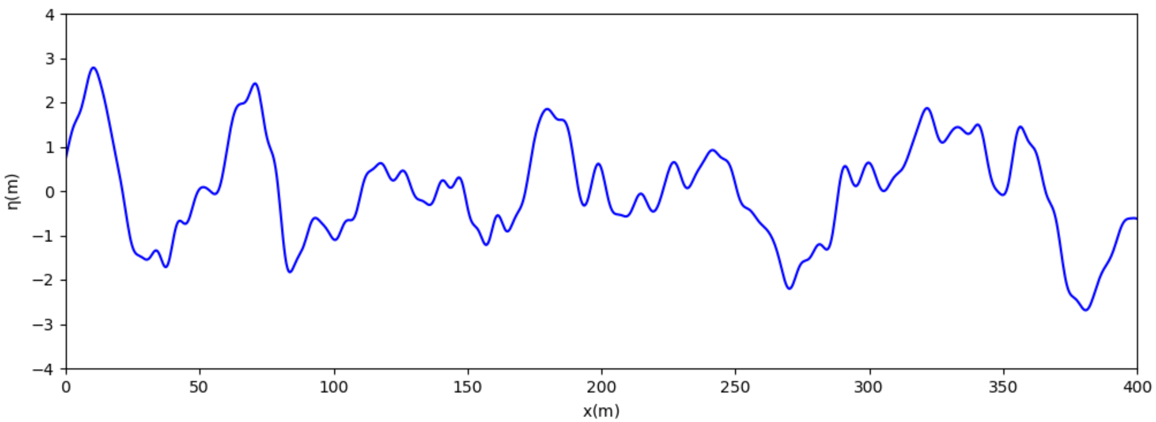
3.4 Reconstructing a wave field

3.5 Prediction of wave climate

Regular vs Irregular Waves



Regular wave:
- single frequency



Irregular wave:
- many frequencies

Wave **energy** depends on $\overline{\eta^2}$

Energy Spectrum

- "**Spectral**" means "**by frequency**"
- A **spectrum** is usually determined by a **Fourier transform**
- This splits a signal up into its **component frequencies**

- **Energy** in a wave is proportional to $\overline{\eta^2}$, where η is surface displacement
- The **energy spectrum**, or **power spectrum**, is the Fourier transform of η^2

Energy Spectrum

$\eta = A\cos(kx - \omega t)$

For **regular** waves (single frequency):

$$E = \frac{1}{2}\rho g A^2 \quad = \rho g \overline{\eta^2(t)} \quad = \frac{1}{8}\rho g H^2$$

For **irregular** waves (many frequencies):

$S(f)\,\mathrm{d}f$ is the "energy" in a small interval $\mathrm{d}f$ near frequency f

$\int_{f_1}^{f_2} S(f)\,\mathrm{d}f$ is the "energy" between frequencies f_1 and f_2

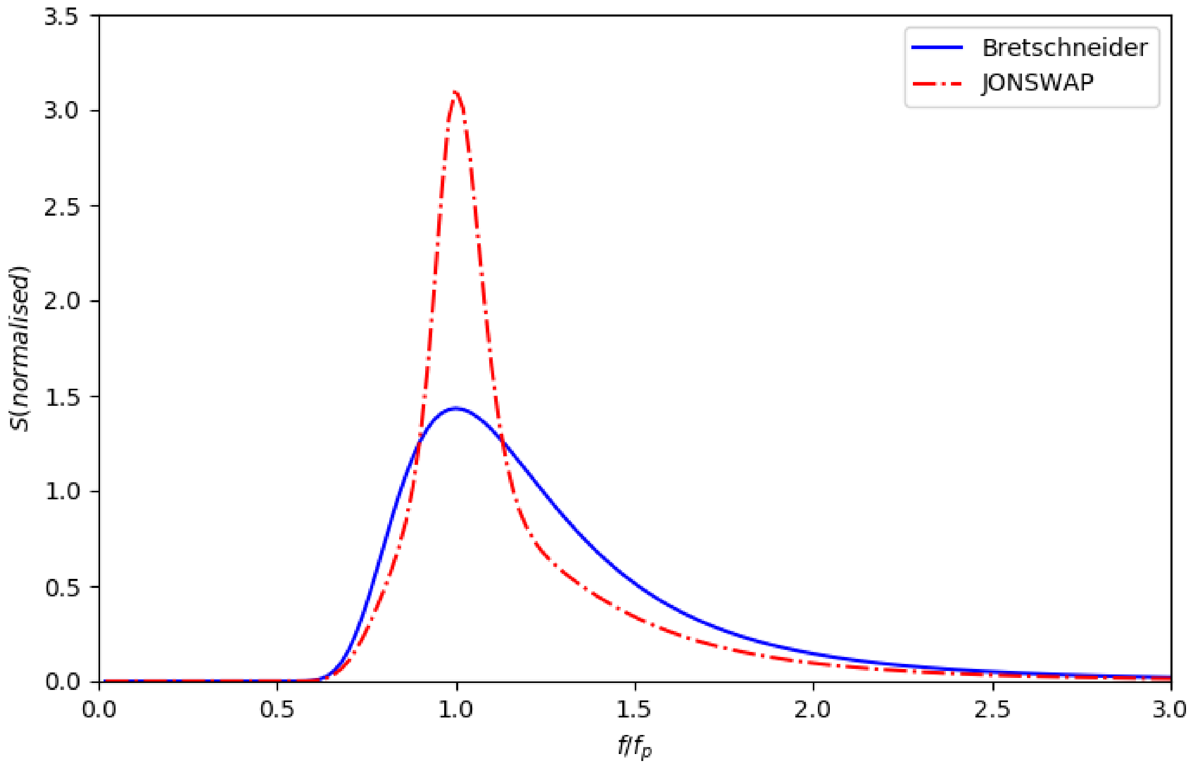
(strictly: energy$/\rho g$)

The **energy spectrum** $S(f)$ is determined by Fourier transforming η^2

Model Spectra

Open ocean: **Bretschneider spectrum**

Fetch-limited seas: **JONSWAP spectrum**



Key parameters: **peak frequency** f_p (=1/(peak period, T_p))

significant wave height H_{m0}

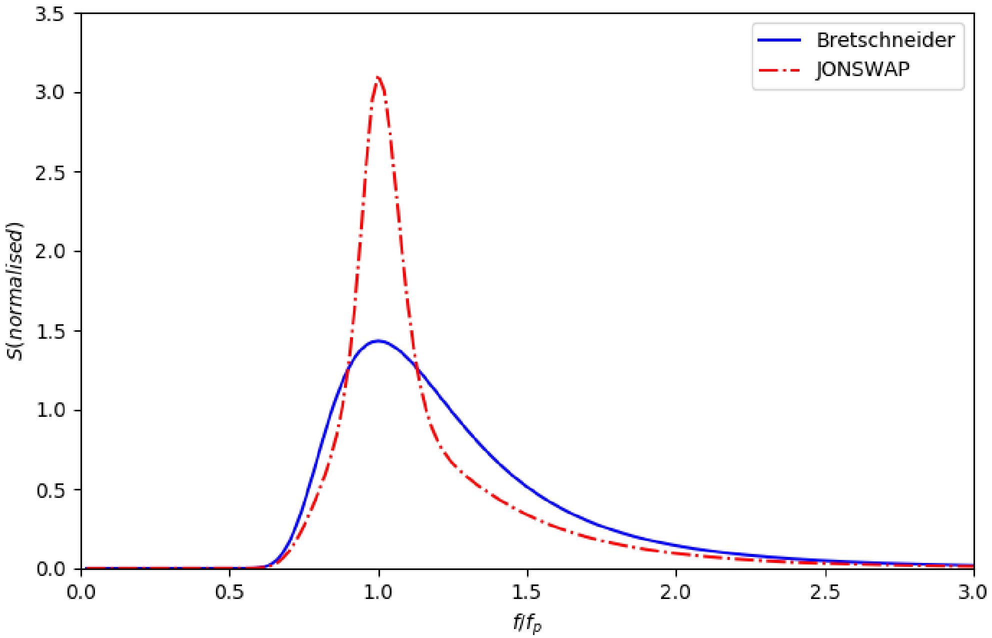
Model Spectra

Bretschneider spectrum:

$$S(f) = \frac{5}{16} H_{m0}^2 \frac{f_p^4}{f^5} \exp\left(-\frac{5}{4}\frac{f_p^4}{f^4}\right)$$

JONSWAP spectrum:

$$S(f) = C\, H_{m0}^2 \frac{f_p^4}{f^5} \exp\left(-\frac{5}{4}\frac{f_p^4}{f^4}\right)\gamma^b$$



$$b = \exp\left\{-\frac{1}{2}\left(\frac{f/f_p - 1}{\sigma}\right)^2\right\}$$

$$\sigma = \begin{cases} 0.07 & f < f_p \\ 0.09 & f > f_p \end{cases}$$

$$\gamma = 3.3$$

Significant Wave Height, H_{m0}

Bretschneider **spectrum**: $$S(f) = \frac{5}{16} H_{m0}^2 \frac{f_p^4}{f^5} \exp\left(-\frac{5}{4}\frac{f_p^4}{f^4}\right)$$

Total energy: $$\frac{E}{\rho g} \equiv \int_0^\infty S(f)\,\mathrm{d}f \quad = \frac{1}{16} H_{m0}^2$$

$$H_{m0} = 4\sqrt{E/\rho\, g} \quad = 4\sqrt{\overline{\eta^2(t)}}$$

Total energy for a **regular wave**: $$\frac{E}{\rho g} = \frac{1}{8} H_{\mathrm{rms}}^2$$

Same energy if $$H_{m0} = \sqrt{2} H_{rms} \quad = 1.414 H_{\mathrm{rms}}$$

Rayleigh distribution: $$H_{1/3} = 1.416 H_{\mathrm{rms}}$$

H_{m0} and $H_{1/3}$ can be used synonymously for H_s …

… but H_{m0} is easier to measure!

Finding Wave Parameters From a Spectrum

- Surface displacement $\eta(t)$ is measured (e.g. wave buoy)
- $\eta^2(t)$ is Fourier-transformed to get energy spectrum $S(f)$
- Height and period parameters are deduced from the peak and the **moments** of the spectrum:

$$m_n = \int_0^\infty f^n S(f)\,\mathrm{d}f$$

Peak period:

$$T_p = \frac{1}{f_p}$$

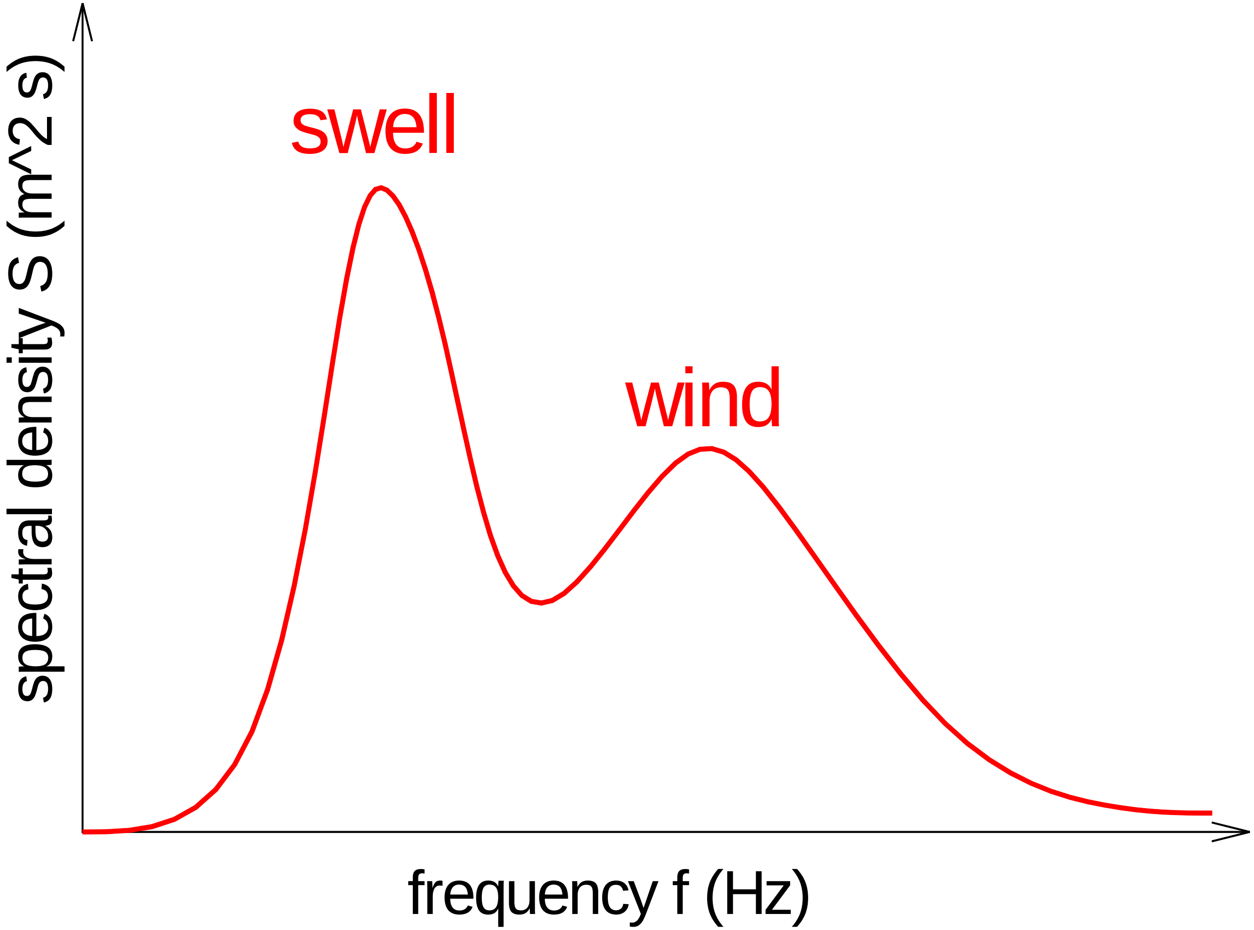
Significant wave height:

$$H_s = H_{m0} = 4\sqrt{m_0}$$

Energy period:

$$T_e = \frac{m_{-1}}{m_0}$$

Multi-Modal Spectra



Statistics and Irregular Waves

3. STATISTICS AND IRREGULAR WAVES

3.1 Measures of height and period

3.2 Probability distribution of wave heights

3.3 Wave spectra

3.4 Reconstructing a wave field

3.5 Prediction of wave climate

Using a Spectrum To Generate a Wave Field

$$\eta(t) = \sum a_i \cos(k_i x - \omega_i t - \phi_i)$$

amplitude

$$S(f_i)\Delta f = E_i = \frac{1}{2} a_i^2$$

$$a_i = \sqrt{2S(f_i)\Delta f}$$

(angular) frequency

$$\omega_i = 2\pi f_i$$

wavenumber

$$\omega_i^2 = g k_i \tanh k_i h$$

random phase

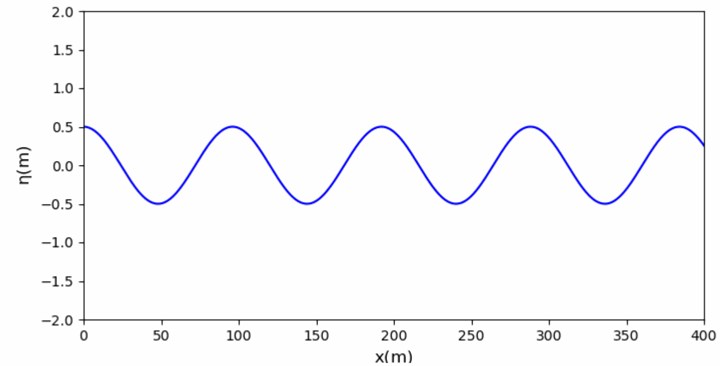
Simulated Wave Field

$T_p = 8\ \mathrm{s}$

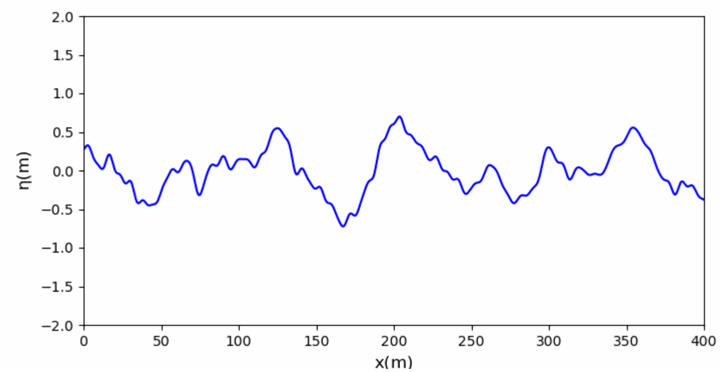
$H_s = 1.0\ \mathrm{m}$

$h = 30\ \mathrm{m}$

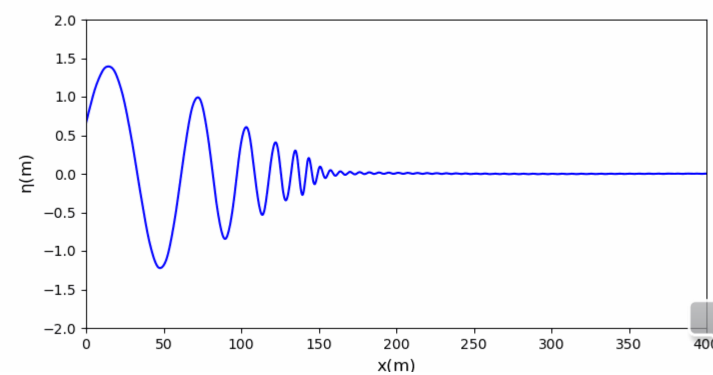
Regular waves:



Irregular waves:



Focused waves:



Example

An irregular wavefield at a deep-water location is characterised by peak period of 8.7 s and significant wave height of 1.5 m.

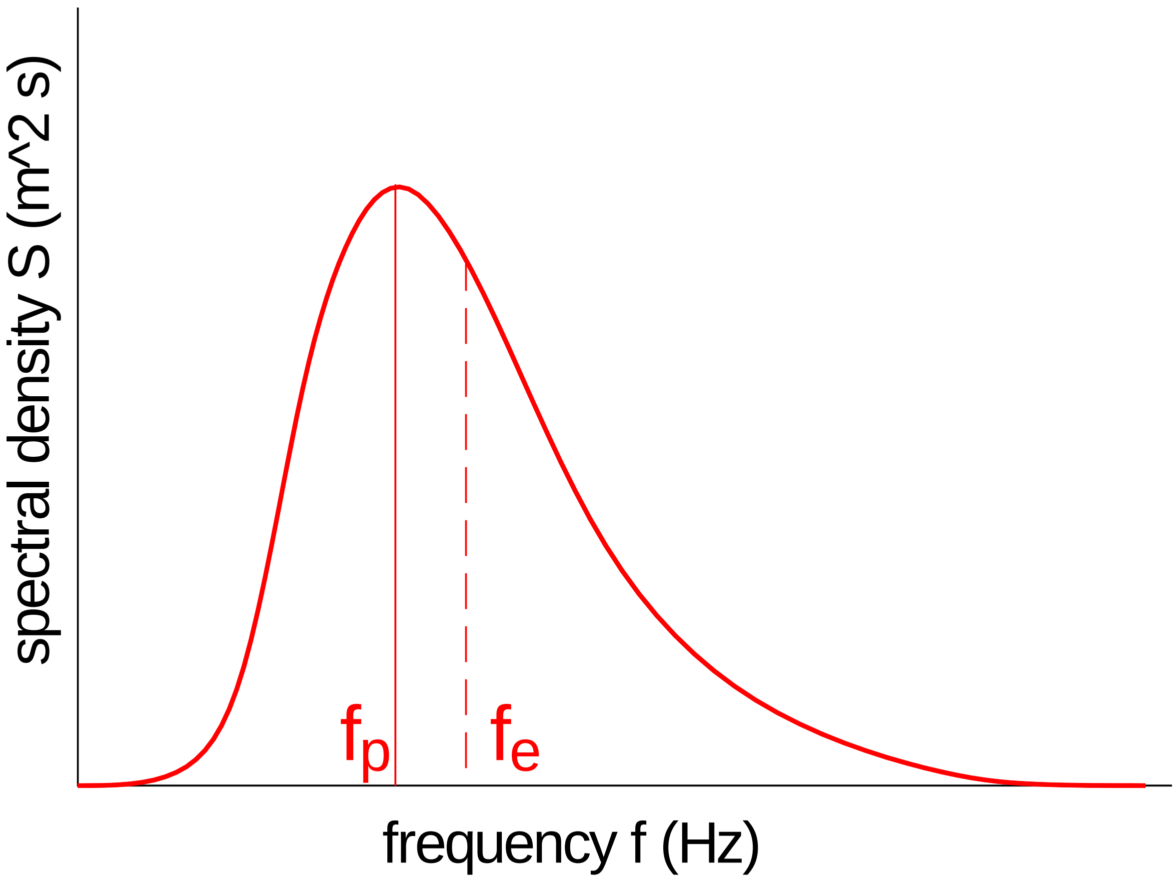
(a) Provide a sketch of a Bretschneider spectrum, labelling both axes with variables and units and indicating the frequencies corresponding to both the peak period and the energy period.

Note: Calculations are not needed for this part.

(b) Determine the power density (in kW m^{-1}) of a regular wave component with frequency 0.125 Hz that represents the frequency range 0.12 to 0.13 Hz of the irregular wave field.

An irregular wavefield at a deep-water location is characterised by peak period of 8.7 s and significant wave height of 1.5 m.

(a) Provide a sketch of a Bretschneider spectrum, labelling both axes with variables and units and indicating the frequencies corresponding to both the peak period and the energy period.



An irregular wavefield at a deep-water location is characterised by peak period of 8.7 s and significant wave height of 1.5 m.

(b) Determine the power density (in kW m^{-1}) of a regular wave component with frequency 0.125 Hz that represents the frequency range 0.12 to 0.13 Hz of the irregular wave field.

$$S(f) = \frac{5}{16} H_s^2 \frac{f_p^4}{f^5} \exp\left(-\frac{5}{4}\frac{f_p^4}{f^4}\right)$$

$T_p = 8.7\ \mathrm{s}$ $\quad$ $H_s = 1.5\ \mathrm{m}$

$$f_p = \frac{1}{T_p} = 0.1149\ \mathrm{Hz}$$

$f = 0.125\ \mathrm{Hz}$ $\quad$ $S(f) = 1.645\ \mathrm{m^2\ s}$

$\Delta f = 0.01\ \mathrm{Hz}$

$E = \rho g \times S(f)\Delta f = 165.4\ \mathrm{J\ m^{-2}}$

$P = Ec_g = E(nc)$ $\quad$ Deep water: $n = \frac{1}{2}$ $\quad$ $T = \frac{1}{f} = 8\ \mathrm{s}$

$$c = \frac{gT}{2\pi} = 12.49\ \mathrm{m\ s^{-1}}$$

$\boldsymbol{P = 1.033\ \mathrm{kW\ m^{-1}}}$

Statistics and Irregular Waves

3. STATISTICS AND IRREGULAR WAVES

3.1 Measures of height and period

3.2 Probability distribution of wave heights

3.3 Wave spectra

3.4 Reconstructing a wave field

3.5 Prediction of wave climate

Terminology

A **wave climate** or **sea state** is a model wave spectrum, usually defined by a representative **height** and **period**, for use in:

- **forecasting** (from a weather forecast in advance)
- **nowcasting** (from ongoing weather)
- **hindcasting** (reconstruction from previous event)

Prediction of Sea State

Require: significant wave height, H_s
significant wave period, T_s (or peak period, T_p)

Depend on: wind speed, U
fetch, F
duration, t
gravity, g

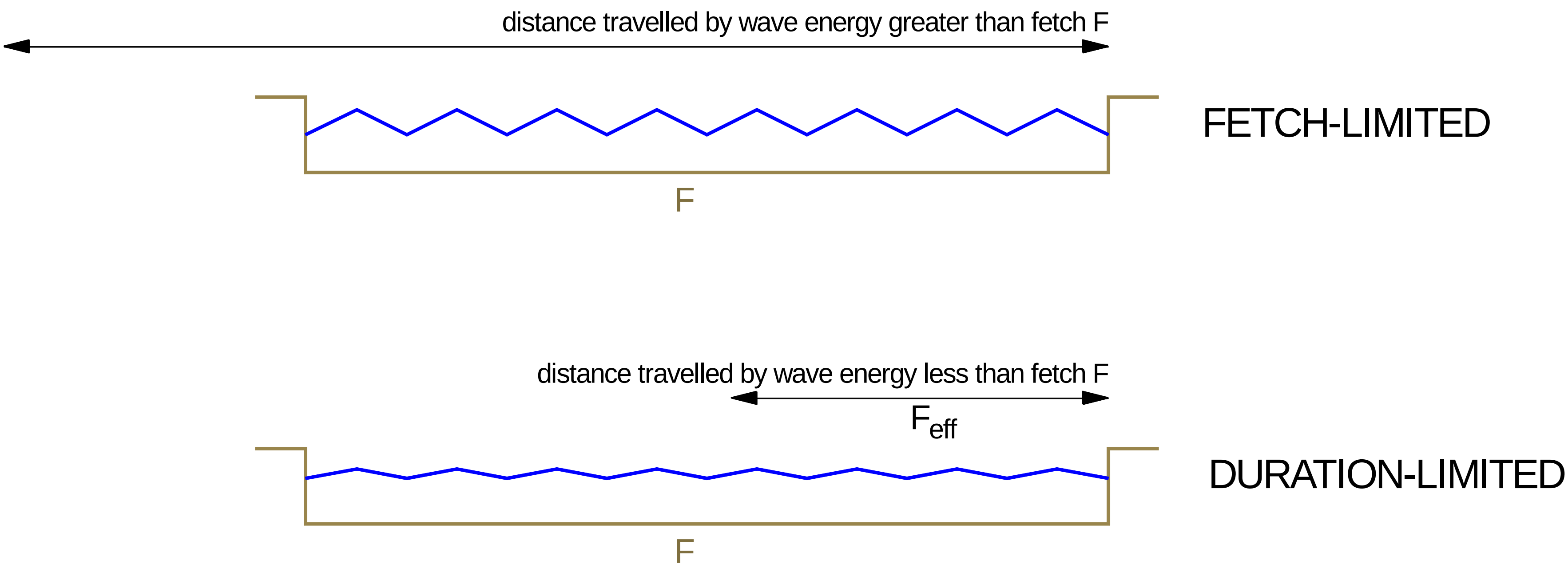
Dimensional analysis:

$$\frac{gH_s}{U^2} = \text{function}\left(\frac{gF}{U^2}, \frac{gt}{U}\right)$$

$$\frac{gT_p}{U} = \text{function}\left(\frac{gF}{U^2}, \frac{gt}{U}\right)$$

Empirical functions: **JONSWAP** or **SMB** curves

Fetch-Limited vs Duration-Limited



JONSWAP Curves

For **fetch-limited** waves:

$$\frac{gH_s}{U^2} = 0.0016\left(\frac{gF}{U^2}\right)^{1/2} \qquad \text{(up to maximum 0.2433)}$$

$$\frac{gT_p}{U} = 0.286\left(\frac{gF}{U^2}\right)^{1/3} \qquad \text{(up to maximum 8.134)}$$

Waves are fetch-limited provided the storm has blown for a minimum time t_{min} given by

$$\left(\frac{gt}{U}\right)_{\min} = 68.8\left(\frac{gF}{U^2}\right)^{2/3} \qquad \text{(up to maximum } 7.15 \times 10^4\text{)}$$

Otherwise the waves are **duration-limited**, and the fetch used to determine height and period is an effective fetch determined by inversion using the actual storm duration t:

$$\left(\frac{gF}{U^2}\right)_{\text{eff}} = \left(\frac{1}{68.8}\frac{gt}{U}\right)^{3/2}$$

JONSWAP Curves (reprise)

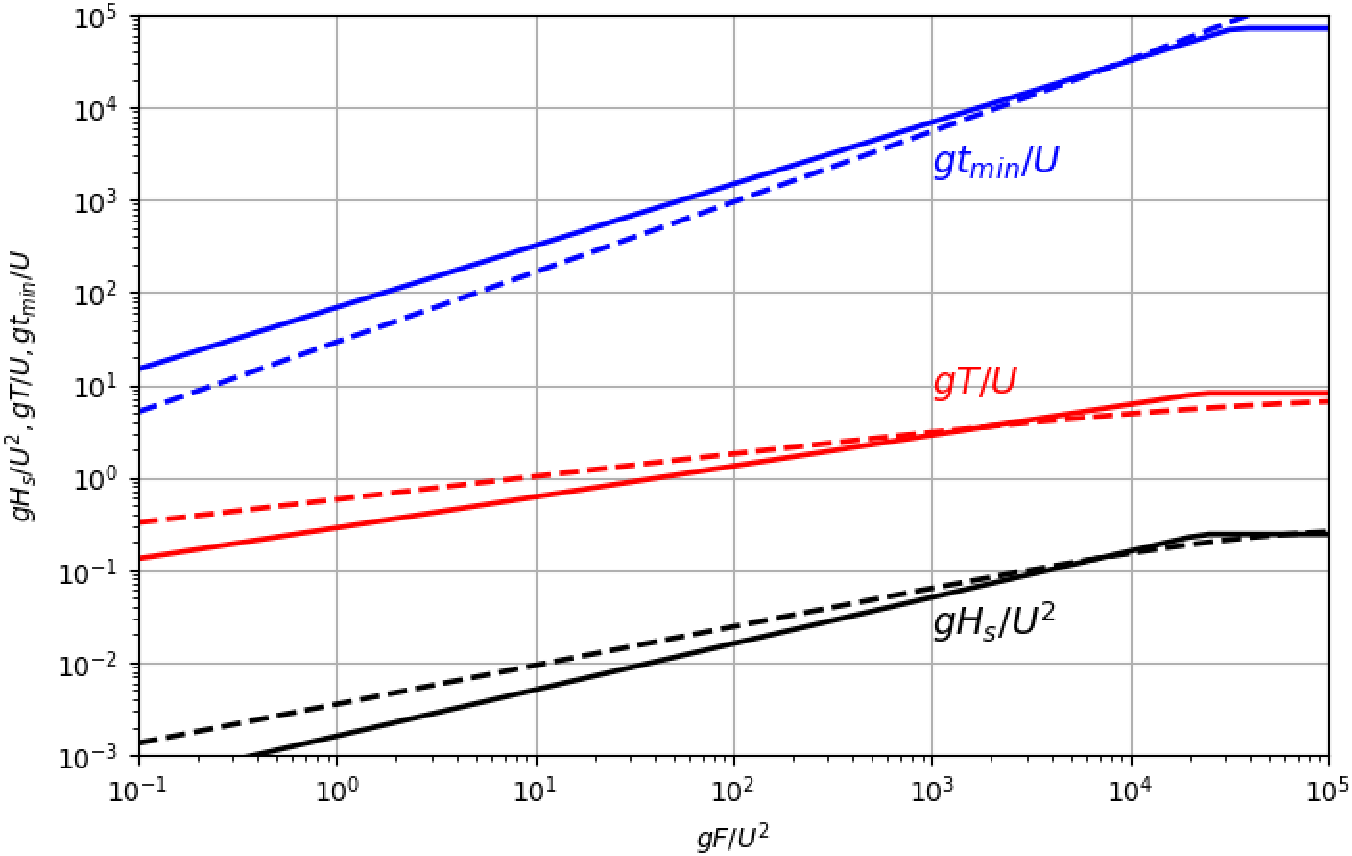
$$\widehat{H}_s = 0.0016\hat{F}^{1/2}$$

$$\hat{T}_p = 0.286\hat{F}^{1/3}$$

$$\hat{t}_{\min} = 68.8\hat{F}^{2/3}$$

$$\hat{F} \equiv \frac{gF}{U^2}\ , \qquad \hat{t} \equiv \frac{gt}{U}\ , \qquad \widehat{H}_s \equiv \frac{gH_s}{U^2}\ , \qquad \hat{T}_p \equiv \frac{gT_p}{U}$$

JONSWAP and SMB Curves



Solid lines: JONSWAP

Dashed lines: SMB

Example

(a) Wind has blown at a consistent $U = 20 \text{ m s}^{-1}$ over a fetch $F = 100 \text{ km}$ for $t = 6 \text{ hrs}$. Determine H_s and T_p using the JONSWAP curves.

(b) If the wind blows steadily for another 4 hours what are H_s and T_p?

(a) Wind has blown at a consistent $U = 20 \text{ m s}^{-1}$ over a fetch $F = 100 \text{ km}$ for $t = 6 \text{ hrs}$. Determine H_s and T_p using the JONSWAP curves.

$$\hat{H}_s = 0.0016\hat{F}^{1/2}$$

$$\hat{T}_p = 0.286\hat{F}^{1/3}$$

$$\hat{t}_{\min} = 68.8\hat{F}^{2/3}$$

$$\hat{F} \equiv \frac{gF}{U^2} \qquad \hat{H}_s \equiv \frac{gH_s}{U^2} \qquad \hat{t} \equiv \frac{gt}{U}$$

$$U = 20 \text{ m s}^{-1}$$

$$F = 10^5 \text{ m}$$

$$t = 6 \times 3600 = 21600 \text{ s}$$

$$\hat{F} \equiv \frac{gF}{U^2} = 2453$$

$$\hat{t} = \frac{gt}{U} = 10590 \qquad \hat{t}_{\min} = 68.8\hat{F}^{2/3} = 12510$$

Duration-limited

$$\hat{F}_{\text{eff}} = \left(\frac{\hat{t}}{68.8}\right)^{3/2} = 1910$$

$$\hat{H}_s = 0.0016\hat{F}_{\text{eff}}^{1/2} = 0.06993 \qquad H_s = \hat{H}_s \times \frac{U^2}{g} = \mathbf{2.851 \text{ m}}$$

$$\hat{T}_p = 0.286\hat{F}_{\text{eff}}^{1/3} = 3.548 \qquad T_p = \hat{T}_p \times \frac{U}{g} = \mathbf{7.233 \text{ s}}$$

(b) If the wind blows steadily for another 4 hours what are H_s and T_p?

$U = 20 \text{ m s}^{-1}$

$F = 10^5 \text{ m}$ $\quad\quad \hat{F} = 2453$

$\hat{t}_{\min} = 12510$

$$\hat{H}_s = 0.0016\hat{F}^{1/2}$$

$$\hat{T}_p = 0.286\hat{F}^{1/3}$$

$$\hat{t}_{\min} = 68.8\hat{F}^{2/3}$$

$$\hat{F} \equiv \frac{gF}{U^2} \quad\quad \hat{H}_s \equiv \frac{gH_s}{U^2} \quad\quad \hat{t} \equiv \frac{gt}{U}$$

$t = 10 \times 3600 = 36000 \text{ s}$

$\hat{t} = \frac{gt}{U} = 17660$ Fetch-limited

$\hat{H}_s = 0.0016\hat{F}^{1/2} = 0.07924$ $\quad\quad H_s = \hat{H}_s \times \frac{U^2}{g} = \mathbf{3.231 \text{ m}}$

$\hat{T}_p = 0.286\hat{F}^{1/3} = 3.857$ $\quad\quad T_p = \hat{T}_p \times \frac{U}{g} = \mathbf{7.863 \text{ s}}$