B. Waves

1. **Linear** wave theory
2. Wave **transformation**
3. **Random** waves and **statistics**
4. **Wave loading** on structures

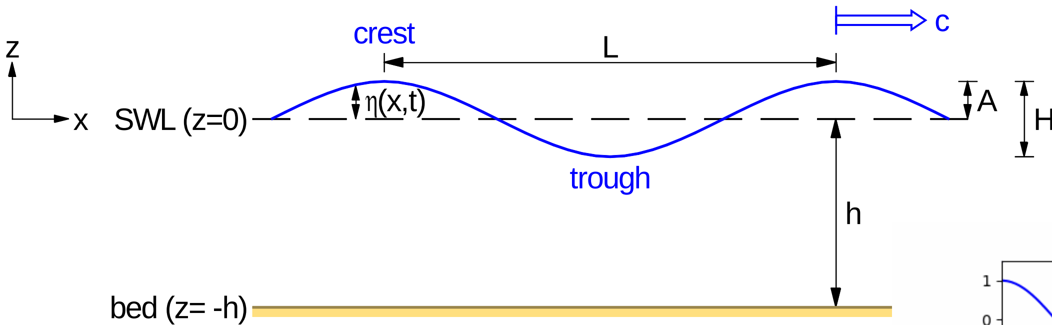
Recommended Books

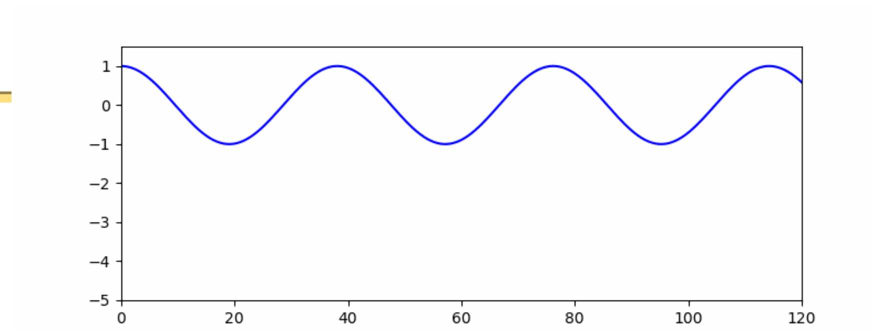
- **Dean and Dalrymple**, *Water Wave Mechanics For Engineers and Scientists*
- **Kamphuis**, *Introduction to coastal engineering and management*

Linear Wave Theory

1. LINEAR WAVE THEORY

Linear Wave Theory





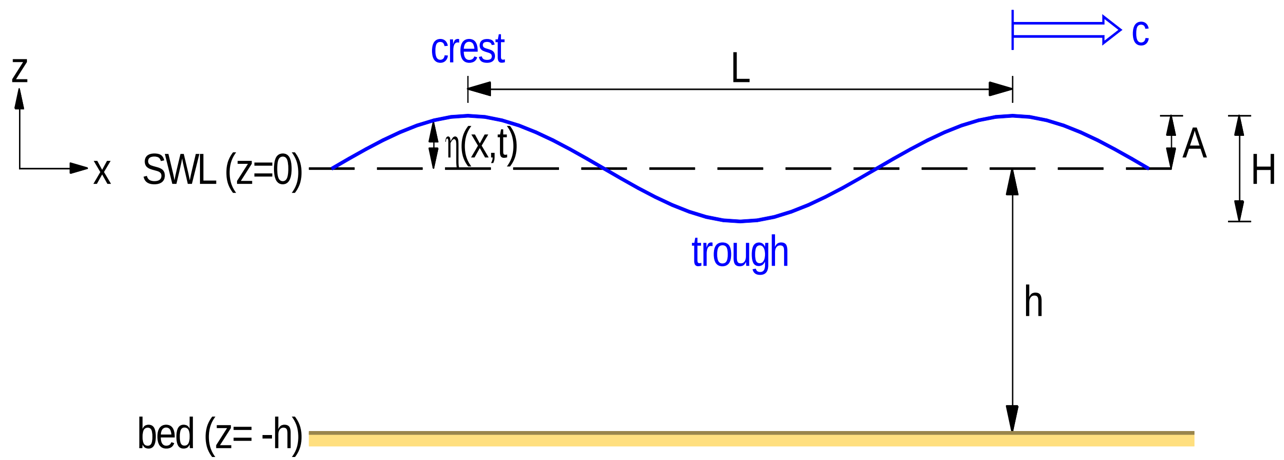
Single-frequency ("monochromatic", "regular") **progressive** wave on **still water**:

$$\eta = A\cos(kx - \omega t)$$

Linear wave theory:

- aka **Airy** wave theory
- assume **amplitude small** (compared with depth and wavelength)
  - neglect powers and products of wave perturbations
  - sum of any such wave fields also a solution

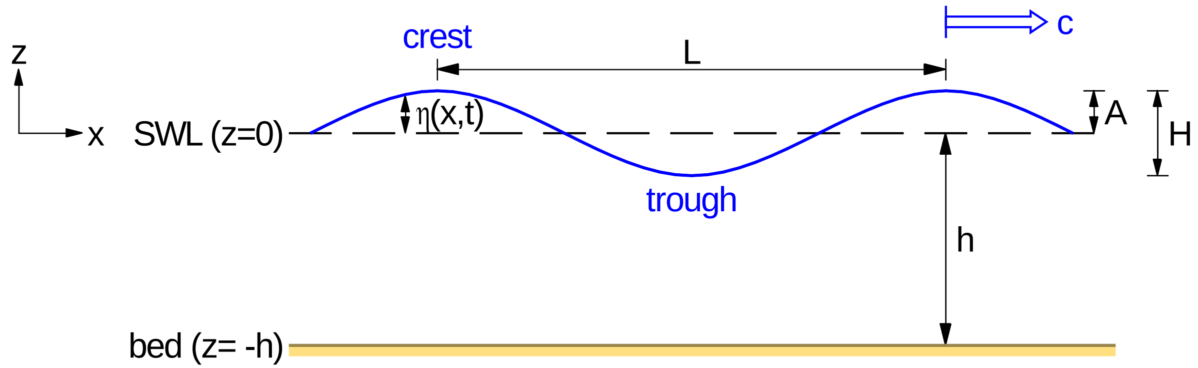
Amplitude and Height



$$\eta = A\cos(kx - \omega t)$$

- **Amplitude** A is the maximum displacement from still-water level (SWL)
- **Wave height** H is the vertical distance between neighbouring crest and trough
- For sinusoidal waves, $\boldsymbol{H = 2A}$
- For **regular** waves, formulae more naturally expressed in terms of A
- For **irregular** waves, H is the more measurable quantity

Wavenumber and Wavelength

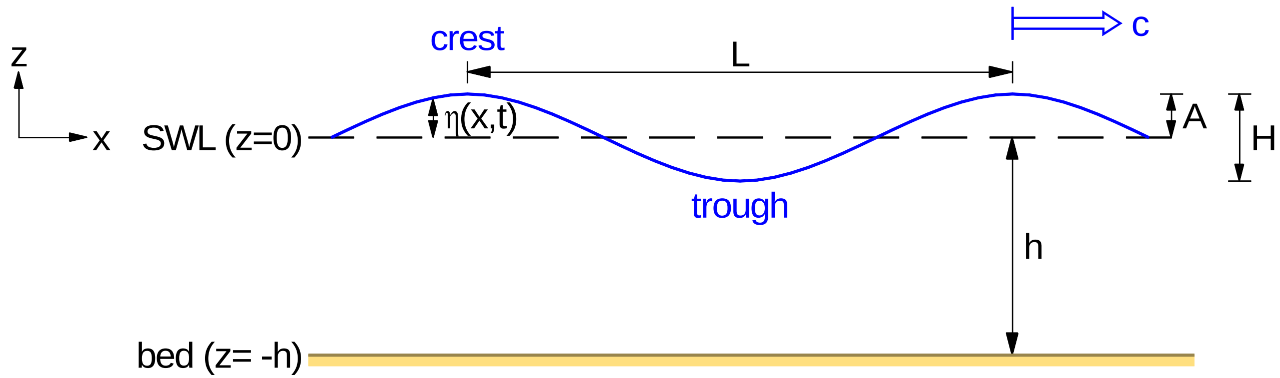


$$\eta = A\cos(kx - \omega t)$$

- k is the **wavenumber**
- **Wavelength** L is the horizontal distance over which wave form repeats:

$$kL = 2\pi \qquad \Rightarrow \qquad L = \frac{2\pi}{k}$$

Frequency and Period



$$\eta = A\cos(kx - \omega t)$$

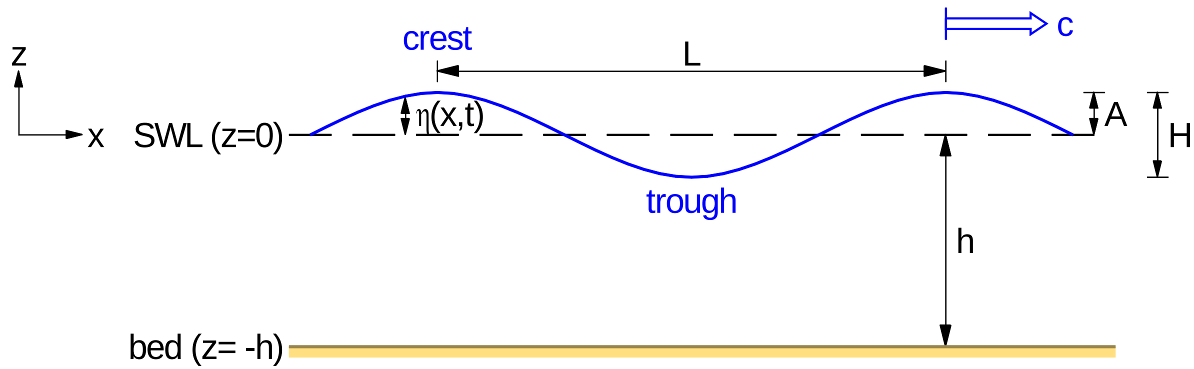
- ω is the wave **angular frequency**
- **Period** T is the time over which the wave form repeats:

$$\omega T = 2\pi \qquad \Rightarrow \qquad T = \frac{2\pi}{\omega}$$

- The actual **frequency** f (cycles per second, or Hertz) is

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Wave Speed

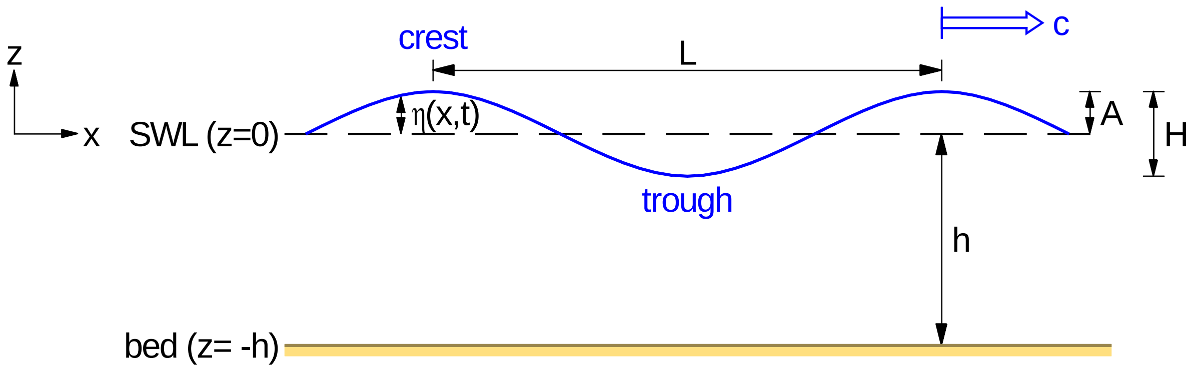


$$\eta = A\cos(kx - \omega t) \qquad = A\cos\left[k\left(x - \frac{\omega}{k}t\right)\right] \qquad = A\cos[k(x - ct)]$$

- c is the **phase speed** or **celerity**
- c is the speed at which the wave form translates

$$\begin{aligned} c &= \frac{\omega}{k} \\ &= \frac{L}{T} \quad \left(\frac{\text{wavelength}}{\text{period}}\right) \\ &= fL \quad (\text{frequency} \times \text{wavelength}) \end{aligned}$$

Summary



Surface elevation: $\eta = A\cos(kx - \omega t)$

Wavenumber k, wavelength L: $L = \dfrac{2\pi}{k}$

Angular frequency ω, period T: $T = \dfrac{2\pi}{\omega}$

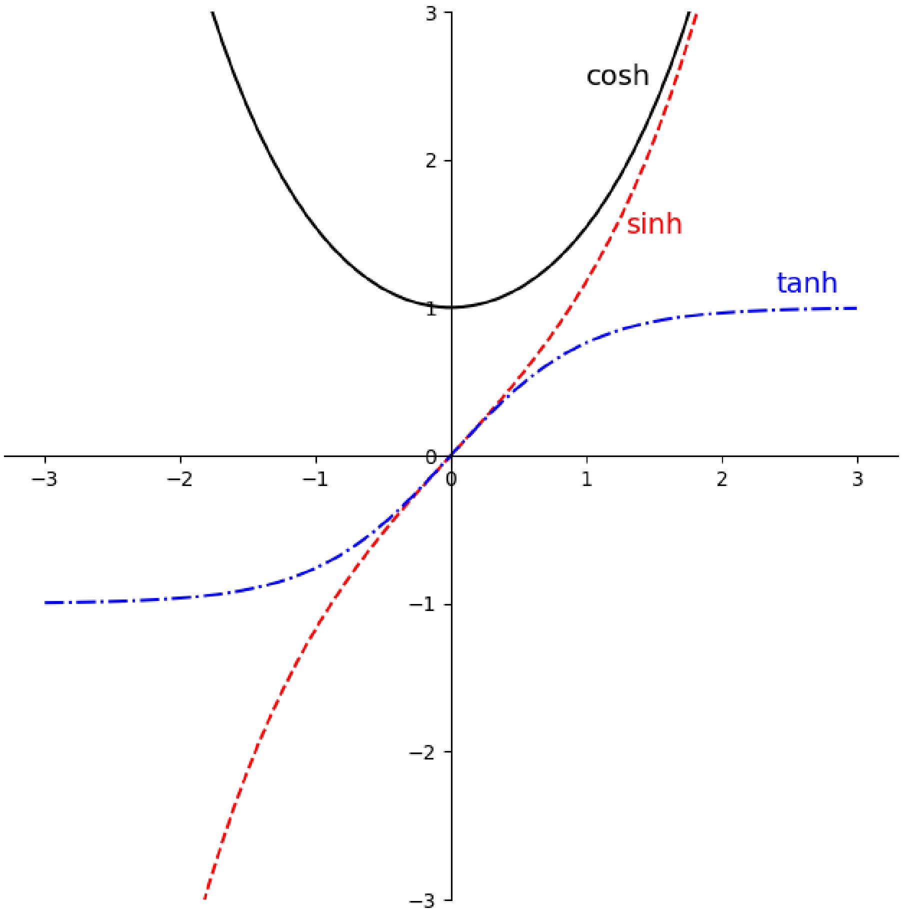
Phase speed (celerity) c: $c = \dfrac{\omega}{k} \quad = \dfrac{L}{T}$

Hyperbolic Functions

$$\sinh x \equiv \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x \equiv \frac{e^{x} + e^{-x}}{2}$$

$$\tanh x \equiv \frac{\sinh x}{\cosh x}$$



Hyperbolic Functions

Trigonometric-like formulae:

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x \quad = \; 2\cosh^2 x - 1$$

$$\sinh 2x = 2\sinh x \cosh x$$

Derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) = \mathrm{sech}^2\, x$$

Asymptotic behaviour:

Small x:

$$\sinh x \sim \tanh x \sim x, \quad \cosh x \to 1 \text{ as } x \to 0$$

Large x:

$$\sinh x \sim \cosh x \sim \frac{1}{2}\mathrm{e}^x, \quad \tanh x \to 1 \text{ as } x \to \infty$$

Fluid-Flow Equations

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Irrotationality:

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

or:

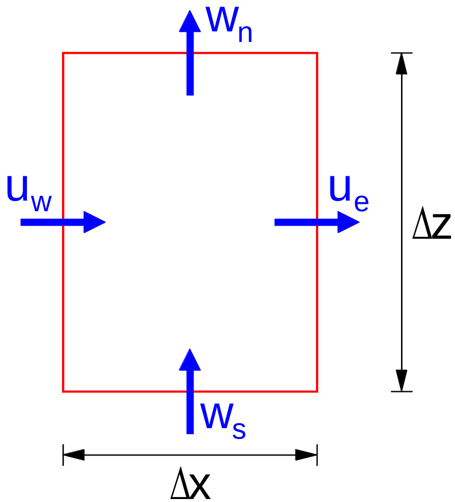
$$u = \frac{\partial \phi}{\partial x}, \qquad w = \frac{\partial \phi}{\partial z}$$

ϕ is a **velocity potential**

Time-dependent **Bernoulli equation**:

$$\rho \frac{\partial \phi}{\partial t} + p + \frac{1}{2}\rho U^2 + \rho g z = C(t), \qquad \text{along a streamline}$$

Continuity



Net volume outflow: $$u_e \Delta z - u_w \Delta z + w_n \Delta x - w_s \Delta x = 0$$

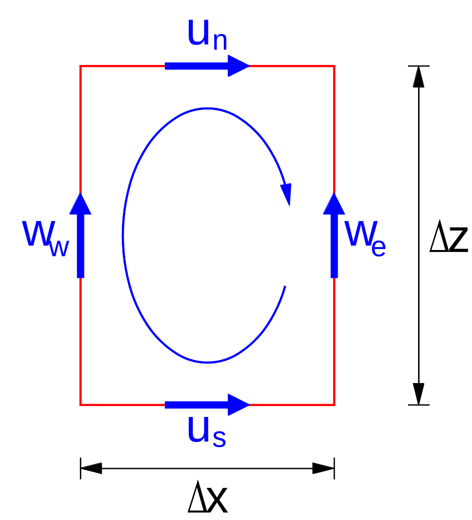
Divide by $\Delta x \Delta z$: $$\frac{u_e - u_w}{\Delta x} + \frac{w_n - w_s}{\Delta z} = 0$$

$$\frac{\Delta u}{\Delta x} + \frac{\Delta w}{\Delta z} = 0$$

$\Delta x, \Delta z \rightarrow 0$: $$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Irrotationality

Pressure forces act normal to surfaces, so can cause no rotation.



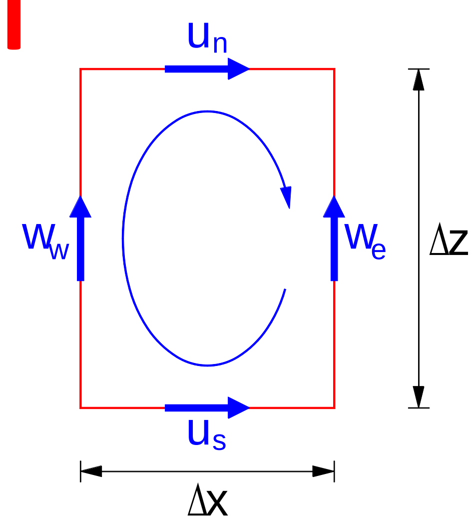
Circulation: $$u_n \Delta x - w_e \Delta z - u_s \Delta x + w_w \Delta z = 0$$

Divide by $\Delta x \Delta z$: $$\frac{u_n - u_s}{\Delta z} - \frac{w_e - w_w}{\Delta x} = 0$$

$$\frac{\Delta u}{\Delta z} - \frac{\Delta w}{\Delta x} = 0$$

$\Delta x, \Delta z \rightarrow 0$: $$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

Velocity Potential



The no-circulation condition makes the following well-defined:

$$d\phi = u\,dx + w\,dz$$

For *any* 2-d function:

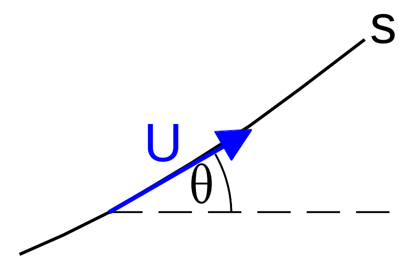
$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial z}\,dz$$

The velocity components are the gradient of the **velocity potential** ϕ:

$$u = \frac{\partial \phi}{\partial x}, \qquad w = \frac{\partial \phi}{\partial z}$$

Aim: solve a single scalar equation for ϕ, then derive everything else from it.

Time-Dependent Bernoulli Equation



mass × acceleration = force

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial s}\right) = -\frac{\partial p}{\partial s} - \rho g \sin\theta$$

$$U = \frac{\partial \phi}{\partial s} \qquad \sin\theta = \partial z/\partial s$$

$$\rho\left(\frac{\partial^2 \phi}{\partial t\,\partial s} + \frac{\partial}{\partial s}\left(\tfrac{1}{2}U^2\right)\right) = -\frac{\partial p}{\partial s} - \rho g\frac{\partial z}{\partial s}$$

$$\frac{\partial}{\partial s}\left[\rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho U^2 + p + \rho g z\right] = 0$$

$$\rho\frac{\partial \phi}{\partial t} + p + \frac{1}{2}\rho U^2 + \rho g z = C(t),$$ along a streamline

Special case: if steady-state then

$$p + \frac{1}{2}\rho U^2 + \rho g z = C,$$ along a streamline

Recap of Fluid-Flow Equations

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Velocity potential

$$u = \frac{\partial \phi}{\partial x}, \quad w = \frac{\partial \phi}{\partial z}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Laplace's equation

Bernoulli equation

$$\rho \frac{\partial \phi}{\partial t} + p + \frac{1}{2}\rho U^2 + \rho g z = C(t)$$

Boundary Conditions

- **Kinematic boundary condition**: no net flow through boundary
- **Dynamic boundary condition**: stress continuous at interface

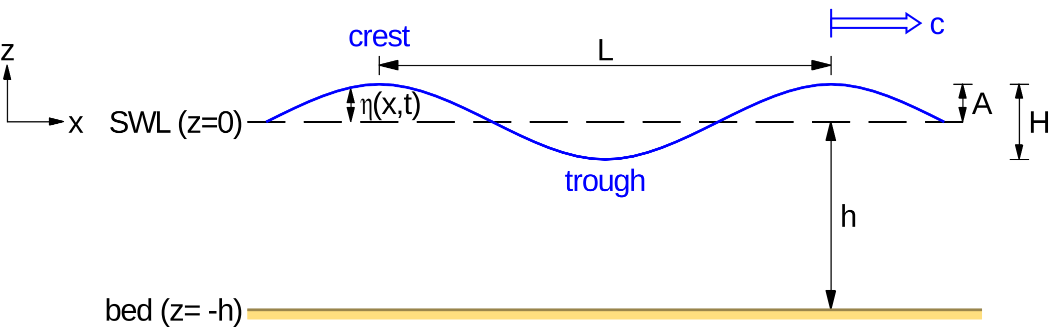
$$z = z_{\mathrm{surf}}(x, t) \xrightarrow{\text{KBC}} \frac{\mathrm{D}}{\mathrm{D}t}(z - z_{\mathrm{surf}}) = 0 \quad \text{on surface}$$

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}$$

$$w - \frac{\partial z_{\mathrm{surf}}}{\partial t} - u\frac{\partial z_{\mathrm{surf}}}{\partial x} = 0 \quad \text{on the surface}$$

$$w = \frac{\partial z_{\mathrm{surf}}}{\partial t} + u\frac{\partial z_{\mathrm{surf}}}{\partial x} \quad \text{on} \quad z = z_{\mathrm{surf}}(x, t)$$

Boundary Conditions



$$w = \frac{\partial z_{\text{surf}}}{\partial t} + u\frac{\partial z_{\text{surf}}}{\partial x} \quad \text{on} \quad z = z_{\text{surf}}$$

KBBC – Kinematic Bed Boundary Condition

$$w = 0 \qquad \text{on} \qquad z = -h$$

KFSBC – Kinematic Free-Surface Boundary Condition

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} \qquad \text{on} \qquad z = \eta(x,t)$$

DFSBC – Dynamic Free-Surface Boundary Condition

$$p = 0 \quad \text{on} \quad z = \eta(x,t)$$

Linearised Equations

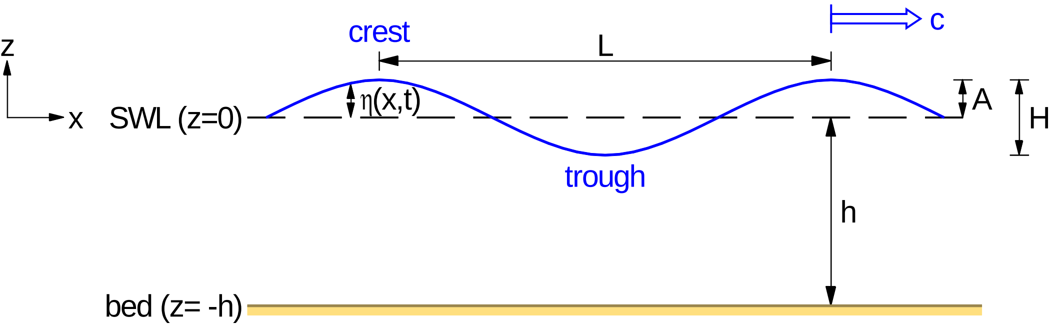
$$y = a + b\varepsilon + c\varepsilon^2 + \cdots$$

- If ε is small, ignore quadratic and higher powers:

$$y = a + b\varepsilon + \cdots$$

- Boundary conditions on $z = \eta(x, t)$ can be applied on $z = 0$

Boundary Conditions



$$w = \frac{\partial z_{surf}}{\partial t} + u\frac{\partial z_{surf}}{\partial x} \quad \text{on} \quad z = z_{surf}$$

$$\rho\frac{\partial \phi}{\partial t} + p + \frac{1}{2}\rho U^2 + \rho g z = C(t)$$

KBBC – Kinematic Bed Boundary Condition

$$w = 0 \qquad \text{on} \qquad z = -h$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{on} \qquad z = -h$$

KBBC – Kinematic Free-Surface Boundary Condition

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} \qquad \text{on} \qquad z = \eta(x,t)$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{on} \qquad z = 0$$

DFSBC – Dynamic Free-Surface Boundary Condition

$$p = 0 \quad \text{on} \quad z = \eta(x,t)$$

$$\frac{\partial \phi}{\partial t} + g\eta = C(t) \qquad \text{on} \qquad z = 0$$

Summary of Equations and BCs

Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

KBBC

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = -h$$

KFSBC

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{on} \quad z = 0$$

DFSBC

$$\frac{\partial \phi}{\partial t} + g\eta = C(t) \quad \text{on} \quad z = 0$$

Solution For Velocity Potential, ϕ

Surface displacement: $\eta = A\cos(kx - \omega t)$

Look for solution by separation of variables: $\phi = X(x,t)Z(z)$

KFSBC: $\dfrac{\partial \phi}{\partial z} = \dfrac{\partial \eta}{\partial t}$ on $z = 0$

$$X \left.\frac{\mathrm{d}Z}{\mathrm{d}z}\right|_{z=0} = A\omega \sin(kx - \omega t)$$

Hence: $X \propto \sin(kx - \omega t)$

WLOG: $X = \sin(kx - \omega t)$ $\qquad \left.\dfrac{\mathrm{d}Z}{\mathrm{d}z}\right|_{z=0} = A\omega$

Laplace's equation: $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial z^2} = 0$

$$-k^2 XZ + X\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = 0$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = k^2 Z$$

General solution: $Z = \alpha \mathrm{e}^{kz} + \beta \mathrm{e}^{-kz}$

Solution For Velocity Potential, ϕ

So far: $\phi = Z(z)\sin(kx - \omega t)$

$$Z = \alpha \mathrm{e}^{kz} + \beta \mathrm{e}^{-kz}$$

KFSBC: $\dfrac{\mathrm{d}Z}{\mathrm{d}z} = A\omega$ on $z = 0$

KBBC: $\dfrac{\mathrm{d}Z}{\mathrm{d}z} = 0$ on $z = -h$

Solution: $Z = \dfrac{A\omega}{k}\dfrac{\cosh k(h+z)}{\sinh kh}$

$$\phi = \frac{A\omega}{k}\frac{\cosh k(h+z)}{\sinh kh}\sin(kx - \omega t)$$

Dispersion Relationship

$$\phi = \frac{A\omega}{k}\frac{\cosh k(h+z)}{\sinh kh}\sin(kx-\omega t)$$

How is wavenumber (k) related to wave angular frequency (ω)?

DFSBC:

$$\frac{\partial \phi}{\partial t} + g\eta = C(t) \quad \text{on} \quad z = 0$$

$$-\frac{A\omega^2}{k}\frac{\cosh kh}{\sinh kh}\cos(kx-\omega t) + Ag\cos(kx-\omega t) = C(t)$$

LHS has zero space average ... so $C(t)$ must be zero

$$-\frac{\omega^2}{k}\frac{\cosh kh}{\sinh kh} + g = 0$$

$$\omega^2 = gk\tanh kh$$

$$\frac{\omega}{k} = \left(\frac{g}{\omega}\right)\frac{\sinh kh}{\cosh kh}$$

$$\phi = \frac{Ag}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx-\omega t)$$

Summary of Solution

Surface displacement: $\eta = A\cos(kx - \omega t)$

Velocity potential: $\phi = \dfrac{Ag}{\omega}\dfrac{\cosh k(h+z)}{\cosh kh}\sin(kx - \omega t)$

Dispersion relation: $\omega^2 = gk\tanh kh$

This is all we need!!!

Velocity: $u \equiv \dfrac{\partial \phi}{\partial x}$ $\quad w \equiv \dfrac{\partial \phi}{\partial z}$

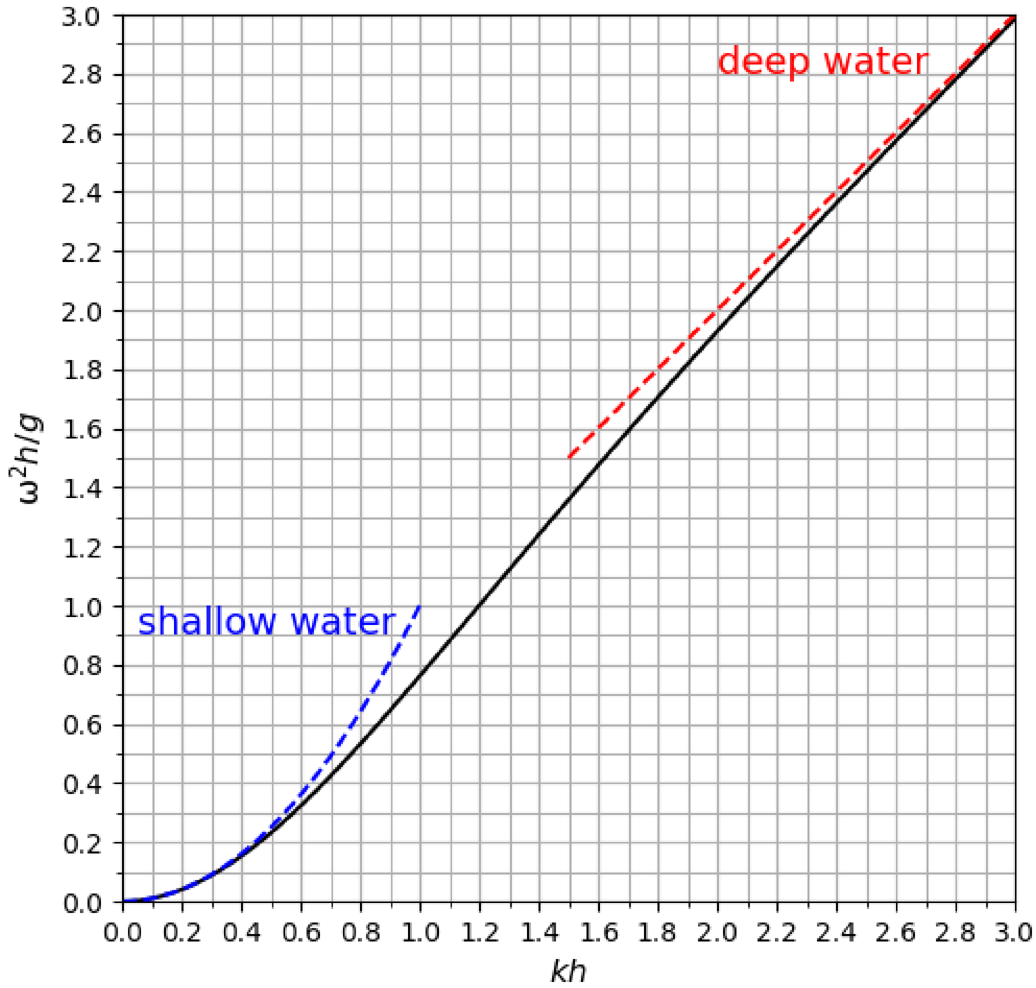
Pressure: $p = -\rho g z - \rho \dfrac{\partial \phi}{\partial t}$

Linear Wave Theory

1. LINEAR WAVE THEORY

Dispersion Relationship

$\omega^2 = gk \tanh kh$ or $\dfrac{\omega^2 h}{g} = kh \tanh kh$



$T = \dfrac{2\pi}{\omega}$ $L = \dfrac{2\pi}{k}$

$c \equiv \dfrac{\omega}{k} = \sqrt{\dfrac{g}{k} \tanh kh}$

Variation of Phase Speed With Depth

$$\omega^2 = gk \tanh kh$$

When waves propagate into **shallower water**:

Period T - and hence ω - are unchanged

Depth h decreases … so wavenumber k increases

Wavelength L decreases

Speed c decreases

This is VERY important !

Solving the Dispersion Relationship

$$\omega^2 = gk \tanh kh$$

1. Know wavelength (L) … find period (T)

$$k = \frac{2\pi}{L}$$

Substitute: gives ω

$$T = \frac{2\pi}{\omega}$$

Solving the Dispersion Relationship

$$\omega^2 = gk \tanh kh$$

2. Know period (T) ... find wavelength (L)

$$\omega = \frac{2\pi}{T}$$

Rewrite as $$\frac{\omega^2 h}{g} = kh \tanh kh$$ $$Y = X \tanh X$$

Iterate $$X = \frac{Y}{\tanh X}$$ or $$X = \frac{1}{2}\left(X + \frac{Y}{\tanh X}\right)$$

Gives $X = kh$ and hence k

$$L = \frac{2\pi}{k}$$

Example

Find, in still water of depth 15 m:

(a) the period of a wave with wavelength 45 m;

(b) the wavelength of a wave with period 8 s.

In each case write down the phase speed (celerity).

Find, in still water of depth 15 m:

(a) the period of a wave with wavelength 45 m;

(b) the wavelength of a wave with period 8 s.

In each case write down the phase speed (celerity).

$h = 15 \text{ m}$

$$\omega^2 = gk \tanh kh$$

wavelength: $L = 45 \text{ m}$

wavenumber: $k = \dfrac{2\pi}{L} = 0.1396 \text{ m}^{-1}$

angular frequency: $\omega = 1.153 \text{ rad s}^{-1}$

period: $T = \dfrac{2\pi}{\omega} = \mathbf{5.449\ s}$

phase speed (celerity): $c = \dfrac{\omega}{k} = \mathbf{8.259\ m\ s^{-1}}$

$\left(\text{or } \dfrac{L}{T}\right)$

Find, in still water of depth 15 m:

(a) the period of a wave with wavelength 45 m;

(b) the wavelength of a wave with period 8 s.

In each case write down the phase speed (celerity).

$h = 15\ \text{m}$

$$\omega^2 = gk \tanh kh$$

period: $T = 8\ \text{s}$

angular frequency: $\omega = \frac{2\pi}{T} = 0.7854\ \text{rad s}^{-1}$

$$\frac{\omega^2 h}{g} = kh \tanh kh$$

$$kh \tanh kh = 0.9432$$

$$kh = \frac{0.9432}{\tanh kh}$$ or $$kh = \frac{1}{2}\left(kh + \frac{0.9432}{\tanh kh}\right)$$

$$kh = 1.152$$

wavenumber: $k = 0.0768\ \text{m}^{-1}$

wavelength: $L = \frac{2\pi}{k} = \mathbf{81.81\ m}$

phase speed (celerity): $c = \frac{\omega}{k} = \mathbf{10.23\ m\ s^{-1}}$

Linear Wave Theory

1. LINEAR WAVE THEORY

1.1 Main wave parameters
1.2 Dispersion relationship
1.3 Wave velocity and pressure
1.4 Wave energy
1.5 Group velocity
1.6 Energy transfer (wave power)
1.7 Particle motion
1.8 Shallow-water and deep-water behaviour
1.9 Waves on currents

Velocity

Surface displacement: $\eta = A\cos(kx - \omega t)$

Velocity potential: $\phi = \dfrac{Ag}{\omega}\dfrac{\cosh k(h+z)}{\cosh kh}\sin(kx - \omega t)$

$$u \equiv \frac{\partial \phi}{\partial x} \qquad = \frac{Agk}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx - \omega t)$$

$$w \equiv \frac{\partial \phi}{\partial z} \qquad = \frac{Agk}{\omega}\frac{\sinh k(h+z)}{\cosh kh}\sin(kx - \omega t)$$

Pressure

Surface displacement: $\eta = A\cos(kx - \omega t)$

Velocity potential: $$\phi = \frac{Ag}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx - \omega t)$$

Bernoulli equation: $$\rho\frac{\partial\phi}{\partial t} + p + \rho g z = 0$$

$$p = -\rho g z - \rho\frac{\partial\phi}{\partial t}$$

$$p = \underbrace{-\rho g z}_{\text{hydrostatic}} + \underbrace{\rho g A\frac{\cosh k(h+z)}{\cosh kh}\cos(kx - \omega t)}_{\text{hydrodynamic (i.e. wave)}}$$

$$= -\rho g z + \rho g \eta \times \frac{\cosh k(h+z)}{\cosh kh}$$

Example

A pressure sensor is located 0.6 m above the sea bed in a water depth $h = 12\ \mathrm{m}$. The pressure fluctuates with period 15 s. A maximum gauge pressure of 124 kPa is recorded.

(a) What is the wave height?

(b) What are the maximum horizontal and vertical velocities at the surface?

A pressure sensor is located 0.6 m above the sea bed in a water depth $h = 12$ m. The pressure fluctuates with period 15 s. A maximum gauge pressure of 124 kPa is recorded.

(a) What is the wave height?

$$p = -\rho g z + \rho g A \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t)$$

$h = 12$ m

$z = -11.4$ m

$T = 15$ s

$p_{\max} = 124000$ Pa

$$124000 = 114630 + 10060A \frac{\cosh(k \times 0.6)}{\cosh kh}$$

$$\omega = \frac{2\pi}{T} = 0.4189 \text{ rad s}^{-1}$$

$$\omega^2 = gk \tanh kh$$

$$\frac{\omega^2 h}{g} = kh \tanh kh$$

$$kh \tanh kh = 0.2147$$

$$kh = \frac{0.2147}{\tanh kh}$$

or

$$kh = \frac{1}{2}\left(kh + \frac{0.2147}{\tanh kh}\right)$$

$$kh = 0.4806$$

$$k = 0.04005 \text{ m}^{-1}$$

$$124000 = 114630 + 10060A \times 0.8949$$

$$A = 1.041 \text{ m}$$

$$H = 2A = \mathbf{2.082\ m}$$

(b) What are the maximum horizontal and vertical velocities at the surface?

$$u = \frac{Agk}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx - \omega t)$$

$$w = \frac{Agk}{\omega}\frac{\sinh k(h+z)}{\cosh kh}\sin(kx - \omega t)$$

$z = 0$ (surface)

$\omega = 0.4189 \text{ rad s}^{-1}$

$kh = 0.4806$

$k = 0.04005 \text{ m}^{-1}$

$A = 1.041 \text{ m}$

$$u_{\max} = \frac{Agk}{\omega} = \mathbf{0.9764\ m\ s^{-1}}$$

$$w_{\max} = \frac{Agk}{\omega}\tanh kh = \mathbf{0.4362\ m\ s^{-1}}$$

Linear Wave Theory

1. LINEAR WAVE THEORY

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1.3 Wave velocity and pressure

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1.6 Energy transfer (wave power)

1.7 Particle motion

1.8 Shallow-water and deep-water behaviour

1.9 Waves on currents

Wave Energy

- **Wave energy density** E is average energy per unit horizontal area.
- Found by integrating over the water column, and averaging over a wave cycle.

- **Kinetic energy**: $\overline{\text{KE}} = \overline{\int_{z=-h}^{\eta} \frac{1}{2}\rho(u^2 + w^2)\,\mathrm{d}z} \quad = \frac{1}{4}\rho g A^2$
- **Potential energy**: $\overline{\text{PE}} = \overline{\int_{z=-h}^{\eta} \rho g z\,\mathrm{d}z} \quad = \frac{1}{4}\rho g A^2 + \text{constant}$

- (Under linear theory) average wave-related KE and PE are the same.
- Total energy:

$$E = \frac{1}{2}\rho g A^2 \qquad = \frac{1}{8}\rho g H^2$$

Kinetic Energy (Appendix A4)

$$\text{KE} = \frac{1}{2}\rho \int_{z=-h}^{\eta} (u^2 + w^2)\,\mathrm{d}z$$

$$u^2 + w^2 = \left(\frac{Agk}{\omega \cosh kh}\right)^2 \{\cosh^2 k(h+z)\cos^2(kx-\omega t) + \sinh^2 k(h+z)\sin^2(kx-\omega t)\}$$

$$\overline{\text{KE}} = \frac{1}{2}\rho \int_{z=-h}^{0} (\overline{u^2 + w^2})\,\mathrm{d}z \qquad = \frac{1}{2}\rho\left(\frac{Agk}{\omega \cosh kh}\right)^2 \times \frac{1}{2}\int_{-h}^{0} \{\cosh^2 k(h+z) + \sinh^2 k(h+z)\}\,\mathrm{d}z$$

$$= \frac{1}{2}\rho\left(\frac{Agk}{\omega \cosh kh}\right)^2 \times \frac{1}{2}\int_{-h}^{0} \cosh 2k(h+z)\;\mathrm{d}z$$

$$= \frac{1}{2}\rho\left(\frac{Agk}{\omega \cosh kh}\right)^2 \times \frac{1}{2}\left[\frac{\sinh 2k(h+z)}{2k}\right]_{-h}^{0}$$

$$= \frac{1}{2}\rho\left(\frac{Agk}{\omega \cosh kh}\right)^2 \times \frac{1}{2} \times \frac{\sinh 2kh}{2k}$$

$$= \frac{1}{2}\rho\left(\frac{Agk}{\omega \cosh kh}\right)^2 \times \frac{1}{2} \times \frac{2\sinh kh \cosh kh}{2k}$$

$$= \frac{1}{4}\frac{\rho A^2 g^2 k \tanh kh}{\omega^2} \qquad \omega^2 = gk\tanh kh$$

$$\overline{\text{KE}} = \frac{1}{4}\rho g A^2$$

Potential Energy (Appendix A4)

$$\text{PE} = \int_{z=-h}^{\eta} \rho g z \, \mathrm{d}z$$

$$= \frac{1}{2}\rho g [z^2]_{-h}^{\eta}$$

$$= \frac{1}{2}\rho g (\eta^2 - h^2)$$

$$= \frac{1}{2}\rho g (A^2 \cos^2(kx - \omega t) + \text{constant})$$

Only the wave component is needed

$$\overline{\text{PE}} = \frac{1}{2}\rho g \times \frac{1}{2} A^2$$

$$\overline{\text{PE}} = \frac{1}{4}\rho g A^2$$

Linear Wave Theory

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Phase and Group Velocities

Phase velocity $c \equiv \dfrac{\omega}{k}$

- velocity at which the **waveform** translates
- really only meaningful for a regular wave, or single frequency component

Group velocity $c_g \equiv \dfrac{\mathrm{d}\omega}{\mathrm{d}k}$

- velocity at which **energy** propagates
- more appropriate for a wave packet comprised of multiple frequency components

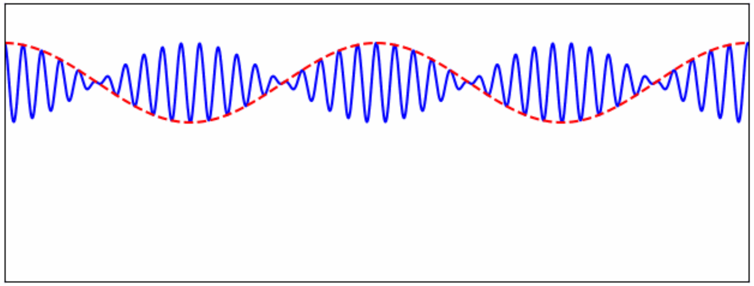
Combination of Frequency Components

Two components: frequencies ω ± Δω and wavenumbers $k \pm \Delta k$

$$\eta = \underbrace{a\cos[(k+\Delta k)x - (\omega+\Delta\omega)t]}_{\text{component 1}} + \underbrace{a\cos[(k-\Delta k)x - (\omega-\Delta\omega)t]}_{\text{component 2}}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\eta = 2a\cos(kx - \omega t)\cos(\Delta k.x - \Delta\omega.t)$$



Amplitude modulation: $A(t) = 2a\cos(\Delta k\, x - \Delta\omega\, t)$

Speed of amplitude envelope: $\frac{\Delta\omega}{\Delta k}$

Group velocity $c_g \equiv \frac{\mathrm{d}\omega}{\mathrm{d}k}$

Group Velocity (Appendix A5)

Group velocity $$c_g \equiv \frac{\mathrm{d}\omega}{\mathrm{d}k}$$

Dispersion relation $$\omega^2 = gk \tanh kh$$

$$2\omega \frac{\mathrm{d}\omega}{\mathrm{d}k} = g \tanh kh + gkh \operatorname{sech}^2 kh$$

$$= \frac{\omega^2}{k} + \frac{\omega^2}{\tanh kh} \frac{h}{\cosh^2 kh}$$

$$= \frac{\omega^2}{k}\left[1 + \frac{kh}{\sinh kh \cosh kh}\right]$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]\frac{\omega}{k}$$

$$c_g = nc \qquad c \equiv \frac{\omega}{k} \qquad n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]$$

$\frac{1}{2} < n < 1$ group velocity < phase velocity

Linear Wave Theory

1. LINEAR WAVE THEORY

Wave Power

Wave power P is the (average) rate of energy transfer per unit length of wave crest.

It can be calculated from the rate of working of pressure forces.

Power $$P = Ec_g$$

$$E = \frac{1}{2}\rho g A^2$$ (energy density)

$$c_g = nc$$ (group velocity)

Wave Power (Appendix A6)

Wave power = (time-averaged) rate of working of pressure forces (pressure × area × velocity)

Per unit length of wave crest: $$\text{power} = \overline{\int_{z=-h}^{\eta} pu \, \mathrm{d}z}$$

pressure (p) × area ($1 \times dz$) × velocity (u)

$$pu = \rho g A \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t) \quad \times \quad \frac{Agk}{\omega} \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t)$$

$$= \frac{\rho g^2 A^2 k}{\omega} \frac{\cosh^2 k(h+z)}{\cosh^2 kh} \cos^2(kx - \omega t)$$

$$\text{power} = \frac{\rho g^2 A^2 k}{\omega \cosh^2 kh} \times \boxed{\int_{-h}^{0} \cosh^2 k(h+z) \; \mathrm{d}z} \times \frac{1}{2}$$

$$\frac{1}{2} \int_{-h}^{0} (\cosh 2k(h+z) + 1) \, \mathrm{d}z$$

$$= \frac{1}{2} \left[\frac{\sinh 2k(h+z)}{2k} + z \right]_{-h}^{0}$$

$$= \frac{1}{2} \left(\frac{\sinh 2kh}{2k} + h \right)$$

Wave Power

$$\text{power} = \frac{\rho g^2 A^2 k}{\omega \cosh^2 kh} \times \frac{1}{2}\left(\frac{\sinh 2kh}{2k} + h\right) \times \frac{1}{2}$$

$$= \frac{1}{2}\rho g A^2 \times \frac{gk}{\omega \cosh^2 kh} \times \frac{\sinh 2kh}{2k}\left(1 + \frac{2kh}{\sinh 2kh}\right) \times \frac{1}{2}$$

$$= \frac{1}{2}\rho g A^2 \times \frac{gk}{\omega \cosh^2 kh} \times \frac{2\sinh kh \cosh kh}{2k}\left(1 + \frac{2kh}{\sinh 2kh}\right) \times \frac{1}{2}$$

$$= \frac{1}{2}\rho g A^2 \times \frac{gk \tanh kh}{\omega^2} \times \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right) \times \frac{\omega}{k}$$

E 1 n c

$$P = E c_g$$

power

$$E = \frac{1}{2}\rho g A^2$$

energy density

$$c_g = nc$$

group velocity

Example

A sea-bed pressure transducer in 9 m of water records a sinusoidal signal with amplitude 5.9 kPa and period 7.5 s.

Find the wave height, energy density and wave power per metre of crest.

A sea-bed pressure transducer in 9 m of water records a sinusoidal signal with amplitude 5.9 kPa and period 7.5 s.

Find the wave height, energy density and wave power per metre of crest.

$h = 9\ \mathrm{m}$

$z = -h$ (sea bed)

$\Delta p_{\mathrm{wave}} = 5900\ \mathrm{Pa}$ (amplitude)

$T = 7.5\ \mathrm{s}$

$$p_{\mathrm{wave}} = \rho g A \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t)$$

$$E = \frac{1}{2} \rho g A^2$$

$$P = E c_g \qquad c_g = nc \qquad n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]$$

$$\omega^2 = gk \tanh kh$$

$$\omega = \frac{2\pi}{T} = 0.8378\ \mathrm{rad\ s^{-1}}$$

$$\frac{\omega^2 h}{g} = kh \tanh kh$$

$$kh \tanh kh = 0.6440$$

$$kh = \frac{0.6440}{\tanh kh} \quad \text{or} \quad kh = \frac{1}{2}\left(kh + \frac{0.6440}{\tanh kh}\right)$$

$$kh = 0.8994$$

$$k = 0.09993\ \mathrm{m^{-1}}$$

A sea-bed pressure transducer in 9 m of water records a sinusoidal signal with amplitude 5.9 kPa and period 7.5 s.
Find the wave height, energy density and wave power per metre of crest.

$h = 9\ \mathrm{m}$

$z = -h$ (sea bed)

$\Delta p_{\mathrm{wave}} = 5900\ \mathrm{Pa}$

$T = 7.5\ \mathrm{s}$

$\omega = 0.8378\ \mathrm{rad\ s^{-1}}$

$kh = 0.8994$

$k = 0.09993\ \mathrm{m^{-1}}$

$$p_{\mathrm{wave}} = \rho g A \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t)$$

$$E = \frac{1}{2}\rho g A^2$$

$$P = E c_g \qquad c_g = nc \qquad n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]$$

$$5900 = 10055A \times \frac{1}{\cosh 0.8994}$$

$$A = 0.8405\ \mathrm{m}$$

$$H = 2A \quad = \mathbf{1.681\ m}$$

$$E = \frac{1}{2}\rho g A^2 \quad = \mathbf{3552\ J\ m^{-2}}$$

$$c = \frac{\omega}{k} \quad = 8.384\ \mathrm{m\ s^{-1}}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.8061$$

$$c_g = nc \quad = 6.758\ \mathrm{m\ s^{-1}}$$

$$P = E c_g \quad = \mathbf{24000\ W\ m^{-1}}$$

Linear Wave Theory

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Particle Motion

Velocity: $$u = A\frac{gk}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx-\omega t) \qquad = \omega A\frac{\cosh k(h+z)}{\sinh kh}\cos(kx-\omega t)$$

$$w = A\frac{gk}{\omega}\frac{\sinh k(h+z)}{\cosh kh}\sin(kx-\omega t) \qquad = \omega A\frac{\sinh k(h+z)}{\sinh kh}\sin(kx-\omega t)$$

Dispersion relation: $$\omega^2 = gk\tanh kh \quad \rightarrow \quad \frac{\omega}{\sinh kh} = \frac{gk}{\omega\cosh kh}$$

$$\frac{dX}{dt} = u \qquad = a\omega\cos(kX_0-\omega t) \qquad\qquad a = A\frac{\cosh k(h+Z_0)}{\sinh kh}$$

$$\frac{dZ}{dt} = w \qquad = b\omega\sin(kX_0-\omega t) \qquad\qquad b = A\frac{\sinh k(h+Z_0)}{\sinh kh}$$

$$X = X_0 - a\sin(kX_0-\omega t) \qquad\qquad \frac{X-X_0}{a} = -\sin(kX_0-\omega t)$$

$$Z = Z_0 + b\cos(kX_0-\omega t) \qquad\qquad \frac{Z-Z_0}{b} = \cos(kX_0-\omega t)$$

$$\sin^2\theta + \cos^2\theta = 1$$

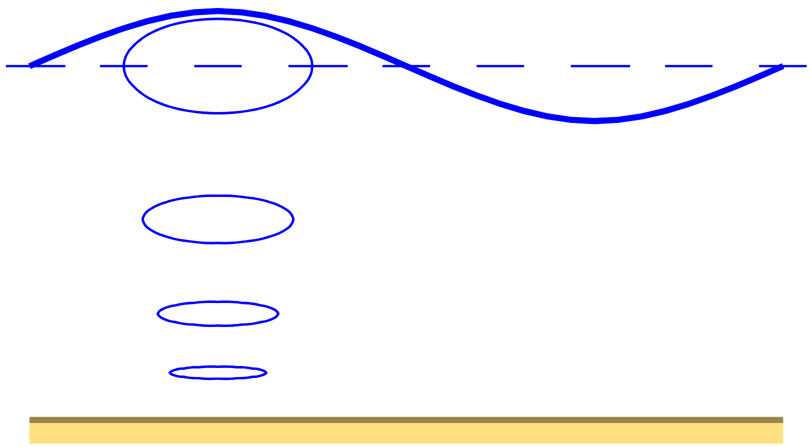
Particle Motion

$$\frac{(X - X_0)^2}{a^2} + \frac{(Z - Z_0)^2}{b^2} = 1$$

$$a = A\frac{\cosh k(h + Z_0)}{\sinh kh}$$

$$b = A\frac{\sinh k(h + Z_0)}{\sinh kh}$$

Ellipse, centre (X_0, Z_0) and semi-axes a and b



Linear Wave Theory

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Shallow-Water / Deep-Water Limits

Dispersion relationship: $\omega^2 = gk \tanh kh$ $\qquad kh = 2\pi \frac{h}{L}$

Asymptotic behaviour:

$\tanh kh \sim kh \qquad (\text{as } kh \to 0)$

$\tanh kh \to 1 \qquad (\text{as } kh \to \infty)$

Shallow water (or long waves): $kh \ll 1$

$\omega^2 \approx k^2 gh$ or $\omega \approx k\sqrt{gh}$

$c = c_g = \sqrt{gh} \qquad n = 1$ (non-dispersive)

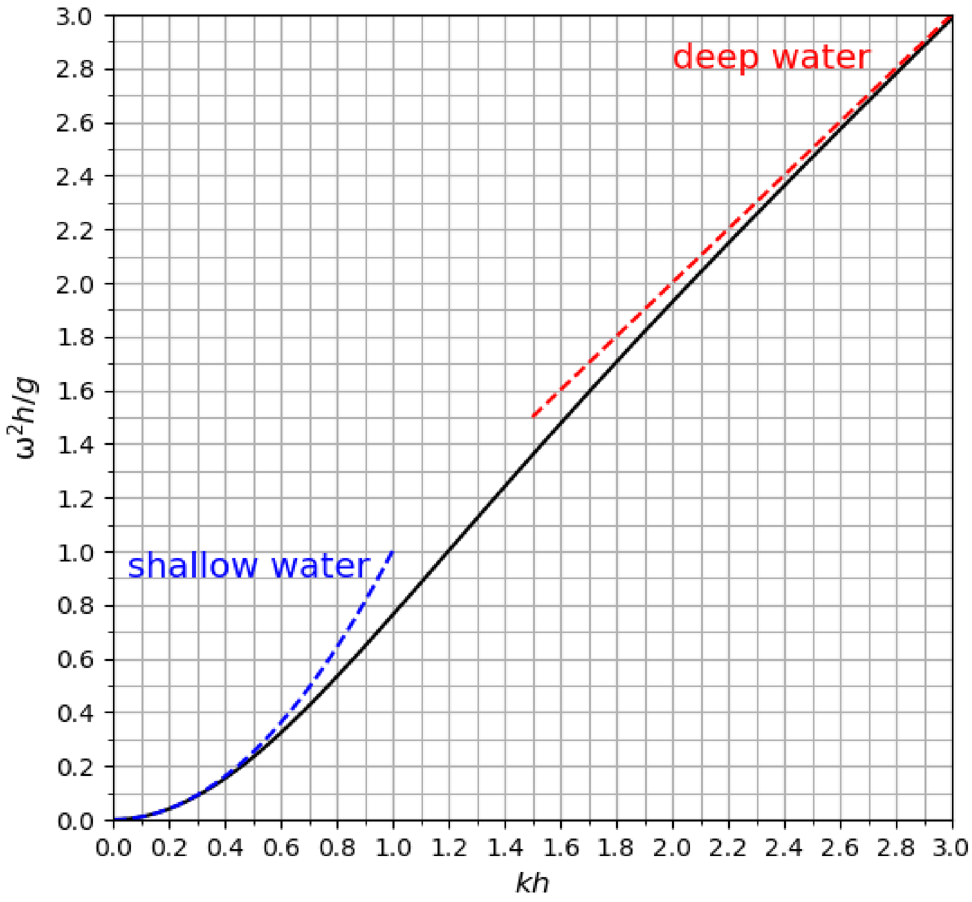
Deep water (or short waves): $kh \gg 1$

$\omega^2 \approx gk$

$$L = \frac{gT^2}{2\pi}$$

$c = \frac{L}{T} = \frac{gT}{2\pi}, \qquad n = \frac{1}{2}, \qquad c_g = \frac{1}{2}c$ (dispersive)

Shallow / Deep Limits



$$\omega^2 = gk \tanh kh$$

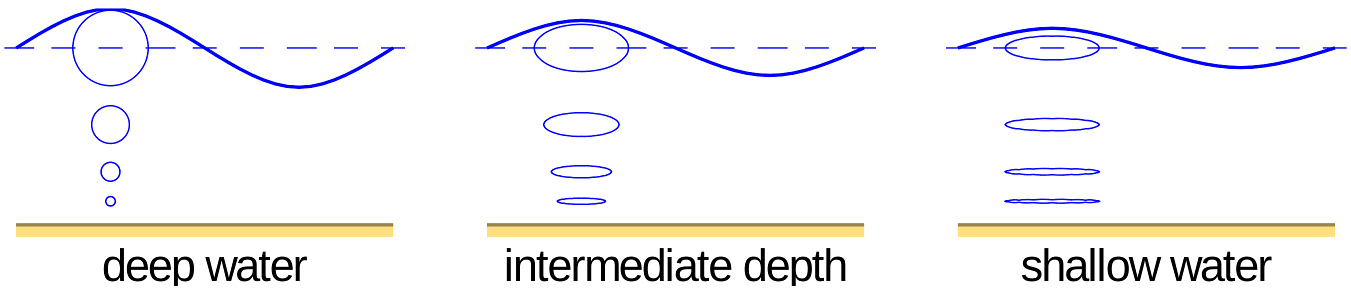
$$\frac{\omega^2 h}{g} = kh \tanh kh$$

Shallow: $kh < \frac{\pi}{10}$ $h < \frac{1}{20} L$

Deep: $kh > \pi$ $h > \frac{1}{2} L$

Shallow / Deep Particle Motions

Ellipses: $$a = A\frac{\cosh k(h+Z_0)}{\sinh kh}, \qquad b = A\frac{\sinh k(h+Z_0)}{\sinh kh}$$



Deep: $kh \gg 1$

$$a = b \approx Ae^{-k|Z_0|}$$

Circles diminishing in size over half a wavelength

Shallow: $kh \ll 1$

$$a \approx \frac{A}{kh}, \qquad \frac{b}{a} \ll 1$$

Highly-flattened ellipses; horizontal excursion almost independent of depth

Shallow / Deep Pressure

$$p = -\rho g z - \rho \frac{\partial \phi}{\partial t} \qquad = \underbrace{-\rho g z}_{\text{hydrostatic}} + \underbrace{\rho g \eta \frac{\cosh k(h+z)}{\cosh kh}}_{\text{hydrodynamic}}$$

Deep: $kh \gg 1$

$$p \approx -\rho g z + \rho g \eta \mathrm{e}^{-k|z|}$$

Perturbation decays over half a wavelength

Shallow: $kh \ll 1$

$$p \approx \rho g(\eta - z)$$

Hydrostatic

Example

(a) Find the deep-water speed and wavelength of a wave of period 12 s.

(b) Find the speed and wavelength of a wave of period 12 s in water of depth 3 m. Compare with the shallow-water approximation.

Reminder of Deep and Shallow Limits

$$\omega^2 = gk \tanh kh$$

Deep water:

$kh \rightarrow \infty$

$\tanh kh \rightarrow 1$

$$\omega^2 = gk$$

$$\left(\frac{2\pi}{T}\right)^2 = g\left(\frac{2\pi}{L}\right)$$

$$L = \frac{gT^2}{2\pi}$$

$$c = \frac{gT}{2\pi}$$

Shallow water:

$kh \rightarrow 0$

$\tanh kh \sim kh$

$$\omega^2 = gk^2h$$

$$\left(\frac{\omega}{k}\right)^2 = gh$$

$$c = \sqrt{gh}$$

$$L = cT$$

(a) Find the deep-water speed and wavelength of a wave of period 12 s.

(b) Find the speed and wavelength of a wave of period 12 s in water of depth 3 m. Compare with the shallow-water approximation.

$T = 12\ \text{s}$

Deep: $c = \dfrac{gT}{2\pi} = \mathbf{18.74\ m\ s^{-1}}$ $\qquad L = \dfrac{gT^2}{2\pi} = \mathbf{224.8\ m}$

Exact, with $h = 3\ \text{m}$:

$$\omega = \frac{2\pi}{T} = 0.5236\ \text{rad s}^{-1} \qquad \omega^2 = gk \tanh kh$$

$$\frac{\omega^2 h}{g} = kh \tanh kh$$

$$kh \tanh kh = 0.08384$$

$$kh = \frac{0.08384}{\tanh kh} \quad \text{or} \quad kh = \frac{1}{2}\left(kh + \frac{0.08384}{\tanh kh}\right)$$

$$kh = 0.2937$$

$k = 0.09790\ \text{m}^{-1}$ $\qquad c = \dfrac{\omega}{k} = \mathbf{5.348\ m\ s^{-1}}$ $\qquad L = \dfrac{2\pi}{k} = \mathbf{64.18\ m}$

Shallow: $c = \sqrt{gh} = \mathbf{5.425\ m\ s^{-1}}$ $\qquad L = cT = \mathbf{65.10\ m}$

Linear Wave Theory

1. LINEAR WAVE THEORY

Waves on Currents

- Waves co-exist with background current U
- Formulae hold in **relative** frame moving with the current:

$$x_r = x - Ut$$

$$\eta = A\cos(kx_r - \omega_r t)$$

$$= A\cos[kx - (\omega_r + kU)t] \quad = A\cos[kx - \omega_a t]$$

$$\omega_a = \omega_r + kU$$

$$c_a = \frac{\omega_a}{k} \quad = c_r + U$$

- Dispersion relationship: $(\omega_a - kU)^2 \;=\; \omega_r^2 \;=\; gk\tanh kh$

Example

An acoustic depth sounder indicates regular surface waves with apparent period 8 s in water of depth 12 m. Find the wavelength and absolute phase speed of the waves when there is:

(a) no mean current;

(b) a current of 3 m s^{-1} in the same direction as the waves;

(c) a current of 3 m s^{-1} in the opposite direction to the waves.

An acoustic depth sounder indicates regular surface waves with apparent period 8 s in water of depth 12 m. Find the wavelength and absolute phase speed of the waves when there is:

(a) no mean current;

(b) a current of 3 m s^{-1} in the same direction as the waves;

(c) a current of 3 m s^{-1} in the opposite direction to the waves.

$$h = 12 \text{ m}$$

$$(\omega_a - kU)^2 = \omega_r^2 = gk \tanh kh$$

$$T_a = 8 \text{ s} \qquad \text{(absolute)}$$

$$\omega_a = \frac{2\pi}{T_a} = 0.7854 \text{ rad s}^{-1}$$

$$k = \frac{(0.7854 - kU)^2}{9.81 \tanh 12k}$$

or

$$k = \frac{1}{2}\left[k + \frac{(0.7854 - kU)^2}{9.81 \tanh 12k}\right]$$

	$U = 0$	$U = +3 \text{ m s}^{-1}$	$U = -3 \text{ m s}^{-1}$
Iteration:	$k = \frac{1}{2}\left[k + \frac{0.7854^2}{9.81 \tanh 12k}\right]$	$k = \frac{1}{2}\left[k + \frac{(0.7854 - 3k)^2}{9.81 \tanh 12k}\right]$	$k = \frac{1}{2}\left[k + \frac{(0.7854 + 3k)^2}{9.81 \tanh 12k}\right]$
k (m^{-1})	0.08284	0.06024	0.1951
L (m) $= \frac{2\pi}{k}$	**75.85**	**104.3**	**32.20**
c_a (m s^{-1}) $= \frac{\omega_a}{k}$	**9.481**	**13.04**	**4.026**