Sediment Transport

- Most natural channels have **mobile** beds
- Sufficiently vigorous fluid motions can stir these into motion
- Many rivers carry a large **sediment load** (e.g. Yellow River)
- Most mobile beds are in **dynamic equilibrium**: 
  - sediment in = sediment out ... on average
- This equilibrium can be seriously disturbed by
  - short-term extreme events (storms; floods)
  - man-made infrastructure (dams; coastal defences)
Bridge Scour
Sediment Transport

Basic questions:

● Does sediment transport occur? (Threshold of motion).
● If so, then at what rate? (Sediment load)
● What net effect does it have on the bed? (Scour/accretion)

Main types of sediment load (volume of sediment per second):

● Bed load
● Suspended load

The combination is total load
Classical Bedforms

Standing waves

Dunes

Antidunes

FLOW

bedform migration

erosion

deposition

FLOW

FLOW
Relevant Properties

- **Particle**
  - Diameter, $d$
  - Specific gravity, $s = \rho_s/\rho$
  - Settling velocity, $w_s$
  - Porosity, $P$
  - Angle of repose, $\phi$

- **Fluid**
  - Density, $\rho$
  - Kinematic viscosity, $\nu$

- **Flow**
  - Bed shear stress, $\tau_b$
  - Mean-velocity profile, $U(z)$
  - Eddy-viscosity profile, $\nu_t(z)$
Inception of Motion

- Inception and magnitude of **bed-load** depends on:
  - bed shear stress $\tau_b$
  - particle diameter $d$ and specific gravity $s$

- Inception of **suspended load** depends on ratio of:
  - settling velocity $w_s$
  - typical turbulent velocity (friction velocity $u_\tau$)
Particle Properties: Diameter $d$

Various types:
- sieve diameter
- sedimentation diameter
- nominal diameter

<table>
<thead>
<tr>
<th>Type</th>
<th>Diameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boulders</td>
<td>$&gt; 256$ mm</td>
</tr>
<tr>
<td>Cobbles</td>
<td>$64$ mm – $256$ mm</td>
</tr>
<tr>
<td>Gravel</td>
<td>$2$ mm – $64$ mm</td>
</tr>
<tr>
<td>Sand</td>
<td>$0.06$ mm – $2$ mm</td>
</tr>
<tr>
<td>Silt</td>
<td>$0.002$ mm – $0.06$ mm</td>
</tr>
<tr>
<td>Clay</td>
<td>$&lt; 0.002$ mm (cohesive)</td>
</tr>
</tbody>
</table>

In practice, there is a range of diameters (typically, lognormally distributed).
Particle Properties: Specific Gravity $s$

$$s = \frac{\rho_s}{\rho}$$

Quartz-like: $\rho_s \approx 2650 \text{ kg m}^{-3}$, $s \approx 2.65$

Anthracite: $\rho_s \approx 1500 \text{ kg m}^{-3}$, $s \approx 1.50$
Particle Properties: Settling Velocity $w_s$

- Used to determine onset and distribution of suspended load.
- Terminal velocity in still fluid.

$\text{drag} = \text{weight} - \text{buoyancy}$

$$c_D \left( \frac{1}{2} \rho w_s^2 \right) \left( \frac{\pi d^2}{4} \right) = (\rho_s - \rho)g \frac{\pi d^3}{6}$$

$$w_s = \left( \frac{4 (s - 1)gd}{3 \frac{c_D}{c_D}} \right)^{1/2}$$

$c_D = f(\text{Re}), \quad \text{Re} = \frac{w_s d}{\nu}$

Small particles (Stokes’ Law):

$$c_D = \frac{24}{\text{Re}} \quad w_s = \frac{1}{18} \frac{(s - 1)gd^2}{\nu}$$

$$\frac{w_s d}{\nu} = \frac{1}{18} \frac{(s - 1)gd^3}{\nu^2} = \frac{1}{18} d^*^3$$

$\quad d^* = d \left( \frac{(s - 1)g}{\nu^2} \right)^{1/3}$

Realistic sizes and shapes:

$$\frac{w_s d}{\nu} = \left[ (25 + 1.2d^*^2)^{1/2} - 5 \right]^{3/2}$$
**Porosity** = fraction of voids (by volume)

Typical uncompacted sediment: $P \approx 0.4$. 
Particle Properties: Angle of Repose $\phi$

**Angle of repose** = limiting angle of slope (in still fluid)

\[ mg \sin \phi = \mu_f (mg \cos \phi) \]

Effective **coefficient of friction** $\mu_f = \tan \phi$

Can be used to estimate the effect of slopes on incipient motion
Flow Properties: Bed Friction

**Bed shear stress** $\tau_b$

- Drag (per unit area) of flow on granular bed.
- Determines inception and magnitude of bed load.

**Friction velocity** $u_\tau$

- Defined (on dimensional grounds) by:
  \[
  \tau_b = \rho u_\tau^2 \quad \text{or} \quad u_\tau = \sqrt{\frac{\tau_b}{\rho}}
  \]
- Determines inception and magnitude of suspended load.
Flow Properties: Mean-Velocity Profile

For a **rough** boundary:

\[ U(z) = \frac{u_\tau}{\kappa} \ln \left( 33 \frac{z}{k_s} \right) \]

\( u_\tau = \) friction velocity;
\( \kappa = \) von Kármán’s constant (\( \approx 0.41 \));
\( z = \) distance from boundary;
\( k_s = \) roughness height (1.0 to 2.5 times particle diameter).
Flow Properties: Eddy-Viscosity Profile

A model for the effective shear stress $\tau$ in a turbulent flow:

$$\tau = \mu_t \frac{dU}{dz} \quad \text{or} \quad \tau = \rho \nu_t \frac{dU}{dz}$$

$\mu_t$ and $\nu_t$ are the dynamic and kinematic eddy viscosities.

At the bed: $\tau = \tau_b \equiv \rho u_T^2$

At the free surface: $\tau = 0$

Assuming linear: $\tau = \rho u_T^2 (1 - z/h)$

From stress and mean-velocity profiles:

$$\rho u_T^2 (1 - z/h) = \rho \nu_t \frac{u_T}{Kz}$$

$$\nu_t = \kappa u_T z (1 - z/h)$$
Formulae For Bed Shear Stress

**Normal flow:** \[ \tau_b = \rho g R_h S \]

**Manning’s formula:** \[ V = \frac{1}{n R_h^{2/3}} S^{1/2} \] (gives \( R_h \) from \( V \) or \( Q \))

**Strickler’s formula:** \[ n = \frac{d^{1/6}}{21.1} \]

Typical values: \( n \approx 0.01 \) to 0.035 m\(^{-1/3} \) s

Via a **friction coefficient:** \[ \tau_b = c_f \left( \frac{1}{2} \rho V^2 \right) \]

**Fully-developed boundary layer (log-law):** \[ c_f = \frac{0.34}{[\ln(12h/k_s)]^2} \]

Typical values: \( c_f \approx 0.003 \) to 0.01
Threshold of Motion

A mobile bed starts to move once the bed stress exceeds a critical stress $\tau_{\text{crit}}$.

Simple model:

$\tau_{\text{crit}} \times c \frac{\pi d^2}{4} = \mu_{\text{frict}} \times (\rho_s - \rho)g \frac{\pi d^3}{6}$

$$\frac{\tau_{\text{crit}}}{(\rho_s - \rho)g d} = \frac{2\mu_{\text{frict}}}{3c}$$

$$\frac{\tau_{\text{crit}}}{(\rho_s - \rho)g d} = \text{dimensionless function of particle shape and size}$$

Shields parameter
Shields’ Diagram (A.F. Shields, 1936)

Dimensionless groups:

\[ \tau^* = \frac{\tau_b}{(\rho_s - \rho)gd} \]

\[ \text{Shields parameter or Shields stress} \]

\[ \text{Re}_p = \frac{u_\tau d}{\nu} \]

\[ \text{particle Reynolds number} \]

\[ \tau_{\text{crit}}^* = f(\text{Re}_p) \]

But this is not helpful in finding the critical \( \tau \) ... because that also appears on the RHS (via \( u_\tau \))
Critical Stress

2 dimensionless groups:

\[ \Pi_1 = \frac{\tau_b}{(\rho_s - \rho)gd} \quad (\tau^*) \]

\[ \Pi_2 = \frac{u_\tau d}{\nu} = \frac{(\sqrt{\frac{\tau_b}{\rho}})d}{\nu} \quad (Re_\rho) \]

Replace \( \Pi_2 \) by

\[ \Pi_2' = \left( \frac{\Pi_2^2}{\Pi_1^2} \right)^{1/3} = \left[ \frac{(s - 1)g}{\nu^2} \right]^{1/3} d \quad \text{... call this } d^* \]

\[ \tau_{\text{crit}}^* = f(d^*) \]

\[ \tau^* = \frac{\tau_b}{(\rho_s - \rho)gd} \quad \text{Shields parameter} \]

\[ d^* = d \left[ \frac{(s - 1)g}{\nu^2} \right]^{1/3} \quad (s = \frac{\rho_s}{\rho}) \]
Finding the Threshold of Motion

\[ \tau_{\text{crit}} = f(d^*) \]

\[ \tau^* = \frac{\tau_b}{(\rho_s - \rho)gd} \]

\[ d^* = d \left[ \frac{(s - 1)g}{v^2} \right]^{1/3} \]

Curve fit (Soulsby):

\[ \tau_{\text{crit}} = \frac{0.30}{1 + 1.2d^*} + 0.055 \left[ 1 - \exp(-0.020d^*) \right] \]
Example

An undershot sluice is placed in a channel with a horizontal bed covered by gravel with a median diameter of 5 cm and density 2650 kg m$^{-3}$. The flow rate is 4 m$^3$ s$^{-1}$ per metre width and initially the depth below the sluice is 0.5 m.

Assuming a critical Shields parameter $\tau_{\text{crit}}^*$ of 0.06 and friction coefficient $c_f$ of 0.01:

(a) find the depth just upstream of the sluice and show that the bed there is stationary;

(b) show that the bed below the sluice will erode and determine the depth of scour.
Example

A long rectangular channel contracts smoothly from 5 m width to 3 m width for a region near its midpoint. The bed is composed of gravel with a median grain size of 10 mm and density 2650 kg m$^{-3}$. The friction coefficient $c_f$ is 0.01 and the critical Shields parameter is 0.05. If the flow rate is $5 \text{ m}^3 \text{s}^{-1}$ and the upstream depth is 1 m:

(a) show that the bed is stationary in the 5 m width and mobile in the 3 m width by assuming that the bed is initially flat;

(b) determine the depth of the scour hole which results just after the contraction.
Inception of Motion in Normal Flow

Assume:

**coarse sediment:** \[ \frac{\tau_b}{(\rho_s - \rho)gd} > 0.056 \]

**normal flow:** \[ \tau_b = \rho g R_h S \]

Mobile if \[ \frac{R_h S}{(\rho_s/\rho - 1)} > 0.056 d \]

For sand: \[ d < 10.8 R_h S \]
Effect of Slopes

Gravitational forces may oppose or assist the initiation of motion

\[ \tau_{\text{crit}} = \frac{\sin(\phi + \beta)}{\sin \phi} \tau_{\text{crit},0} \]

(\( \beta \) positive for upslope flow)

On side slopes the gravitational force always favours motion:

\[ \tau_{\text{crit}} = \cos \beta \sqrt{1 - \frac{\tan^2 \beta}{\tan^2 \phi}} \tau_{\text{crit},0} \]
Bed Load

- **Bed load** consists of particles sliding, rolling or saltating, but remaining essentially in contact with the bed.

- It is the dominant form of sediment transport for larger particles (settling velocity too large for suspension).

- The **bed-load flux** $q_b$ is the volume of non-suspended sediment crossing unit width of bed per unit time.
# Dimensional Analysis

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Dimensions</th>
</tr>
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<tbody>
<tr>
<td>Bed load flux</td>
<td>$q_b$</td>
<td>$L^2T^{-1}$</td>
</tr>
<tr>
<td>Bed shear stress</td>
<td>$\tau_b$</td>
<td>$ML^{-1}T^{-2}$</td>
</tr>
<tr>
<td>Reduced gravity</td>
<td>$(s-1)g$</td>
<td>$LT^{-2}$</td>
</tr>
<tr>
<td>Particle diameter</td>
<td>$d$</td>
<td>$L$</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho$</td>
<td>$ML^{-3}$</td>
</tr>
<tr>
<td>Kinematic viscosity</td>
<td>$\nu$</td>
<td>$L^2T^{-1}$</td>
</tr>
</tbody>
</table>

6 variables, 3 dimensions $\Rightarrow$ 3 dimensionless groups

3 scales: $\rho$, $(s-1)g$, $d$

\[
\Pi_1 = \frac{q_b}{\sqrt{(s-1)gd^3}} \quad \Pi_2 = \frac{\tau_b}{\rho(s-1)gd} \quad \Pi_3 = \frac{\nu}{\sqrt{(s-1)gd^3}}
\]

\[
\Pi_3' = \frac{1}{\Pi_3^{2/3}} = d \left[\frac{(s-1)g}{\nu^2}\right]^{1/3}
\]
Dimensionless Groups

\[ q^* = \frac{q_b}{\sqrt{(s - 1)gd^3}} \]  
\text{dimensionless bed-load flux}

\[ \tau^* = \frac{\tau_b}{\rho(s - 1)gd} \]  
\text{dimensionless bed shear stress}  
\text{(Shields parameter)}

\[ d^* = d \left[ \frac{(s - 1)g}{v^2} \right]^{1/3} \]  
\text{dimensionless particle diameter}
## Bed-Load Models

<table>
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<td>$q^* = 12(\tau^* - \tau_{crit}^<em>)\sqrt{\tau^</em>}$</td>
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<td>Van Rijn (1984)</td>
<td>$q^* = \frac{0.053}{d^{0.3}} \left(\frac{\tau^*}{\tau_{crit}} - 1\right)^{2.1}$</td>
</tr>
<tr>
<td>Einstein-Brown (Brown, 1950)</td>
<td>$q^* = \begin{cases} \frac{K \exp(-0.391/\tau^<em>)}{0.465} &amp; \tau^</em> &lt; 0.182 \ \frac{40K\tau^<em>^3}{0.182} &amp; \tau^</em> \geq 0.182 \end{cases}$</td>
</tr>
<tr>
<td>Yalin (1963)</td>
<td>$q^* = 0.635r\sqrt{\tau^*}[1 - \frac{1}{\sigma r}\ln(1 + \sigma r)]$</td>
</tr>
</tbody>
</table>

$$q^* = \frac{q_b}{\sqrt{(s - 1)gd^3}} \quad \tau^* = \frac{\tau_b}{\rho(s - 1)gd} \quad d^* = d \left[\frac{(s - 1)g}{\nu^2}\right]^{1/3}$$
Calculating Bed Load

\[ q^* = f(\tau^*, d^*) \]

\[ q^* = \frac{q_b}{\sqrt{(s - 1)gd^3}} \quad \text{dimensionless bed-load flux} \]

\[ \tau^* = \frac{\tau_b}{\rho(s - 1)gd} \quad \text{dimensionless bed shear stress} \]

\[ d^* = d \left[ \frac{(s - 1)g}{v^2} \right]^{1/3} \quad \text{dimensionless particle diameter} \]

To find bed-load flux:

- from particle and fluid properties, find \( d^* \)
- from formula or graph, find \( \tau^*_\text{crit} \)
- from flow hydraulics, find \( \tau_b \) and hence \( \tau^* \)
- if \( \tau^* > \tau^*_\text{crit} \) (or \( \tau > \tau^*_\text{crit} \)), find \( q^* \) by chosen model
- invert to get absolute bed-load flux per unit width, \( q_b \)
- if required, multiply by width to get bed-load flux, \( Q_b \)
Suspended Load

- **Suspended load** consists of finer particles carried in suspension by turbulent fluid flow.

- Significant suspended load only occurs if turbulent velocity fluctuations are larger than the settling velocity:
  \[ \frac{u_\tau}{w_s} > 1 \]

- For coarser sediment, suspended load does not occur and all sediment motion is bed load.
Concentration

- Concentration $C$ is the volume of sediment per total volume.

- Sediment settles, so concentrations are larger near the bed.

- Hence, upward-moving eddies tend to carry more sediment than downward-moving ones.

- This leads to a net upward diffusion of material.

- Equilibrium when **downward settling = upward diffusion**.
Sediment flux = volume flux $\times$ amount per unit volume

$= (velocity \times area) \times concentration$

Diffusion:

Net upward flux of sediment through a horizontal area $A$:

$$\frac{1}{2} u' A (C - l \frac{dC}{dz}) - \frac{1}{2} u' A (C + l \frac{dC}{dz}) = -u' l \frac{dC}{dz} A$$

Upward diffusive flux: $-K \frac{dC}{dz} A$

$K = \text{eddy diffusivity}$

Settling:

Downward settling flux: $(w_s A) C$
A dynamic equilibrium exists when the net upward flux due to diffusion equals the net downward flux due to settling:

\[ -K \frac{dC}{dz} = w_s C \]
Eddy Diffusivity

Diffusive flux of sediment (per unit area): \(-K \frac{dC}{dz}\)

Sediment is diffused by the same turbulent eddies as those that transfer momentum:

\[
\tau = \mu_t \frac{dU}{dz} = \nu_t \frac{d(\rho U)}{dz}
\]

flux of momentum (per unit area)
momentum (per unit volume)

Hence, common to take

\[
K = \nu_t = \kappa \tau u \left(1 - \frac{z}{h}\right)
\]
Diffusion Equation For Concentration (2)

\[-K \frac{dC}{dz} = w_s C\]

where

\[K = v_t = \kappa u_\tau z(1 - z/h)\]

\[-\kappa u_\tau z(1 - z/h) \frac{dC}{dz} = w_s C\]

\[\frac{dC}{C} = - \frac{w_s}{\kappa u_\tau} \frac{1}{z(1 - z/h)} \, dz\]

\[\int_{c_{\text{ref}}}^{C} \frac{dC}{C} = - \frac{w_s}{\kappa u_\tau} \int_{z_{\text{ref}}}^{Z} \left( \frac{1}{z} + \frac{1}{h - z} \right) \, dz\]

Semi-empirical formulae for \(c_{\text{ref}}\) and \(z_{\text{ref}}\)
Solution

\[
[\ln C]_{C_{\text{ref}}}^{C} = -\frac{w_s}{\kappa u_\tau} [\ln z - \ln(h - z)]_{z_{\text{ref}}}
\]

\[
\ln C - \ln C_{\text{ref}} = -\frac{w_s}{\kappa u_\tau} \left( \ln \frac{z}{h - z} - \ln \frac{z_{\text{ref}}}{h - z_{\text{ref}}} \right)
\]

\[
\ln C - \ln C_{\text{ref}} = -\frac{w_s}{\kappa u_\tau} \left( \ln \frac{1}{h/z - 1} - \ln \frac{1}{h/z_{\text{ref}} - 1} \right)
\]

\[
\ln C - \ln C_{\text{ref}} = \frac{w_s}{\kappa u_\tau} \left( \ln(h/z - 1) - \ln(h/z_{\text{ref}} - 1) \right)
\]

\[
\ln \frac{C}{C_{\text{ref}}} = \ln \left( \frac{h/z - 1}{h/z_{\text{ref}} - 1} \right) \frac{w_s}{\kappa u_\tau}
\]

Rouse profile:

\[
\frac{C}{C_{\text{ref}}} = \left( \frac{h/z - 1}{h/z_{\text{ref}} - 1} \right)^{\frac{w_s}{\kappa u_\tau}}
\]

Rouse number:

\[
\frac{w_s}{\kappa u_\tau}
\]
Rouse Distribution

Rouse profile:

\[
\frac{C}{C_{\text{ref}}} = \left( \frac{h/z - 1}{h/z_{\text{ref}} - 1} \right)^{\frac{w_s}{\kappa u_\tau}}
\]

Rouse number:

\[
\frac{w_s}{\kappa u_\tau}
\]

\(w_s\) = settling velocity of the particle

\(u_\tau\) = friction velocity of the flow \(= \sqrt{\tau_b/\rho}\)

\(\kappa\) = von Kármán’s constant \((\approx 0.41)\)
Calculation of Suspended Load

Volume flow rate of water: \( u \ dA \)

per unit span: \( u \ dz \) (through depth \( dz \))

Volume flux of sediment = concentration × volume flux of water

\[ = Cu \ dz \]

Suspended load:

\[ q_s = \int_{z_{ref}}^{h} Cu \ dz \]

\[ u(z) = \frac{u_\tau}{\kappa} \ln\left(33 \frac{z}{k_s}\right) \]

\[ \frac{C}{C_{ref}} = \left(\frac{h/z - 1}{h/z_{ref} - 1}\right) \frac{w_s}{\kappa u_\tau} \]
Using Cheng’s formula estimate the settling velocity of a sand particle of diameter 1 mm in
(a) air;
(b) water.

Cheng’s formula for settling velocity:

\[
\frac{w_s d}{v} = \left[ (25 + 1.2 d^*^2)^{1/2} - 5 \right]^{3/2}
\]

\[
d^* = d \left( \frac{(s - 1)g}{v^2} \right)^{1/3}
\]
Find the critical Shields parameter and critical absolute stress $\tau_{\text{crit}}$ for a sand particle of diameter 1 mm in water.

Critical shear stress

\[ \tau^* = \frac{0.30}{1 + 1.2d^*} + 0.055 \left[1 - \exp\left(-0.020d^*\right)\right] \]

\[ \tau^* = \frac{\tau_b}{(\rho_s - \rho)gd} \]

\[ d^* = d \left[ \frac{(s - 1)g}{v^2} \right]^{1/3} \]
Assuming a friction coefficient $c_f = 0.005$, estimate the velocities $V$ at which
(a) incipient motion;
(b) incipient suspended load;
occur in water for sand particles of density 2650 kg m$^{-3}$ and diameter 1 mm.

\[
\tau^* = \frac{\tau_b}{(\rho_s - \rho)gd}
\]
A wide channel of slope 1:800 has a fine sandy bed with $d_{50} = 0.5$ mm. The discharge is 5 m$^3$ s$^{-1}$ per metre width. The specific gravity of the bed material is 2.65.

(a) Estimate Manning’s $n$ using Strickler’s formula.

(b) Find the depth of flow; (assume normal flow).

(c) Find the bed shear stress.

(d) Show that the bed is mobile and calculate the bed-load flux (per metre width) using: (i) Meyer-Peter and Müller; (ii) Nielsen models.

(e) Find the particle settling velocity and show that suspended load will occur.

(f) Estimate the suspended-load flux (per metre width), explaining your method and stating any assumptions made.
# Bed-Load Formulae

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\[ q^* = \frac{q_b}{\sqrt{(s - 1)gd^3}} \]  

\[ \tau^* = \frac{\tau_b}{\rho(s - 1)gd} \]  

\[ d^* = d \left[\frac{(s - 1)g}{v^2}\right]^{1/3} \]  

(dimensionless bed-load flux)

(dimensionless bed shear stress)

(dimensionless particle diameter)