Q1.
$S = 1.5 \times 10^{-4}$
$n = 0.016 \text{ m}^{-1/3} \text{ s}$
$R = 0.35 \text{ m}$ (radius of semi-circle; half-width of rectangle)
$Q = 0.12 \text{ m}^3 \text{ s}^{-1}$

To determine whether normal depth lies above or below the semi-circular section, first find the normal flow rate for that section only:

$$A = \frac{1}{2} \times \pi R^2 = 0.1924 \text{ m}^2$$

$$P = \pi R \quad = 1.100 \text{ m}$$

$$R_h = \frac{A}{P} \quad = \frac{\pi R^2/2}{\pi R} \quad = \frac{R}{2} \quad = 0.175 \text{ m}$$

$$V = \frac{1}{n} R_h^{2/3} S^{1/2} \quad = 0.2395 \text{ m s}^{-1}$$

$$Q = VA \quad = 0.0461 \text{ m}^3 \text{ s}^{-1}$$

This is less than the required flow rate, so the actual flow depth will in the rectangular section: distance d into it, say. Then, adding semi-circular and rectangular contributions:

$$A = 0.1924 + 0.7d$$

$$P = 1.010 + 2d$$

$$R_h \equiv \frac{A}{P}$$

$$V = \frac{1}{n} R_h^{2/3} S^{1/2}$$

$$Q = VA \quad = \frac{S^{1/2}}{n} \frac{A^{5/3}}{P^{2/3}}$$

Substituting values/formulae:

$$Q = 0.7655 \frac{(0.1924 + 0.7d)^{5/3}}{(1.100 + 2d)^{2/3}}$$

This can be solved by many numerical methods, including simple repeated trial on d to try to get the desired flow rate $Q = 0.12 \text{ m}^3 \text{ s}^{-1}$; (the following uses precise linear interpolation at each step, but doing it "by eye" is fine):

d (m)	RHS
0.0	0.04606
0.2	0.09318
0.3138	0.1213
0.3085	**0.1200**

The total depth from the invert is then

$$h_n = 0.35 + 0.3085 \quad = 0.6585 \text{ m}$$

Similarly for the critical depth: first find the Froude number if water (at flow rate $0.12\ \text{m}^3\ \text{s}^{-1}$) just fills the semi-circular section:

$$\text{Fr}^2 = \frac{V^2}{g\bar{h}} \quad = \frac{Q^2/A^2}{gA/b_s} \quad = \frac{Q^2 b_s}{gA^3} \quad = \frac{0.12^2 \times 0.7}{9.81 \times 0.1924^3} \quad = 0.1443$$

This is subcritical (deep). The critical depth must, therefore, occur within the semicircular part.

Require

$$\frac{Q^2 b_s}{gA^3} = 1$$

where, parameterising b_s and A by the half-angle θ at the centre,

$$b_s = 2R\sin\theta$$

$$A = 2 \times \left(\frac{1}{2}R^2\theta - \frac{1}{2}R\sin\theta\, R\cos\theta\right) \quad = R^2\left(\theta - \frac{1}{2}\sin 2\theta\right)$$

Hence,

$$\frac{Q^2(2R\sin\theta)}{gR^6\left(\theta - \frac{1}{2}\sin 2\theta\right)^3} = 1$$

Again, this can be solved by repeated trial or, for example, rearranged for iteration:

$$\theta = \frac{1}{2}\sin 2\theta + \left(\frac{2Q^2}{gR^5}\sin\theta\right)^{1/3}$$

Iterating from, e.g., $\theta = 1$ (taking care that θ is measured in radians):

$$\theta = 1.164\ \text{rad}$$

The corresponding depth is

$$h_c = R - R\cos\theta \quad = 0.2115\ \text{m}$$

Answer: normal depth 0.659 m;   critical depth 0.212 m

Q2.

(a) Because of the narrow point, the depth initially rises rapidly with flow rate, making it accurate to measure at low flow rates.

(b) From the discharge equation

$$Q = \frac{8}{15} c_d \tan\left(\frac{\theta}{2}\right) \sqrt{2g}\, h^{5/2}$$

With substituted values $c_d = 0.59$, $\theta = 120°$, this gives (in m-s units):

$$Q = 2.414 h^{5/2}$$

At maximum depth, inflow = outflow, so, with $Q = 250$ L $\mathrm{s}^{-1} = 0.25$ m^3 s^{-1},

$$h = \left(\frac{0.25}{2.414}\right)^{2/5} \quad = 0.4037 \text{ m}$$

Answer: 0.404 m

(c) In the final state, $Q = 1$ L $\mathrm{s}^{-1} = 0.001$ m^3 s^{-1} and

$$h_{\text{final}} = \left(\frac{0.001}{2.414}\right)^{2/5} \quad = 0.04435 \text{ m}$$

By continuity:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = Q_{\text{in}} - Q_{\text{out}}$$

$$A_{ws} \frac{\mathrm{d}h}{\mathrm{d}t} = 0 - 2.414 h^{5/2}$$

With $A_{ws} = 15$ m^2 this rearranges as

$$-6.214 h^{-5/2}\, \mathrm{d}h = \mathrm{d}t$$

Integrating from $t = 0$ ($h = 0.4037$ m) to $t = T$ ($h = 0.04435$ m):

$$-\int_{0.4037}^{0.04435} 6.214 h^{-5/2}\, \mathrm{d}h = \int_0^T \mathrm{d}t$$

$$\Rightarrow \qquad \left[4.143 h^{-3/2}\right]_{0.4037}^{0.04435} = T$$

$$\Rightarrow \qquad T = 427.4 \text{ s}$$

Answer: 427 s

Q3.
$Q = 6 \text{ m}^3 \text{ s}^{-1}$
$b = 4$ m (main channel); $b_m = 3$ m (constriction)
$S = 0.02$
$n = 0.025 \text{ m}^{-1/3}$ s

(a) For a long channel the approach flow will be normal. A hydraulic transition at the venturi will occur if the approach-flow head is less than the critical head at the throat of the venturi.

In normal flow,

$$Q = VA$$

where

$$V = \frac{1}{n} R_h^{2/3} S^{1/2}, \qquad A = bh, \qquad R_h = \frac{bh}{b + 2h} \quad = \frac{h}{1 + 2h/b}$$

$$\Rightarrow \qquad Q = \frac{1}{n}\left(\frac{h}{1 + 2h/b}\right)^{2/3} S^{1/2} bh$$

$$\Rightarrow \qquad \frac{nQ}{b\sqrt{S}} = \frac{h^{5/3}}{(1 + 2h/b)^{2/3}}$$

$$\Rightarrow \qquad h = \left(\frac{nQ}{b\sqrt{S}}\right)^{3/5} (1 + 2h/b)^{2/5}$$

Here, with lengths in metres,

$$h = 0.4509(1 + 0.5h)^{2/5}$$

Iteration (from, e.g., $h = 0.4509$) gives

$$h_n = 0.4924 \text{ m}$$

The corresponding velocity is

$$V_n = \frac{Q}{bh_n} \quad = 3.046 \text{ m s}^{-1}$$

In the vicinity of the restricted section the approach-flow head equals the specific energy

$$H_a = E_a \quad = h_n + \frac{V_n^2}{2g} \quad = 0.9653 \text{ m}$$

At the venturi throat the critical depth is

$$h_c = \left(\frac{q_m^2}{g}\right)^{1/3} \quad = \left(\frac{Q^2}{b_m^2 g}\right)^{1/3} \quad = 0.7415 \text{ m}$$

and the critical head is

$$H_c = \frac{3}{2} h_c \quad = 1.112 \text{ m}$$

Critical head (1.112 m) exceeds available head (0.9653 m), so a hydraulic transition must take place.

(b) The Froude number in the approach flow is

$$\mathrm{Fr_n} = \frac{V_n}{\sqrt{gh_n}} \quad = 1.386$$

This is greater than 1 (so, supercritical). However, the flow upstream of a forced hydraulic transition is subcritical. Hence, a hydraulic jump must occur on the upstream side of the contraction to go from supercritical to subcritical flow. (No such jump is required downstream, where this is simple GVF from supercritical to supercritical flow.)

(c) Gradually-varied flow will occur between the upstream depth at the venturi and the sequent depth in the hydraulic jump.

The depth just upstream of the restricted section is the subcritical depth with total head $H = H_c = 1.112$ m and the main channel width ($b = 4$ m):

$$H = h + \frac{V^2}{2g} \quad = h + \frac{Q^2}{2gb^2h^2}$$

For h in metres:

$$1.112 = h + \frac{0.1147}{h^2}$$

Rearranging for the deep (subcritical) solution:

$$h = 1.112 - \frac{0.1147}{h^2}$$

Iteration (from, e.g., h =1.112) gives

$$h_0 = 0.9965 \text{ m}$$

The sequent depth after the hydraulic jump is

$$h_J = \frac{h_n}{2}\left(-1 + \sqrt{1 + 8\mathrm{Fr}_n^2}\right) \quad = \frac{0.4924}{2}\left(-1 + \sqrt{1 + 8 \times 1.386^2}\right) \quad = 0.7499 \text{ m}$$

The GVF is subcritical; hence, integrate in the upstream direction. The depth step is

$$\Delta h = \frac{h_J - h_0}{2} \quad = -0.1233 \text{ m}$$

GVF equation:

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 - S_f}{1 - \mathrm{Fr}^2}$$

For the direct-step method, invert the GVF equation

$$\frac{\mathrm{d}x}{\mathrm{d}h} = \frac{1 - \mathrm{Fr}^2}{S_0 - S_f} \qquad \text{and} \qquad \Delta x \approx \left(\frac{\mathrm{d}x}{\mathrm{d}h}\right)\Delta h$$

For the working, write the derivative as a function of h; (all lengths in metres).

$$\mathrm{Fr} = \frac{V}{\sqrt{gh}} \quad = \frac{Q}{b\sqrt{gh^3}} \qquad \Rightarrow \qquad \mathrm{Fr}^2 = \frac{Q^2}{b^2 gh^3} \quad = \frac{0.2294}{h^3}$$

$$S_f = \left(\frac{nQ}{bh^{5/3}}\right)^2 \left(1 + \frac{2h}{b}\right)^{4/3} \quad = 1.406 \times 10^{-3} \frac{(1 + 0.5h)^{4/3}}{h^{10/3}}$$

Working formulae:

$$\Delta x = \left(\frac{\mathrm{d}x}{\mathrm{d}h}\right)_{\mathrm{mid}} \Delta h$$

where

$$\frac{\mathrm{d}x}{\mathrm{d}h} = \frac{1 - \frac{0.2294}{h^3}}{\left[20 - 1.406 \frac{(1 + 0.5h)^{4/3}}{h^{10/3}}\right] \times 10^{-3}}, \qquad \Delta h = -0.1233$$

i	h_i	x_i	h_{mid}	$(\mathrm{d}x/\mathrm{d}h)_{\mathrm{mid}}$	Δx
0	0.9965	0			
			0.9349	42.15	–5.197
1	0.8732	–5.197			
			0.8116	36.69	–4.524
2	0.7499	–9.721			

Answer: 9.72 m (upstream)