This section will look at the following wave transformations as waves approach a coastline:
Refraction – change of *direction* as oblique waves move into shallower water;
Shoaling – change of *height* as waves move into shallower water;
Breaking – *collapse* of waves after steepening;
Diffraction – *spreading* of waves into a region of shadow;
Reflection – reversal of direction at a boundary, often combining to form a *standing wave*.

2.1 Refraction

Refraction is the bending of waves toward the normal as they move into shallower water.

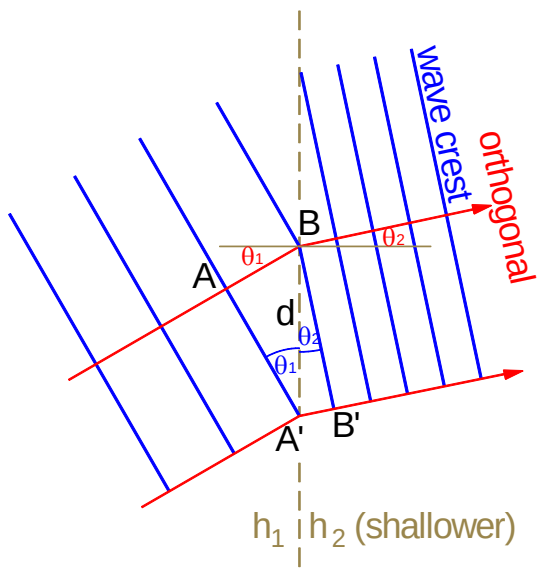
Waves may approach a coastline at an angle θ, defined as the angle between wave crests and the depth contours, or between the corresponding *normals* (for the wave, also called *rays* or *orthogonals*).

As waves move toward a coastline:
- their period remains constant;
- their wavelength changes (because the depth changes);

and hence:
- their speed changes.

Refraction (which occurs with other wave phenomena, notably in optics) is the change in direction with wave speed.

2.1.1 Snell's Law



Consider a discrete depth change $h_1 \rightarrow h_2$, with corresponding phase-speed change $c_1 \rightarrow c_2$. Look at the region where two successive wave crests cross the depth change. From the diagram, since the period of waves is not changed, distances AB and A′ B′ between successive wave crests are covered in the same time T:

$$T = \frac{\text{distance}}{\text{speed}} \quad = \frac{d \sin\theta_1}{c_1} \quad = \frac{d \sin\theta_2}{c_2}$$

Hence,

$$\frac{\sin\theta_1}{c_1} \quad = \frac{\sin\theta_2}{c_2}$$

In reality, depth (and hence speed) change as waves approach the shore is gradual rather than sudden. However, it can still be imagined as a series of such small steps. Thus, refraction is governed by *Snell's law*:

$$\frac{\sin\theta}{c} = \text{constant}$$

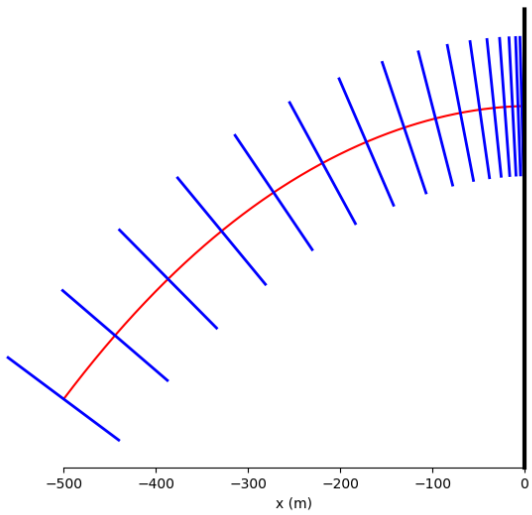
or, in terms of wavenumber (since $c = \omega/k$, and frequency ω is unchanged):

$$k \sin\theta = \text{constant}$$

or

$$(k \sin\theta)_1 = (k \sin\theta)_2$$

Thus, wave direction changes approaching the coast may be determined from the change in celerity c or wavenumber k with depth. Since the depth, and hence speed, tend to zero at the coastline, obliquely-approaching waves eventually turn to meet the coast at 90º. (Waves never "miss" the beach!)

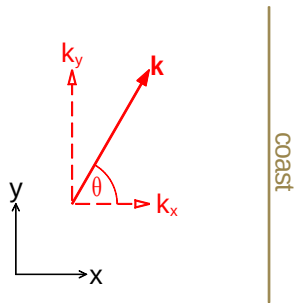


2.1.2 (Optional) Ray-Tracing Derivation of Snell's Law

This was taught in previous years, and hence appears in some exam papers. It is part of an impressively powerful, but considerably more mathematical approach to wave motion.

The phase function (e.g. what goes inside the sine or cosine for a harmonic wave) of a disturbance travelling in the direction $\mathbf{k} = (k_x, k_y, 0)$ is

$$\Omega = \mathbf{k} \bullet \mathbf{x} - \omega t$$



From this, $\mathbf{k}$ is the gradient of the phase; i.e. its components are the spatial partial derivatives:

$$\mathbf{k} = \nabla\Omega \equiv \left(\frac{\partial\Omega}{\partial x}, \frac{\partial\Omega}{\partial y}, \frac{\partial\Omega}{\partial z}\right)$$

Since $\mathbf{k}$ is the gradient of a function it is irrotational (essentially because the second partial derivatives commute), so that

$$\frac{\partial k_y}{\partial x} - \frac{\partial k_x}{\partial y} = 0$$

Taking x as toward the coast and y as the along-shore direction parallel to the coast (strictly, parallel to the depth contours) quantities do not vary in the y direction, so $\partial k_x / \partial y = 0$. Hence,

$$\frac{\partial k_y}{\partial x} = 0$$

or

$$k_y (= k \sin\theta) \quad \text{is constant}$$

as before[1].

[1] There are many other ways of deriving Snell's Law, including Fermat's Principle of Least Time. This is worth looking up.

Example.
A straight coastline borders a uniformly-sloping sea bed. Regular waves are observed to cross the 8 m depth contour at an angle of 14° to the coastline-normal, with wavelength 45 m. Find:

(a) the wave period;
(b) the wavelength in deep water;
(c) the direction in deep water.

(a) In depth $h = 8$ m,

$$L = 45 \text{ m}$$

$$k = \frac{2\pi}{L} \quad = 0.1396 \text{ m}^{-1}$$

Dispersion relation

$$\omega^2 = gk \tanh kh$$

gives

$$\omega = 1.051 \text{ rad s}^{-1}$$

$$T = \frac{2\pi}{\omega} \quad = 5.978 \text{ s}$$

Answer: 5.98 s

(b) Wavelength in deep water:

$$L_0 = \frac{gT^2}{2\pi} \quad = 55.80 \text{ m}$$

Answer: 55.8 m

(c) The wavenumber in deep water is

$$k_0 = \frac{2\pi}{L_0} \quad = 0.1126 \text{ m}^{-1}$$

Snell's Law:

$$(k \sin\theta)_0 = (k \sin\theta)_{8\text{ m}}$$

$$\Rightarrow \quad 0.1126 \sin\theta_0 = 0.1396 \sin 14°$$

$$\Rightarrow \quad \theta_0 = 17.45°$$

Answer: 17.5°

2.2 Shoaling

Shoaling is the change in height of waves as they move into shallower water.

This occurs because of a change in group velocity (energy transmission velocity) as the depth decreases. If there is no bottom friction, no wave breaking and no reflection then the energy transmission shoreward must remain the same in all water depths.

Wave energy (per unit surface area) and power (per unit length of crest) are given by

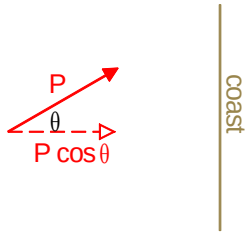
$$E = \frac{1}{2}\rho g A^2 \qquad = \frac{1}{8}\rho g H^2$$

$$P = E c_g$$

where H is the wave height and c_g is the group velocity. For regular waves:

$$c_g = nc\,, \qquad \text{where} \qquad n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right], \qquad c = \frac{\omega}{k}$$

ω and k are related by the usual dispersion relation.



Shoaling behaviour is determined on the basis that the *shoreward component of energy transport is constant*; i.e.

$$P \cos\theta = \text{constant}$$

(An alternative approach assumes that the *power between two orthogonals is constant*. See the diagram for refraction in the previous section:

$$P \times (\text{length of crest}) = Pd\cos\theta$$

for two orthogonals crossing a depth contour distance d apart.)

Since energy $\propto H^2$ the wave height approaching the shore satisfies

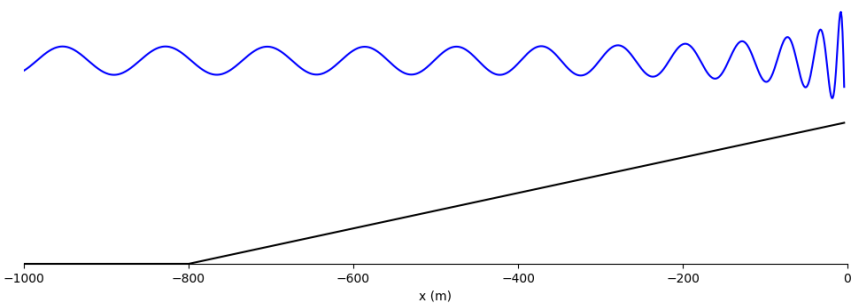
$$H^2 c_g \cos\theta = \text{constant}$$

i.e.

$$(H^2 nc\cos\theta)_1 = (H^2 nc\cos\theta)_2$$

If subscript 1 denotes reference (often deep-water) conditions, then at location 2:

$$H_2 = H_1 \underbrace{\left(\frac{\cos\theta_1}{\cos\theta_2}\right)^{1/2}}_{\substack{\text{refraction}\\ \text{coefficient}\\ K_R}} \underbrace{\left(\frac{(nc)_1}{(nc)_2}\right)^{1/2}}_{\substack{\text{shoaling}\\ \text{coefficient}\\ K_S}}$$



Note that as the wave moves into shallower water:

- the wavelength decreases and an oblique wave will turn toward the normal;
- at intermediate depths the height may actually increase or decrease depending on group velocity and refraction; however, in sufficiently shallow water, $\cos\theta \to 1$, $n \to 1$ and $c \sim \sqrt{gh}$, so eventually wave height must increase; the resulting steepening of the wave ultimately leads to wave breaking (see later).

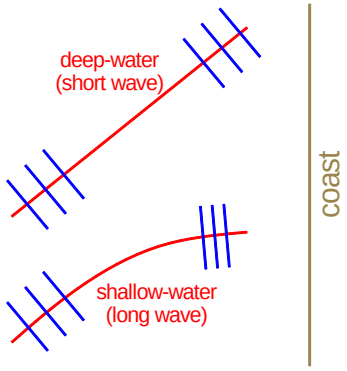
Example.
Waves propagate towards a long straight coastline that has a very gradual bed slope normal to the coast. In water depth of 20 m, regular waves propagate at heading $\theta = 40°$ relative to the bed slope.

(a) Sketch the shape of a wave ray from the 20 m depth contour to the 5 m depth contour for a wave that is of deep-water type in both depths and, separately, for a wave that is of shallow water type in both depths. Calculations are not required.

(b) For a wave with period $T = 8$ s and height 1.2 m at 20 m depth, calculate the wave heading and wave height at the 5 m depth contour.

(a)
Deep-water waves (by definition) do not feel the effect of the bottom, so are unchanged on moving from 20 m to 5 m depth.

By contrast, shallow-water waves are bent toward the normal as they move into shallower water.



(b)

$$T = 8 \text{ s}$$

Hence

$$\omega = \frac{2\pi}{T} \quad = 0.7854 \text{ rad s}^{-1}$$

The dispersion relation is

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \quad \frac{\omega^2 h}{g} = kh \tanh kh$$

This may be iterated as either

$$kh = \frac{\omega^2 h/g}{\tanh kh} \qquad \text{or} \qquad kh = \frac{1}{2}\left(kh + \frac{\omega^2 h/g}{\tanh kh}\right)$$

	$h = 5$ m	$h = 20$ m
$\dfrac{\omega^2 h}{g}$	0.3144	1.258
Iteration:	$kh = \dfrac{1}{2}\left(kh + \dfrac{0.3144}{\tanh kh}\right)$	$kh = \dfrac{1.258}{\tanh kh}$
kh	0.5918	1.416
k	0.1184 m^{-1}	0.07080 m^{-1}
$c = \dfrac{\omega}{k}$	6.633 m s^{-1}	11.09 m s^{-1}
$n = \dfrac{1}{2}\left[1 + \dfrac{2kh}{\sinh 2kh}\right]$	0.8999	0.6674
θ	?	40º
H	?	1.2 m

Refraction:

$$(k \sin\theta)_{5\text{ m}} = (k \sin\theta)_{20\text{ m}}$$

$$0.1184 \sin\theta = 0.07080 \sin 40°$$

Hence,

$$\theta = 22.60°$$

From the shoaling equation:

$$(H^2 nc \cos\theta)_{5\text{ m}} = (H^2 nc \cos\theta)_{20\text{ m}}$$

Hence:

$$H_{5\text{ m}} = H_{20\text{ m}} \sqrt{\frac{(nc\cos\theta)_{20\text{ m}}}{(nc\cos\theta)_{5\text{ m}}}} = 1.2 \times \sqrt{\frac{0.6674 \times 11.09 \times \cos 40°}{0.8999 \times 6.633 \times \cos 22.60°}}$$
$$= 1.217\text{ m}$$

Answer: heading 22.6°; wave height 1.22 m

2.3 Breaking

As waves move into shallow water, shoaling causes wavelength to decrease, but wave height ultimately to increase; i.e. the wave steepens. According to linear theory, H would become unbounded, but, in reality, linear theory breaks down, waves become unstable and break, dissipating their energy in turbulence and subsequently running up the beach at much reduced depth accompanied by foam (the post-breaking or *swash* zone).

Because wave loading on structures depends significantly on wave height, design and siting of nearshore structures requires knowledge of the breaking point.

2.3.1 Breaking Criteria

Wave breaking depends on the ratio of wave height (H) to wavelength (L) and/or depth (h), and on the slope of the beach (m).

<u>Miche Criterion (Steepness, or Height-to-Wavelength Ratio)</u>

An early criterion was that of Miche (1944):

$$\left(\frac{H}{L}\right)_b = 0.14 \tanh(kh)_b$$

where subscript b denotes breaking conditions. Note that wavelength L and the corresponding wavenumber $k = 2\pi/L$ are the local conditions, i.e. at the breaking point.

A useful interpretation is that waves break when fluid particles moving at the wave crest speed $u_{\max}$ begin to catch up with the wave speed c:

$$\frac{u_{\max}}{c} = \frac{(Agk/\omega)}{(\omega/k)} \quad = \frac{Agk^2}{\omega^2} \quad = \frac{Ak}{\tanh kh} \quad = \frac{\pi H/L}{\tanh kh}$$

Thus, $u_{\max}/c$ reaches some critical value when

$$\frac{H}{L} = \text{constant } \times \tanh(kh)$$

with the constant being determined empirically.

In deep water ($\tanh kh \to 1$) the Miche criterion yields a critical steepness

$$\left(\frac{H}{L}\right)_b = 0.14\,, \qquad \text{or about } \frac{1}{7}$$

In shallow water ($\tanh kh \sim kh$) this gives, with $k = 2\pi/L$,

$$\left(\frac{H}{L}\right)_b = 0.14 \times \frac{2\pi h}{L}$$

or a *breaker depth index* (see below)

$$\gamma_b = \left(\frac{H}{h}\right)_b = 0.88$$

This is slightly on the high side.

Breaker Height Index

The *breaker height index* is

$$\Omega_b = \frac{H_b}{H_0} \qquad \left(\frac{\text{wave height at breaking}}{\text{deep-water height}}\right)$$

The deep-water wave height H_0 can be determined from height at any known location via the shoaling equation (see example below).

A commonly-used correlation is

$$\Omega_b = 0.56\left(\frac{H_0}{L_0}\right)^{-1/5}$$

where deep-water wave height H_0 is established from shoaling and the deep-water wavelength L_0 is calculated from the wave period: $L_0 = gT^2/2\pi$.

Breaker Depth Index

The *breaker depth index* is

$$\gamma_b = \left(\frac{H}{h}\right)_b \qquad \left(\frac{\text{wave height at breaking}}{\text{water depth at breaking}}\right)$$

McCowan (1894) gave a critical value $\gamma_b = 0.78$, which is still widely used for shallow-water breaking on mild slopes. However, use of a constant value fails to recognise the role of beach slope, *m*. A reanalysis of data by Weggel (1972) produced the following empirical formulae for the critical breaker depth index:

$$\gamma_b = b - a\frac{H_b}{gT^2} \qquad = b - a\frac{H_b}{2\pi L_0}$$

where

$$a = 43.8(1 - e^{-19m}), \qquad b = \frac{1.56}{1 + e^{-19.5m}}$$

Again, this bases the measure of steepness H_b/L_0 on the *deep-water* wavelength L_0. Note that this gives $\gamma_b = 0.78$ when the beach slope $m \to 0$, consistent with the earlier approximation.

2.3.2 Types of Breakers

The type of breaker which ensues depends on the beach slope *m* compared with the wave steepness H/L_0 (again based on the deep-water wavelength) and has been found to correlate well with the *Irribarren number* (aka *surf-similarity parameter*):

$$\xi_0 = \frac{m}{\sqrt{H_0/L_0}} \qquad \text{or} \qquad \xi_b = \frac{m}{\sqrt{H_b/L_0}} \qquad \left(\frac{\text{beach slope}}{\sqrt{\text{wave steepness}}}\right)$$

(Note the two possibilities, depending on whether the breaking height H_b or deep-water height H_0 is used.)

Then:

$\xi_0 < 0.5$	spilling breakers
$0.5 < \xi_0 < 3.3$	plunging breakers
$3.3 < \xi_0$	surging or collapsing breakers

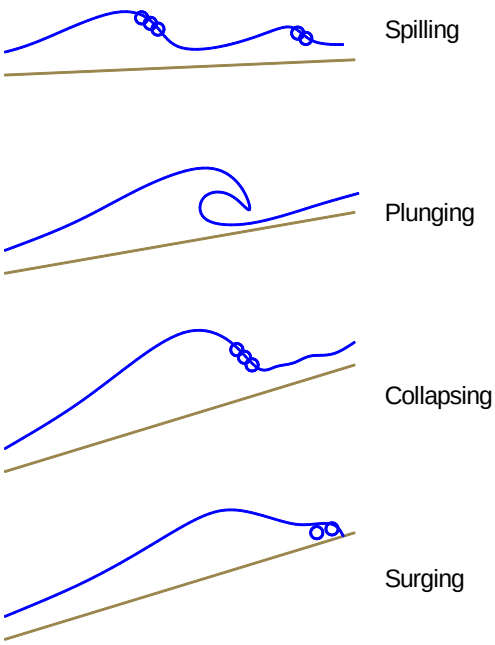
If ξ_b is used instead then the limiting numbers are 0.4 and 2.0 (Battjes, 1974).

Spilling breakers occur for steep waves or mildly-sloping beaches (typically less than 1 in 100). Foam spills down the front of the crest ("white horses"). Gentle, slowly-breaking waves.

Plunging breakers occur on moderately steep beaches. The steepening wave front curls over the preceding trough and crashes down into it. Sudden, violent, dissipative breaking.

Collapsing breakers occur for very long waves or very steep beaches. Crest never fully rolls over but the steep front face collapses from the bottom.

Surging breakers occur for long-period, low-steepness waves on steep beaches. The wave may not break, but forms a front, with extensive run-up on the beach. Lack of dissipation means these are quite reflective.



Example.
Waves propagate towards a long straight coastline that has a constant bed slope of 1 in 100. Consider the *x*-axis to be normal to the coastline and the *y*-axis parallel to the coastline. Waves propagate at an angle θ to the *x*-axis.

(a) A wave with period 7 s and height 1.2 m crosses the 36 m depth contour at angle $\theta = 22°$.
   (i) Determine the direction, height and power per metre width of wave crest at the 4 m depth contour.
   (ii) Explain how height changes between these depths.

(b) A wave with period 7 s and height 1.0 m crosses the 4 m depth contour at angle $\theta = 0°$. Determine the breaking wave height and breaking depth from their corresponding indices and identify the type of breaker expected.

(c) Further along the coast, waves propagate over the outflow of a river. In water depth of 14 m, measurements indicate a period of 7 s and depth-averaged flow velocity of 0.8 m s^{-1} against the wave direction. Determine the wavelength.

(a)
(i)

$$T = 7 \text{ s}$$

Hence

$$\omega = \frac{2\pi}{T} \quad = 0.8976 \text{ rad s}^{-1}$$

The dispersion relation is

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \quad \frac{\omega^2 h}{g} = kh \tanh kh$$

This may be iterated as either

$$kh = \frac{\omega^2 h/g}{\tanh kh} \qquad \text{or} \qquad kh = \frac{1}{2}\left(kh + \frac{\omega^2 h/g}{\tanh kh}\right)$$

	$h = 4$ m	$h = 36$ m
$\frac{\omega^2 h}{g}$	0.3285	2.957
Iteration:	$kh = \frac{1}{2}\left(kh + \frac{0.3285}{\tanh kh}\right)$	$kh = \frac{2.957}{\tanh kh}$
kh	0.6065	2.973
k	0.1516 m^{-1}	0.08258 m^{-1}

$c = \dfrac{\omega}{k}$	5.921 m s^{-1}	10.87 m s^{-1}
$n = \dfrac{1}{2}\left[1 + \dfrac{2kh}{\sinh 2kh}\right]$	0.8956	0.5156
θ	?	22º
H	?	1.2 m

Refraction:

$$(k \sin\theta)_{4\text{ m}} = (k \sin\theta)_{36\text{ m}}$$

$$0.1516 \sin\theta = 0.08258 \sin 22°$$

Hence,

$$\theta = 11.77°$$

From the shoaling equation:

$$(H^2 nc \cos\theta)_{4\text{ m}} = (H^2 nc \cos\theta)_{36\text{ m}}$$

Hence:

$$H_{4\text{ m}} = H_{36\text{ m}} \sqrt{\frac{(nc\cos\theta)_{36\text{ m}}}{(nc\cos\theta)_{4\text{ m}}}} = 1.2 \times \sqrt{\frac{0.5156 \times 10.87 \times \cos 22°}{0.8956 \times 5.921 \times \cos 11.77°}}$$
$$= 1.201\text{ m}$$

The power per metre of crest is

$$P = \frac{1}{8}\rho g H^2 (nc) = \frac{1}{8} \times 1025 \times 9.81 \times 1.201^2 \times (0.8956 \times 5.921)$$
$$= 9614\text{ W m}^{-1}$$

Answer: heading 11.8°; wave height 1.20 m; power = 9.61 kW m^{-1}

(ii) The height initially reduces (because of the group velocity) and then increases again.

(b) At 4 m depth we have

$$H = 1.0\text{ m}$$

$$\theta = 0°$$

and from the above:

$$n = 0.8956$$

$$c = 5.921\text{ m s}^{-1}$$

In deep water,

$$n = \frac{1}{2}$$

$$c = \frac{gT}{2\pi} \quad = 10.93 \text{ m s}^{-1}$$

$$L_0 = \frac{gT^2}{2\pi} \quad = 76.50 \text{ m}$$

whilst the deep-water height can be deduced from the shoaling equation:

$$(H^2nc)_0 = (H^2nc)_{4\text{ m}}$$

whence

$$H_0 = H_{4\text{ m}}\sqrt{\frac{(nc)_{4\text{ m}}}{(nc)_0}} \quad = 1.0 \times \sqrt{\frac{0.8956 \times 5.921}{0.5 \times 10.93}} \quad = 0.9851 \text{ m}$$

Then the given breaking criterion ("breaker height index") gives

$$H_b = 0.56H_0\left(\frac{H_0}{L_0}\right)^{-1/5} \quad = 0.56 \times 0.9851 \times \left(\frac{0.9851}{76.50}\right)^{-1/5} \quad = 1.317 \text{ m}$$

From the formula sheet the breaker depth index is

$$\gamma_b \equiv \left(\frac{H}{h}\right)_b = b - a\frac{H_b}{gT^2}$$

where

$$a = 43.8(1 - \mathrm{e}^{-19m}) \quad = 43.8\left(1 - \mathrm{e}^{-19/100}\right) \quad = 7.579$$

$$b = \frac{1.56}{1 + \mathrm{e}^{-19.5m}} \quad = \frac{1.56}{1 + \mathrm{e}^{-19.5/100}} \quad = 0.8558$$

Hence,

$$\frac{1.317}{h_b} = 0.8558 - 7.579 \times \frac{1.317}{9.81 \times 7^2}$$

giving breaking depth

$$h_b = 1.577 \text{ m}$$

The surf-similarity parameter (Irribarren number) is

$$\xi_0 = \frac{m}{\sqrt{H_0/L_0}} \quad = \frac{0.01}{\sqrt{0.9851/76.50}} \quad = 0.0881$$

This is significantly less than 0.5, so we expect spilling breakers.

Answer: breaking height = 1.32 m; breaking depth = 1.58 m; spilling breakers

(c) Given

$$h = 14 \text{ m}$$

$$T_a = 7 \text{ s}$$

$$U = -0.8 \text{ m s}^{-1}$$

Then

$$\omega_a = \frac{2\pi}{T_a} \quad = \ 0.8976 \text{ rad s}^{-1}$$

Dispersion relation:

$$(\omega_a - kU)^2 = \omega_r^2 \ = gk \tanh kh$$

Rearrange as

$$k = \frac{(\omega_a - kU)^2}{g \tanh kh}$$

or here, in m-s units:

$$k = \frac{(0.8976 + 0.8k)^2}{9.81 \tanh 14k}$$

to get

$$k = 0.1087 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 57.80 \text{ m}$$

Answer: 57.8 m

Example. (Exam 2017, part)
Waves propagate towards a straight shoreline. The wave heading is equal to the angle formed between wave crests and the bed contours. The bed slope is less than 1 in 100. Waves are measured in 30 m depth and wave conditions at 6 m depth are required to inform design of nearshore structures.

Regular waves are measured with period of 7 s and height of 3 m.

(a) Determine the water depth in which waves with this period can be considered as deep-water waves.

(b) For 30 m depth, determine the breaking height by the Miche criterion and briefly describe this type of breaking wave.

(c) If the heading is zero degrees, calculate wave height in 6 m depth. State your assumptions.

(d) If the heading of the measured conditions is 30°, calculate the wave heading and height in 6 m depth. Hence calculate the change of wave power per unit width of wave crest (kW m^{-1}) between the two depths.

(a) Given

$$T = 7 \text{ s}$$

Then

$$\omega = \frac{2\pi}{T} \quad = 0.8976 \text{ rad s}^{-1}$$

Deep-water waves if

$$kh > \pi$$

From the dispersion relation

$$\omega^2 = gk \tanh kh$$

$$\frac{\omega^2 h}{g} = kh \tanh kh$$

Hence, one has deep-water waves if

$$\frac{\omega^2 h}{g} > \pi \tanh \pi$$

i.e. if

$$h > \pi \tanh \pi \times \frac{g}{\omega^2} \quad = 38.11$$

Answer: deep-water waves if $h > 38.1$ m

(b) Need wave conditions for $h = 30$ m (not deep water, from part (a)).

$$\frac{\omega^2 h}{g} = 2.464$$

Rearranging the dispersion relation:

$$kh = \frac{2.464}{\tanh kh}$$

which iterates to give

$$kh = 2.498$$

$$k = \frac{2.498}{h} \quad = 0.08327 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 75.46 \text{ m}$$

Needed later in part (c):

$$c = \frac{\omega}{k} \quad = 10.78 \text{ m s}^{-1}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.5338$$

If the waves are to break at a depth of 30 m then, from the Miche criterion:

$$\frac{H_b}{L} = 0.14 \tanh kh$$

Hence,

$$H_b = L \times 0.14 \times \tanh kh \quad = 75.46 \times 0.14 \times \tanh 2.498 \quad = 10.42 \text{ m}$$

The surf-similarity parameter (Irribarren number) is

$$\xi_b = \frac{m}{\sqrt{H_b/L_0}}$$

where L_0 is the deep-water wavelength:

$$L_0 = \frac{gT^2}{2\pi} \quad = \frac{9.81 \times 7^2}{2\pi} \quad = 76.50 \text{ m}$$

Since the beach slope m is less than 0.01, then

$$\xi_b < \frac{0.01}{\sqrt{10.42/76.50}} \quad = 0.0271$$

This is (much) less than 0.4; hence the breaking-wave type is “spilling breakers”.

Answer: 10.4 m; spilling breakers

(c) Need wave parameters for $h = 6$ m. As before:

$$\omega = 0.8976 \text{ rad s}^{-1}$$

$$\frac{\omega^2 h}{g} = 0.4928$$

Rearranging the dispersion relation for iteration:

$$kh = \frac{1}{2}\left(kh + \frac{0.4928}{\tanh kh}\right)$$

which iterates to give

$$kh = 0.7651$$

$$k = \frac{0.7651}{h} \quad = 0.1275 \text{ m}^{-1}$$

$$c = \frac{\omega}{k} \quad = 7.040 \text{ m s}^{-1}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.8476$$

Shoaling equation with no refraction:

$$(H^2 nc)_{6\text{ m}} = (H^2 nc)_{30\text{ m}}$$

Hence:

$$H_{6\text{ m}} = H_{30\text{ m}}\sqrt{\frac{(nc)_{30\text{ m}}}{(nc)_{6\text{ m}}}} = 3 \times \sqrt{\frac{0.5338 \times 10.78}{0.8476 \times 7.040}} = 2.946 \text{ m}$$

Answer: 2.95 m; assumes negligible energy dissipation

(d) Refraction:

$$(k \sin\theta)_{6\text{ m}} = (k \sin\theta)_{30\text{ m}}$$

$$0.1275 \sin\theta = 0.08327 \sin 30°$$

Hence,

$$\theta = 19.06°$$

From the shoaling equation:

$$(H^2 nc \cos\theta)_{6\text{ m}} = (H^2 nc \cos\theta)_{30\text{ m}}$$

Hence:

$$H_{6\text{ m}} = H_{30\text{ m}}\sqrt{\frac{(nc\cos\theta)_{30\text{ m}}}{(nc\cos\theta)_{6\text{ m}}}} = 3 \times \sqrt{\frac{0.5338 \times 10.78 \times \cos 30°}{0.8476 \times 7.040 \times \cos 19.06°}} = 2.820 \text{ m}$$

Hence the change of power is

$$\Delta P = \left(\frac{1}{8}\rho g H^2 nc\right)_{30\text{ m}} - \left(\frac{1}{8}\rho g H^2 nc\right)_{6\text{ m}}$$

$$= \frac{1}{8} \times 1025 \times 9.81 \times (3^2 \times 0.5338 \times 10.78 - 2.82^2 \times 0.8476 \times 7.040)$$

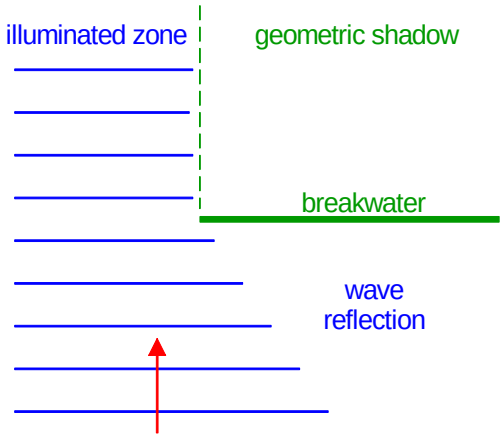
$$= 5451\text{ W m}^{-1}$$

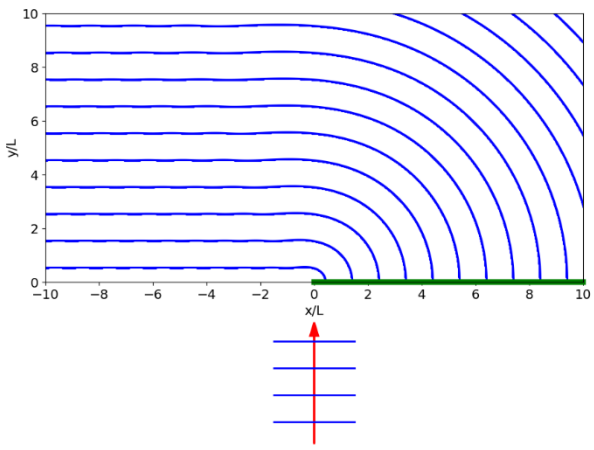
Answer: power change = 5.45 kW per metre crest

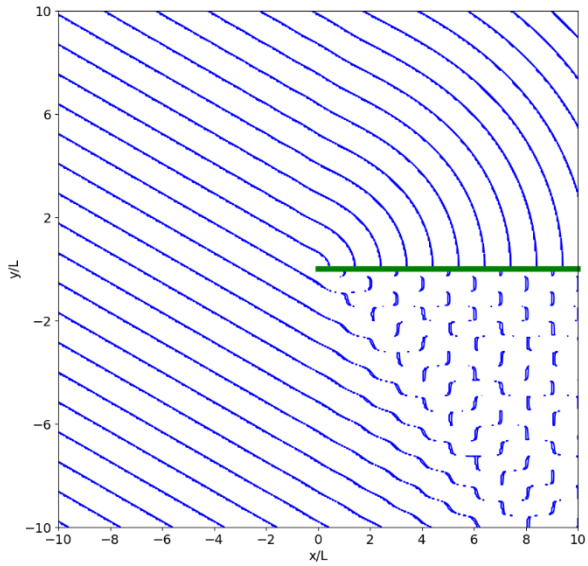
2.4 Diffraction

When an obstacle, for example a harbour breakwater or offshore structure, impedes the passage of a wave it creates a region of geometric shadow. Since the water waves still cause oscillations in the "illuminated zone" (note the analogy with light waves!) disturbances must propagate into the region of shadow to avoid discontinuities in the water surface.

Diffraction is the spreading of waves into a region of geometric shadow.







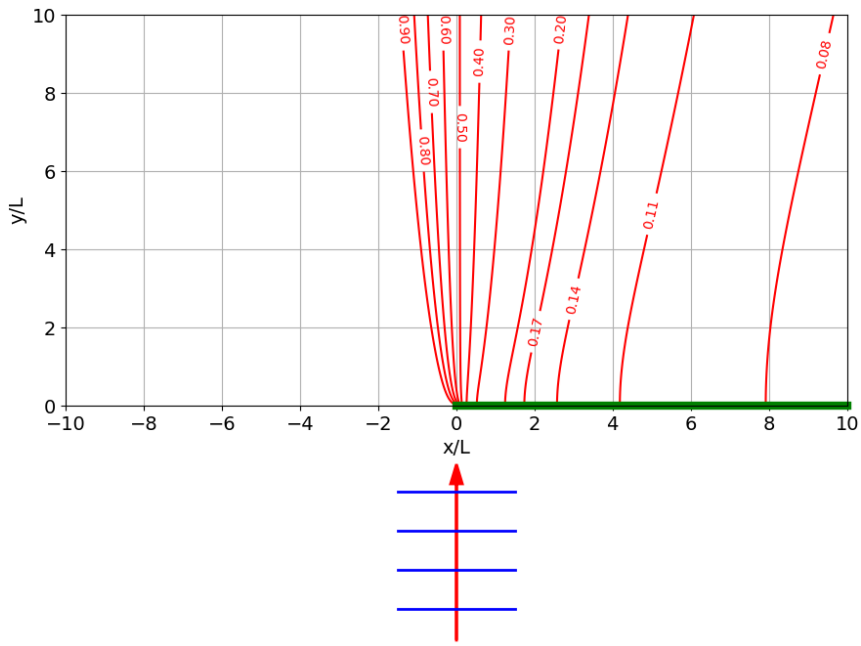
2.4.1 Diffraction Coefficients

Diffraction behaviour is difficult to calculate (see the optional summary at the end of this section if you are interested) and data is usually presented either as charts (e.g. Shore Protection Manual (1984), or USACE Coastal Engineering Manual), tables, or computer software in terms of *diffraction coefficients*:

$$K_D(\mathbf{x}) = \frac{H(\mathbf{x})}{H_0}$$

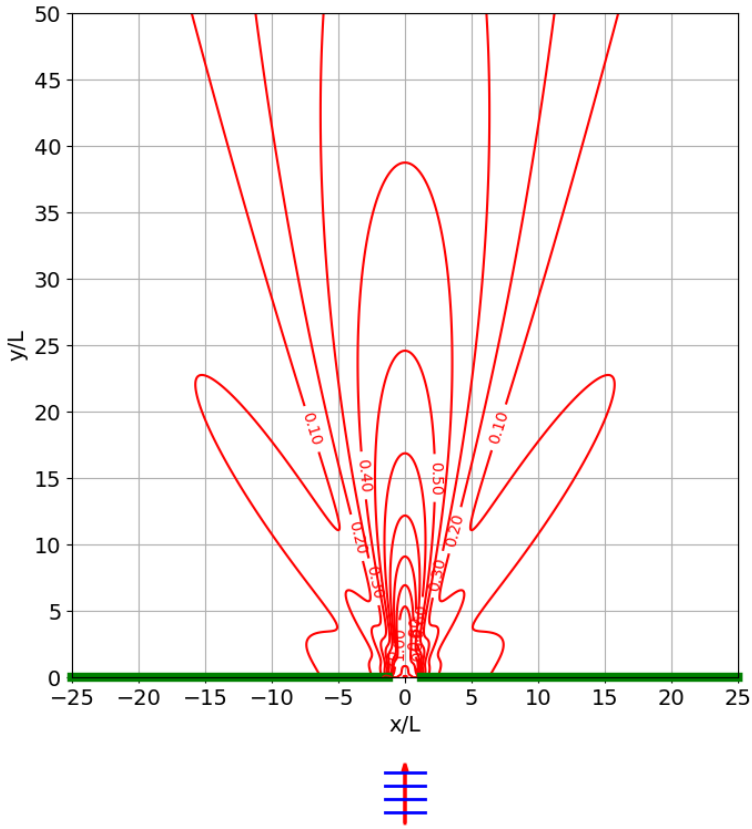
the ratio of wave heights (or amplitudes) at a particular point to those in an incident plane wave.

Semi-infinite plane breakwater; 90° wave incidence

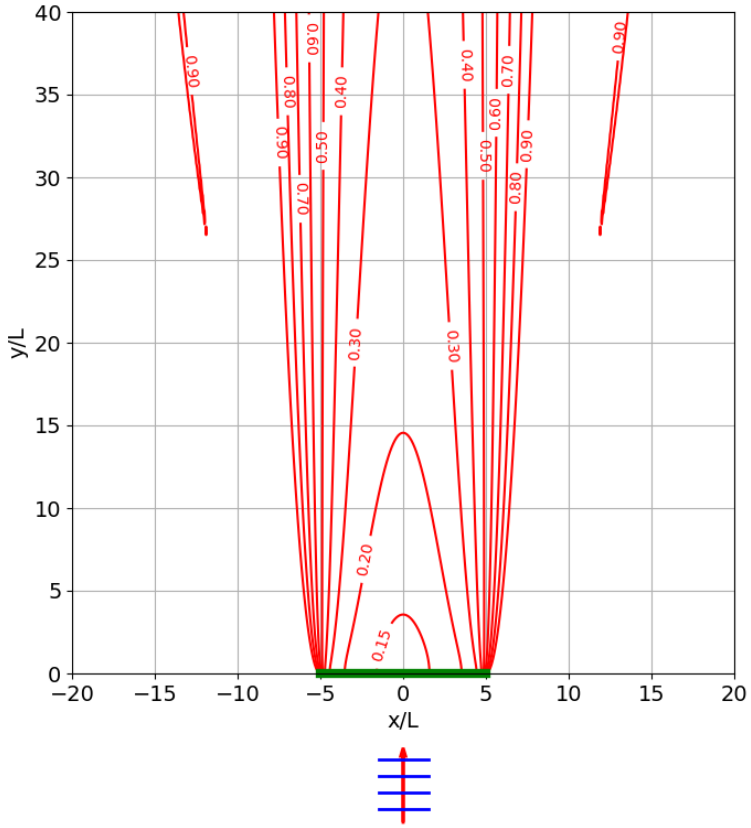


Note that the wave height along the line behind the breakwater tip is exactly half that in the incident wave. This continues (as an approximation) for other angles of incidence.

Breakwater with a 2*L* gap



Offshore finite breakwater; length 10*L*



2.4.2 (Optional) Calculation of Diffraction Coefficients

For the interested, diffraction patterns are the result of solving for a *complex* velocity potential of the form

$$\phi = \frac{\cosh k(h+z)}{\cosh kh} F(x,y) e^{-\mathrm{i}\omega t}$$

(The actual physical solution would just be the real part of this complex number). For an incident plane wave with wavenumber vector **k** and unit amplitude, ϕ will satisfy the Laplace equation if F satisfies a Helmholtz equation of the form

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -k^2 F$$

with boundary conditions

$$F \sim \mathrm{e}^{\mathrm{i}\mathbf{k}\bullet\mathbf{x}} \quad \text{at infinity}$$

$$\frac{\partial F}{\partial n} = 0 \quad \text{on solid boundaries}$$

The first boundary condition defines the incident unit-amplitude plane wave and the latter the kinematic condition of no velocity through the boundary.

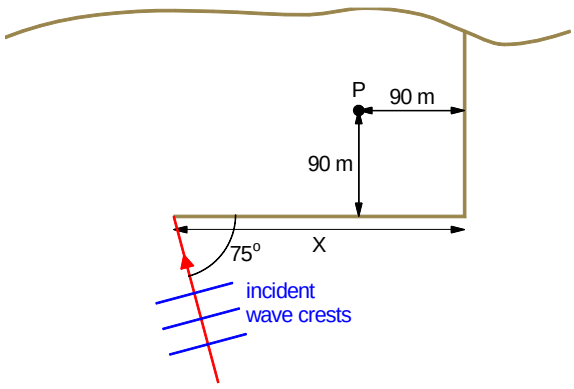
As a complex number, the amplitude of F gives the diffraction coefficient (ratio of wave height at a point to that in the incident wave), whilst its phase can be used to find the wave crests.

Penney and Price (1952)[2] give the exact solution for oblique incidence on a semi-infinite plane breakwater (based on a classical solution for light waves by Sommerfeld, 1896) and approximate, but fairly accurate, solutions for finite-length breakwaters and breakwater gaps. The diagrams in these notes have been plotted from the solutions in that paper.

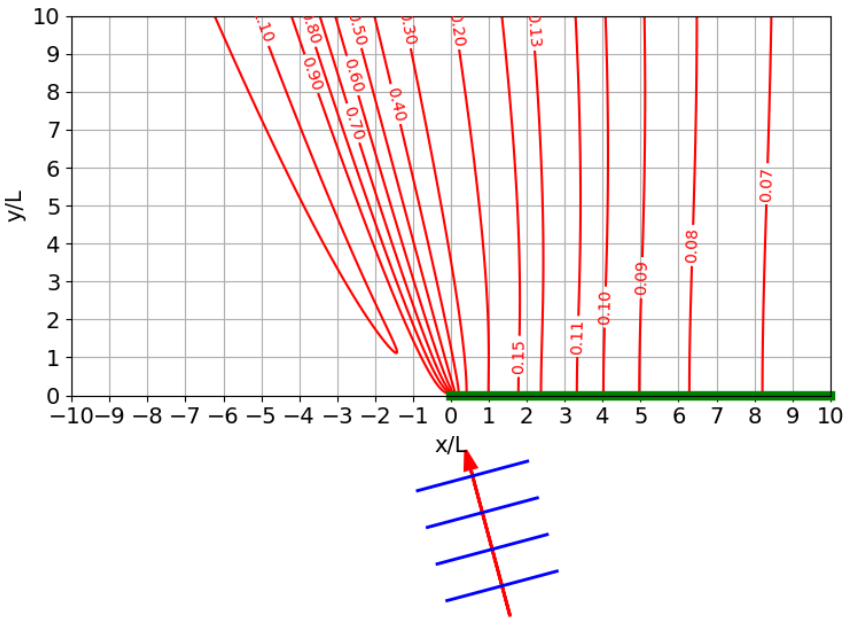
Further chart-based solutions are available from the Shore Protection Manual[3] or the US Army Corps of Engineers Coastal Engineering Manual[4].

Example.

A harbour is to be protected by an L-shaped breakwater as sketched. Determine the length X of the outer arm necessary for the wave height at point P to be 0.3 m when incident waves have a height of 3 m and a period of 5 s.



The depth is everywhere uniform at 5 m. The diagram for the appropriate approach angle is shown. Neglect reflections within the harbour.



[2] Penney, W.G. and Price, A.T., 1952, The diffraction theory of sea waves and the shelter afforded by breakwaters, Phil. Trans. Roy. Soc. London, Ser A, 244, pp. 236-253.

[3] Downloadable from, e.g., http://resolver.tudelft.nl/uuid:98791127-e7ae-40a1-b850-67d575fa1289

[4] Downloadable from, e.g., Chapter 7 in: https://www.publications.usace.army.mil/Portals/76/Publications/EngineerManuals/EM_1110-2-1100_Part-02.pdf

We require

$$\frac{H}{H_{in}} = \frac{0.3}{3} \quad = 0.1$$

From the diffraction diagram the 0.1 contour is almost vertical, with horizontal distance from the point $4L$. Hence, the total arm length is

$$X = 4L + 90$$

So, we need the wavelength L for the specified wave.

Given

$$h = 5 \text{ m}$$
$$T = 5 \text{ s}$$

Then,

$$\omega = \frac{2\pi}{T} \quad = 1.257 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \quad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \quad 0.8053 = kh \tanh kh$$

Iterate as either

$$kh = \frac{0.8053}{\tanh kh} \qquad \text{or (better here)} \qquad kh = \frac{1}{2}(kh + \frac{0.8053}{\tanh kh})$$

to get

$$kh = 1.037$$

$$k = \frac{1.037}{h} \quad = 0.2074 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 30.30 \text{ m}$$

Finally, for the arm length:

$$X = 90 + 4 \times 30.3 \quad = 211.2 \text{ m}$$

Answer: arm length 211 m

2.5 Reflection

2.5.1 Combining Incident and Reflected Waves

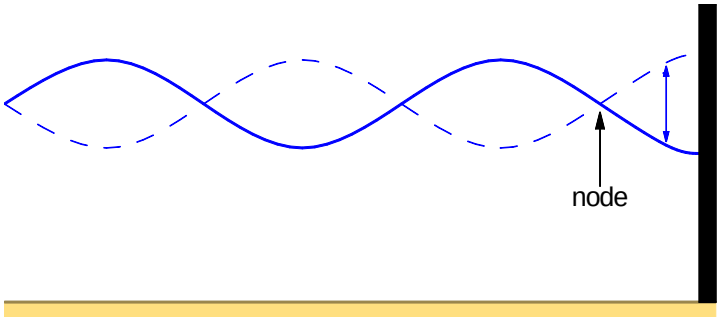
When a wave reaches a rigid impermeable vertical wall it is reflected, with boundary condition

$$u = 0 \quad \text{at the wall}$$

Consider the superposition of two *progressive* waves with similar frequencies and wavenumbers, but travelling in opposite directions: an incoming one (coming from the right) and an outgoing one. For example, the water surface elevation:

$$\begin{aligned} \eta &= A\cos(kx - \omega t) + A\cos(-kx - \omega t) \\ &= A[\cos(kx - \omega t) + \cos(kx + \omega t)] \\ &= 2A\cos kx \cos \omega t \end{aligned}$$

This produces a *standing wave* (at any point in space it oscillates harmonically in time) with *double the amplitude*, with *nodes* (points of zero displacement) at $kx = \pi/2, 3\pi/2, 5\pi/2, \ldots$ corresponding to 1/4, 3/4, 5/4 … wavelengths back from the reflective wall.



The corresponding velocity superposition gives

$$\begin{aligned} u &= \frac{Agk}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx - \omega t) + \frac{Ag(-k)}{\omega}\frac{\cosh[(-k)(h+z)]}{\cosh[(-k)h]}\cos(-kx - \omega t) \\ &= \frac{Agk}{\omega}\frac{\cosh k(h+z)}{\cosh kh}[\cos(kx - \omega t) - \cos(kx + \omega t)] \\ &= \frac{2Agk}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\sin kx \sin \omega t \end{aligned}$$

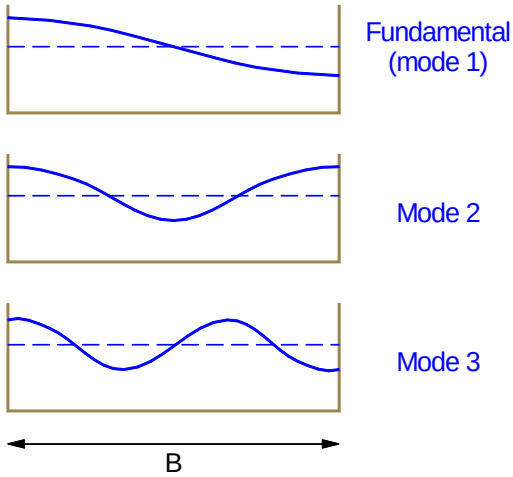
Again, this corresponds to a standing wave. The points of zero horizontal velocity are $kx = n\pi$ and hence this could describe reflection at any of these points.

Hence:

- (Complete) reflection can be represented by the superposition of two equal and opposite progressive waves to form a standing wave.
- The standing wave has the same wavelength and frequency but twice the amplitude.
- There are nodes (positions of zero surface displacement η) separated by distance π/k or $L/2$ (i.e. half a wavelength), with velocity nodes half way between them.
- The point of reflection corresponds to a point of zero velocity and double-amplitude displacement. This is important when we consider wave forces on structures.

2.5.2 Seiching (Basin Oscillations)

Just like musical instruments, standing waves may occur in enclosed basins (reflection from both ends) as a resonance condition if the wind-speed and basin dimensions are favourable. The first few modal shapes are illustrated below.



The basin length is a whole number of half-wavelengths:

$$B = n_s(\frac{1}{2}L)$$

or

$$L = \frac{2B}{n_s}$$

The enclosure means that these are typically shallow-water waves with speed

$$\frac{L}{T} = \sqrt{gh}$$

with seiching period (*Merian formula*):

$$T = \frac{2B}{n_s\sqrt{gh}}$$

Only the fundamental mode ($n_s = 1$) is usually established.

Example.
Lake Baikal in Siberia contains about one fifth of the world's fresh-water resources. It is 636 km long, with an average depth of 744 m. Find the fundamental period for seiching.

In the fundamental mode, the length of the lake is half a wavelength. Hence the wavelength is

$$L = 2 \times 636000 \quad = 1272000 \text{ m}$$

The depth is considerably less than 1/20th of a wavelength, so these are shallow-water waves with speed

$$c = \sqrt{gh} \quad = \sqrt{9.81 \times 744} \quad = 85.43 \text{ m s}^{-1}$$

Then,

$$\text{period} = \frac{\text{wavelength}}{\text{speed}} \quad = \frac{L}{c} \quad = 14890 \text{ s}$$

or

$$\frac{14890}{3600} = 4.136 \text{ hrs}$$

Answer: seiching period = 4.1 hours