Real wave fields are not regular, but a combination of many components of different amplitude, frequency and direction, which may be assumed to follow some statistical distribution. For design, models must reflect the appropriate probability distributions of heights and spectral distribution of frequencies (and, ideally, direction).

3.1 Measures of Wave Height and Period

At a fixed location, a depth-monitoring device (such as a wave buoy, pressure sensor or acoustic sounder) measures sea-surface elevation $\eta(t)$. From this, successive wave heights may be measured. Suppose that a set of N wave heights H_i are measured and put in descending order. The following are common measures of the *sea state*.

$H_{\max}$	Largest wave height in the sample	$H_{\max} = \max\limits_i (H_i)$
H_{av}	Mean wave height	$H_{\text{av}} = \dfrac{1}{N}\sum H_i$
H_{rms}	Root-mean-square wave height	$H_{\text{rms}} = \sqrt{\dfrac{1}{N}\sum H_i^2}$
$H_{1/3}$	Average of the highest $N/3$ waves	$H_{1/3} = \dfrac{1}{N/3}\sum\limits_1^{N/3} H_i$
H_{m0}	Estimate based on the rms surface elevation	$H_{m0} = 4\left(\overline{\eta^2}\right)^{1/2}$
H_s	Significant wave height	Either $H_{1/3}$ or H_{m0}

For a typical probability distribution of wave heights (Rayleigh distribution) $H_{1/3} = 1.416\, H_{\text{rms}}$, whilst for a typical spectral distribution of frequencies (Bretschneider spectrum – see later) $H_{m0} = \sqrt{2}\, H_{\text{rms}}$. Thus, $H_{1/3}$ and H_{m0} are almost equal and either may be taken to define the *significant wave height* H_s, depending on what is used to measure sea state. The "significant wave height" (allegedly) corresponds to "what an experienced mariner would judge the height of waves in a storm".

There are also common measures of wave period:

T_s	Significant wave period (average over highest $N/3$ waves)
T_p	Peak period (corresponds to the peak frequency of the energy spectrum)
T_e	Energy period (period of a regular wave with same significant wave height and power density; used in wave-energy prediction; derived from the energy spectrum)
T_z	Mean zero up-crossing period

3.2 Probability Distribution of Wave Heights

For a narrow-banded sea state (i.e. a small range of frequencies) deep-water waves have been observed to follow a *Rayleigh Distribution*, for which the probability that wave heights *exceed* H is given by

$$P(\text{height} > H) = \exp[-(H/H_{\text{rms}})^2]$$

Thus,

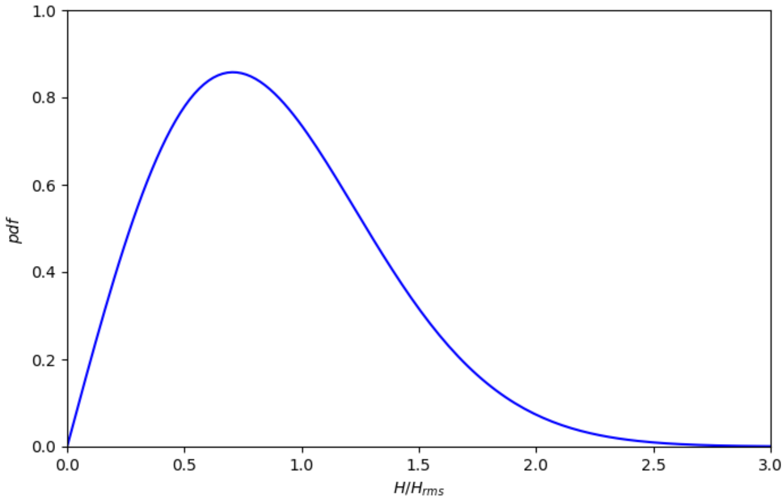
cumulative distribution function: $F(H) = P(\text{height} < H) \quad = 1 - \mathrm{e}^{-(H/H_{\text{rms}})^2}$

probability density function: $f(H) = \dfrac{\mathrm{d}F}{dH} \quad = 2\dfrac{H}{H_{\text{rms}}^2}\mathrm{e}^{-(H/H_{\text{rms}})^2}$

By definition of a probability density function, the probability that an individual wave height lies between H_1 and H_2 is

$$P(H_1 < H < H_2) = \int_{H_1}^{H_2} f(H)\,\mathrm{d}H \quad = F(H_2) - F(H_1)$$

$$= \mathrm{e}^{-(H_1/H_{\text{rms}})^2} - \mathrm{e}^{-(H_2/H_{\text{rms}})^2}$$



The only parameter of this distribution is the root-mean-square height, H_{rms}.

Exercise: verify from probability theory, i.e.

$$E(H^2) \equiv \int_0^\infty H^2 f(H)\,\mathrm{d}H$$

that $E(H^2) = H_{\text{rms}}^2$.

Other key wave statistics may be determined by the usual rules for probability distributions:

$$H_{\text{av}} \equiv E(H) \quad = \int_0^\infty H\, f(H)\, \mathrm{d}H$$

$$H_p = \text{average of highest fraction } p \text{ of waves} \quad = \frac{1}{p} \int_{F^{-1}(1-p)}^\infty H\, f(H)\, \mathrm{d}H$$

With this we find[1]:

$$H_{\text{av}} = \frac{\sqrt{\pi}}{2} H_{\text{rms}} = 0.886\, H_{\text{rms}}$$

$$H_{1/3} = 1.416\, H_{\text{rms}}$$

$$H_{1/10} = 1.800\, H_{\text{rms}}$$

$$H_{1/100} = 2.359\, H_{\text{rms}}$$

Example:
Near a pier, 400 consecutive wave heights are measured. Assume that the sea state is narrow-banded.

(a) How many waves are expected to exceed $2H_{\text{rms}}$?
(b) If the significant wave height is 2.5 m, what is H_{rms}?
(c) Estimate the wave height exceeded by 80 waves.
(d) Estimate the number of waves with a height between 1.0 m and 3.0 m.

---

[1] A certain amount of fiddly mathematics gives for the average of the highest fraction p:

$$\frac{H_p}{H_{\text{rms}}} = \sqrt{\ln(1/p)} + \frac{\sqrt{\pi}}{2p} \operatorname{erfc}(\sqrt{\ln(1/p)})$$

Here erfc() is the *complementary error function*, which most modern computer languages will provide as a library function. Here it is in Python:

```
from math import sqrt, log, erfc, pi

def prob( p ):
    x = sqrt( log( 1 / p ) )
    return x + sqrt( pi ) / ( 2 * p ) * erfc( x )

p = float( input( "Enter p: " ) )
print( "H/Hrms = ", prob( p ) )
```

3.3 Wave Spectra

For regular waves of a single frequency, the wave energy (per unit surface area) is given by

$$E = \frac{1}{2}\rho g A^2 \qquad = \rho g \overline{\eta^2(t)} \qquad = \frac{1}{8}\rho g H^2$$

i.e. the wave energy is proportional to the square of the amplitude A (or height H) of the harmonically-varying surface displacement $\eta(t)$.

Real wave fields contain many frequencies. The *energy spectrum* or *power spectral density* $S(f)$ is such that the amount of energy (divided by ρg) in the small frequency interval $\mathrm{d}f$ is

$$S(f)\,\mathrm{d}f$$

or, equivalently, that the energy from waves between frequencies f_1 and f_2 is (ρg times)

$$\int_{f_1}^{f_2} S(f)\,\mathrm{d}f$$

(This is rather like a continuous probability distribution). Note that we can just as well work in wave angular frequency ω, where $\omega = 2\pi f$. In that case,

$$S(\omega)\,\mathrm{d}\omega$$

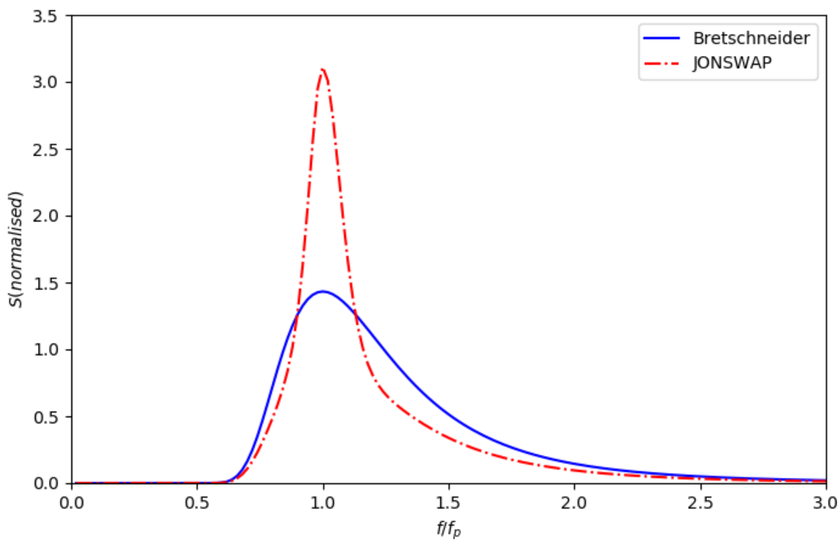
is the energy (divided by ρg) between wave angular frequencies ω_1 and ω_2.

3.3.1 Bretschneider Spectrum

Various model spectra are used in design. The *Bretschneider spectrum* is recommended for use in open-ocean conditions:

$$S(f) = \frac{5}{16} H_{m0}^2 \frac{f_p^4}{f^5} \exp\left(-\frac{5}{4}\frac{f_p^4}{f^4}\right)$$

where H_{m0} is a measure of significant wave height H_s based on the total energy (see below) and f_p is the peak frequency ($= 1/T_p$, where T_p is the peak period).



If we integrate over all frequencies we obtain total energy (strictly, energy divided by ρg):

$$\frac{E}{\rho g} \equiv \int_0^\infty S(f)\,\mathrm{d}f \quad = \frac{1}{16} H_{m0}^2$$

Hence

$$H_{m0} = 4\sqrt{E/\rho g} \quad = 4\sqrt{\overline{\eta^2(t)}}$$

For a *regular* wave with height H_{rms} (there is only one wave height, so it is the rms value):

$$\frac{E}{\rho g} = \frac{1}{8} H_{\mathrm{rms}}^2$$

Thus, the complete spectrum with height parameter H_{m0} will have the same energy density as a regular wave with parameter H_{rms} provided

$$H_{m0} = \sqrt{2} H_{rms} \quad = 1.414 H_{\mathrm{rms}}$$

But, for a Rayleigh distribution of wave heights, we have already seen that

$$H_s = H_{1/3} \quad = 1.416 H_{\mathrm{rms}}$$

Hence, in practice, either H_{m0} and $H_{1/3}$ can be used synonymously for H_s.

3.3.2 Use of Spectral Data to Determine height and Period Parameters

From, e.g., wave-buoy data for $\eta(t)$, the energy spectral density $S(f)$ is proportional to $|\tilde{\eta}|^2$, where $\tilde{\eta}$ is the Fourier transform of η. The **peak period** T_p can be derived from the peak frequency f_p in $S(f)$:

$$T_p = \frac{1}{f_p}$$

The **significant wave height** can be taken as

$$H_s = H_{m0} = 4\sqrt{m_0}$$

where m_0 is the *zeroth moment*: the area under the $S(f)$ curve. Other moments can be defined:

$$m_n = \int_0^\infty f^n S(f)\,\mathrm{d}f$$

and found from an experimentally-derived spectrum by numerical integration[2]. A particularly important one is m_{-1}, since this can be used to determine the **energy period** T_e (the period of a regular wave with the same significant wave height and power density, which is widely used in wavepower prediction):

$$T_e = \frac{m_{-1}}{m_0}$$

and the **zero up-crossing period** T_z, which can be estimated from the moments of the wave spectrum by:

[2] For the Bretschneider spectrum, some moderate mathematics produces $m_n = \frac{1}{16} H_{m0}^2 f_p^n \left(\frac{5}{4}\right)^{\frac{n}{4}} \Gamma(1-\frac{n}{4})$, where $\Gamma()$ is the gamma function (a generalisation of a factorial function).

$$T_z = \sqrt{\frac{m_0}{m_2}}$$

For the Bretschneider spectrum these give

$$T_e = 0.857T_p \quad \text{(energy period)}$$

$$T_z = 0.710T_p \quad \text{(zero up-crossing period)}$$

3.3.3 The JONSWAP Spectrum

Another widely-used spectrum recommended for fetch-limited conditions (based on extensive wave data from the North Sea) is the JONSWAP spectrum

$$S(f) = CH_{m0}^2 \frac{f_p^4}{f^5} \exp\left(-\frac{5}{4}\frac{f_p^4}{f^4}\right)\gamma^b$$

Here, the peak of the spectrum is enhanced (i.e. a greater proportion of the total energy is clustered around the peak frequency) by the factor γ^b, where γ may be fitted to real measurements, but is typically 3.3 and

$$b = \exp\left\{-\frac{1}{2}\left(\frac{f/f_p - 1}{\sigma}\right)^2\right\}, \qquad \sigma = \begin{cases} 0.07 & f < f_p \\ 0.09 & f > f_p \end{cases}$$

C is the constant required to get the correct total energy (e.g.by numerical integration). A JONSWAP spectrum is recommended for seas with more limited fetch.

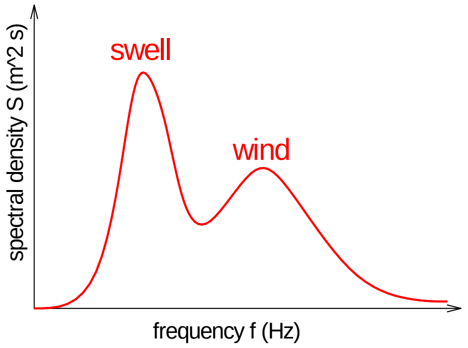
For a JONSWAP spectrum (with $\gamma = 3.3$) numerical integration gives other design periods:

$$T_e = 0.903T_p \quad \text{(energy period)}$$

$$T_z = 0.778T_p \quad \text{(zero up-crossing period)}$$

3.3.4 Multi-Modal Spectra

Real sea states may contain waves from multiple sources – often waves of lower frequency from a far-off storm ("swell") and higher-frequency waves from a local storm ("wind"). Complex statistical techniques can be used to extract the separate contributions from the combined spectrum.



3.4 Constructing a Representative Wave Field From a Spectrum

For most spectra there is negligible energy associated with frequencies less than $0.5\,f_p$ or greater than $3f_p$, where f_p is the peak frequency. If we break this or a larger frequency range up into discrete intervals of length Δf, we can simulate a realistic spectrum (either in a numerical simulation, or in a wave tank with programmable wave paddle) as a sum of individual harmonic components:

$$\eta(t) = \sum a_i \cos(k_i x - \omega_i t - \phi_i)$$

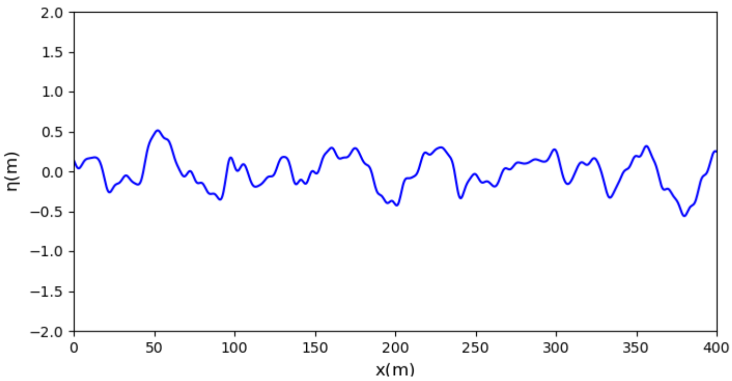
where ω_i and k_i are the wavenumbers associated with frequency f_i, the ϕ_i are random phases, and the correct amount of energy (E_i) at this frequency occurs if we take

$$S(f_i)\Delta f = E_i = \frac{1}{2}a_i^2$$

or

$$a_i = \sqrt{2S(f_i)\,\Delta f}$$

The wave tanks at the University of Manchester are equipped with programmable wave paddles that can create such realistic random wave fields for a given spectrum.



Note that, as different wavenumbers travel with different speeds, this wave form is not propagated unchanged (as it would be for a regular wave), but evolves with time.

If, instead of choosing random phases ϕ_i we choose them deliberately such that waves travelling at different speeds arrive at the same point at the same time it is possible to generate *focused wave groups*, with the focusing producing a very large-amplitude disturbance at a single instant. These worst-case scenarios are used to simulate extreme-wave events.

Example.
An irregular wavefield at a deep-water location is characterised by peak period of 8.7 s and significant wave height of 1.5 m.

(a) Provide a sketch of a Bretschneider spectrum, labelling both axes with variables and units and indicating the frequencies corresponding to both the peak period and the energy period.
Note: Calculations are not needed for this part.

(b) Determine the power density (in kW m^{-1}) of a regular wave component with frequency 0.125 Hz that represents the frequency range 0.12 to 0.13 Hz of the irregular wave field.

Example.
Wave measurements are obtained from a stationary sensor located in deep water. The measured surface elevation of an irregular wave can be modelled as the sum of four regular wave components:

Period (s)	6	7	8	9
Amplitude (m)	0.8	1.2	0.8	0.4

(a) In the context of modelling an irregular wave, explain the meaning of the following terms:
(i) significant wave height;
(ii) significant wave period;
(iii) duration-limited.

(b) Obtain the total power conveyed by these deep-water wave components per metre width of wave crest if conditions were measured:
(i) with zero current;
(ii) with an opposing current of 1.0 m s^{-1}.

3.5 Prediction of Wave Climate

Models for wave spectra generally require one to specify a representative wave height (e.g. H_s) and period (T_s or T_p).

Waves are generated by wind stress on the water surface, whilst gravity provides the restoring force. Thus, wave height and period are expected to be functions of:

- wind speed U (conventionally the wind speed at 10 m above the surface);
- fetch F (the distance over which the wind blows);
- duration t of the storm;
- gravity, g.

By dimensional analysis,

$$H_s \sim U, F, t, g$$

This gives 5 variables, 2 independent dimensions (length and time), and hence 3 dimensionless Π groups:

$$\frac{gH_s}{U^2} = \text{function}(\frac{gF}{U^2}, \frac{gt}{U})$$

Similarly

$$\frac{gT_p}{U} = \text{function}(\frac{gF}{U^2}, \frac{gt}{U})$$

Extensive wave data has led to empirical forms for these. Two of the commonest are SMB (Sverdrup, Monk and Bretschneider) and JONSWAP (JOint North Sea WAve Project).

These correlations can be used to predict wave climate (usually to construct a wave spectrum) from a weather forecast (*forecasting*), ongoing weather (*nowcasting*) and reconstructing wave climate from measured wind records (*hindcasting*).

The following are standard correlations for *deep-water* waves.

3.5.1 JONSWAP (Hasselman et al., 1973)

If the wind has blown long enough for wave energy to propagate right across the fetch then the wave parameters become functions only of the fetch F and cease to be dependent on the storm duration t. These are called *fetch-limited* waves and an empirical correlation is

$$\frac{gH_s}{U^2} = 0.0016 \left(\frac{gF}{U^2}\right)^{1/2} \qquad \text{(up to maximum 0.2433)}$$

$$\frac{gT_p}{U} = 0.286 \left(\frac{gF}{U^2}\right)^{1/3} \qquad \text{(up to maximum 8.134)}$$

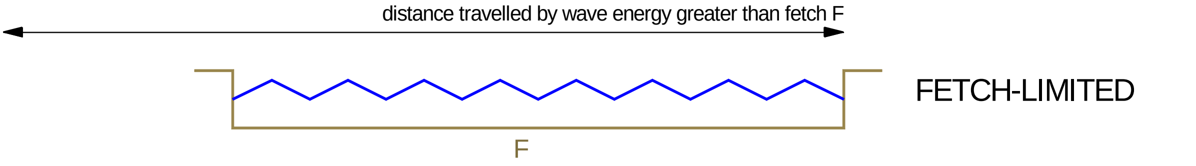
The (fairly rare) "maximum" conditions correspond to a "fully-developed sea" – one for which energy dissipation equals energy input and wave conditions become independent of fetch.

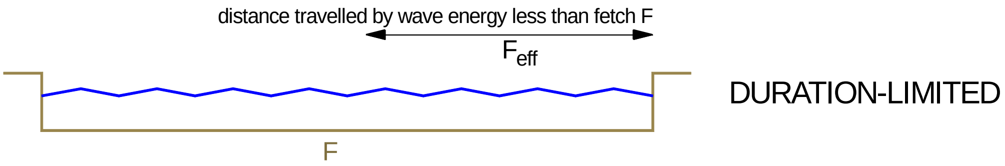
The minimum duration for fetch-limited waves, $t_{\min}$ is given, in non-dimensional form by

$$\left(\frac{gt}{U}\right)_{\min} = 68.8\left(\frac{gF}{U^2}\right)^{2/3} \qquad (\text{fully}-\text{developed sea:}\ 7.15\times 10^4)$$

If the storm duration $t < t_{\min}$ then the wave conditions are said to be *duration-limited* and the non-dimensional fetch gF/U^2 in the equations for H_s and T_P has to be replaced by an *effective fetch* F_{eff} determined by rearranging the last equation in terms of the *actual* duration t:

$$\left(\frac{gF}{U^2}\right)_{\text{eff}} = \left(\frac{1}{68.8}\frac{gt}{U}\right)^{3/2}$$





Note:
(1) The period parameter predicted here is the peak period T_p, since this is what is required in the Jonswap spectrum. If required, the significant wave period can be estimated by

$$T_s \approx 0.945 T_p$$

(2) It is a considerable nuisance to have to keep writing the Π groups out in full. The JONSWAP equations are conveniently written (ignoring the fully-developed limit) as

$$\hat{H}_s = 0.0016\hat{F}^{1/2}$$

$$\hat{T}_p = 0.286\hat{F}^{1/3}$$

$$\hat{t}_{\min} = 68.8\hat{F}^{2/3}$$

where

$$\hat{F} \equiv \frac{gF}{U^2}, \qquad \hat{t} \equiv \frac{gt}{U}, \qquad \hat{H}_s \equiv \frac{gH_s}{U^2}, \qquad \hat{T}_p \equiv \frac{gT_p}{U}$$

3.5.2 SMB (Bretschneider, 1970)

This alternative correlation has slightly(!) more complex formulae. Note that the representative period here is T_s, the significant wave period, rather than T_p, the peak period. In non-dimensional form:

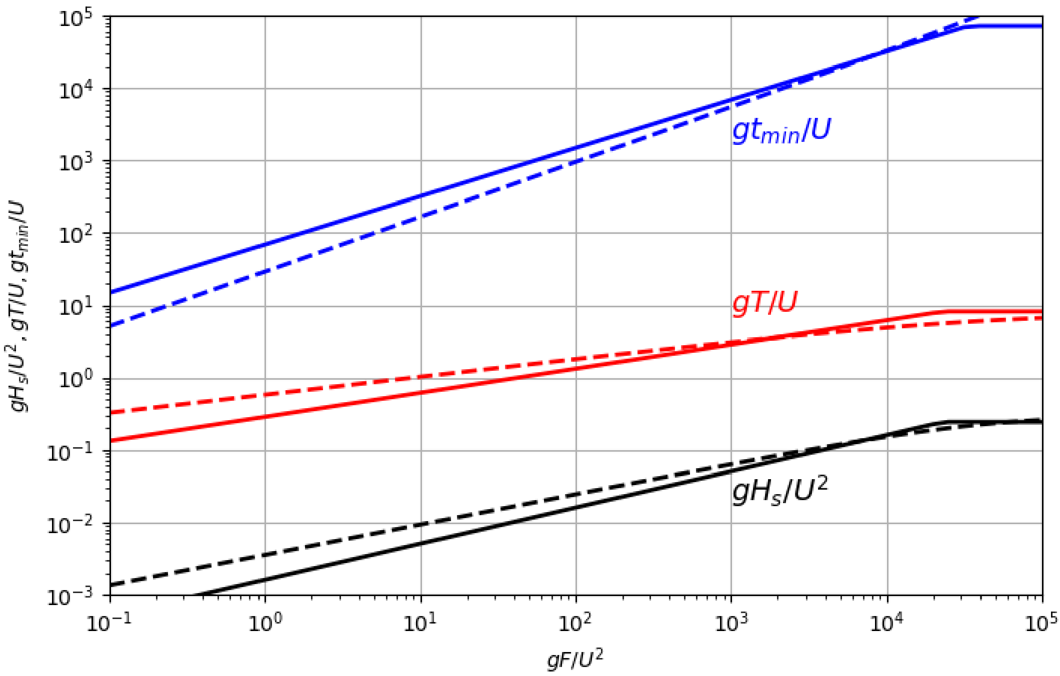
$$\hat{H}_s = 0.283 \tanh\{0.0125\hat{F}^{0.42}\}$$

$$\hat{T}_s = 7.54 \tanh\{0.077\hat{F}^{0.25}\}$$

The minimum storm duration for fetch-limited waves is given by

$$\hat{t}_{\min} = K \exp\left\{\sqrt{\left[A(\ln \hat{F})^2 - B \ln \hat{F} + C\right]} + D \ln \hat{F}\right\}$$

where $K = 6.5882$, $A = 0.0161$, $B = 0.3692$, $C = 2.2024$, $D = 0.8798$.



Prediction curves; solid: JONSWAP; dashed: SMB

Example.

(a) Wind has blown at a consistent $U = 20$ m s^{-1} over a fetch $F = 100$ km for $t = 6$ hrs. Determine H_s and T_p using the JONSWAP curves.

(b) If the wind blows steadily for another 4 hours what are H_s and T_p?