## Section 1.2

**Example.**
Find, in still water of depth 15 m:
(a) the period of a wave with wavelength 45 m;
(b) the wavelength of a wave with period 8 s.
In each case write down the phase speed (celerity).

(a) Given

$$h = 15 \text{ m}$$
$$L = 45 \text{ m}$$

Then,

$$k = \frac{2\pi}{L} \quad = 0.1396 \text{ m}^{-1}$$

Dispersion relation:

$$\begin{aligned} \omega^2 &= gk \tanh kh \\ &= 9.81 \times 0.1396 \tanh(0.1396 \times 15) \\ &= 1.329 \text{ (rad s}^{-1})^2 \end{aligned}$$

$$\Rightarrow \qquad \omega = \sqrt{1.329} \quad = 1.153 \text{ rad s}^{-1}$$

$$\Rightarrow \qquad T = \frac{2\pi}{\omega} \quad = 5.449 \text{ s}$$

Phase speed:

$$c = \frac{\omega}{k} \left(\text{or } \frac{L}{T}\right) \quad = 8.259 \text{ m s}^{-1}$$

**Answer:** period = 5.45 s; phase speed = 8.26 m $s^{-1}$.

(b) Given

$$h = 15 \text{ m}$$
$$T = 8 \text{ s}$$

Then,

$$\omega = \frac{2\pi}{T} \quad = 0.7854 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$\Rightarrow \qquad 0.9432 = kh \tanh kh$

Iterate as either

$$kh = \frac{0.9432}{\tanh kh} \qquad \text{or} \qquad kh = \frac{1}{2}(kh + \frac{0.9432}{\tanh kh})$$

to get

$$kh = 1.152$$

$$k = \frac{1.152}{h} \quad = 0.0768 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 81.81 \text{ m}$$

$$c = \frac{\omega}{k} \ \left(\text{or } \frac{L}{T}\right) \quad = 10.23 \text{ m s}^{-1}$$

**Answer:** wavelength = 81.8 m; phase speed = 10.2 m $s^{-1}$.

**Section 1.3**

**Example.**
A pressure sensor is located 0.6 m above the sea bed in a water depth $h = 12$ m. The pressure fluctuates with period 15 s. A maximum gauge pressure of 124 kPa is recorded.
(a) What is the wave height?
(b) What are the maximum horizontal and vertical velocities at the surface?

(a) Total pressure:

$$p = -\rho g z + \rho g A \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t)$$

At the sensor, $z = -12 + 0.6 = -11.4$ m. At any depth the maximum pressure is recorded when $\cos(\ldots) = 1$. Hence (using seawater density, $\rho = 1025 \text{ kg m}^{-3}$):

$$124000 = -1025 \times 9.81 \times (-12) + \frac{1025 \times 9.81 \times A \times \cosh(k \times 0.6)}{\cosh kh} \qquad (*)$$

To continue we need the wave properties $kh$ and $k$. Given

$h = 12$ m
$T = 15$ s

Then,

$$\omega = \frac{2\pi}{T} = 0.4189 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \quad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \quad 0.2147 = kh \tanh kh$$

The LHS is small, so iterate as

$$kh = \frac{1}{2}\left(kh + \frac{0.2147}{\tanh kh}\right)$$

to get

$$kh = 0.4806$$

and

$$k = \frac{0.4806}{h} = 0.04005 \text{ m}^{-1}$$

Returning to (*):

$$124000 = 114630 + 8999A$$

$$\Rightarrow \quad A = 1.041 \text{ m}$$

$\Rightarrow \qquad H = 2A \quad = 2.082$ m

**Answer:** wave height = 2.08 m.

(b) Velocities:

$$u = \frac{Agk}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx - \omega t)$$

$$w = \frac{Agk}{\omega}\frac{\sinh k(h+z)}{\cosh kh}\sin(kx - \omega t)$$

Hence, at the surface ($z = 0$) the maximum values of each (when $\cos(\ldots)$ or $\sin(\ldots)$ is $+1$) are

$$u_{\text{max}} = \frac{Agk}{\omega} \quad = \frac{1.041 \times 9.81 \times 0.04005}{0.4189} \quad = 0.9764 \text{ m s}^{-1}$$

$$w_{\text{max}} = \frac{Agk}{\omega}\tanh(kh) = 0.9764 \times \tanh 0.4806 \quad = 0.4362 \text{ m s}^{-1}$$

**Answer:** 0.976 m s$^{-1}$ (horizontally); 0.436 m s$^{-1}$ (vertically).

## Section 1.6

**Example.**
A sea-bed pressure transducer in 9 m of water records a sinusoidal signal with amplitude 5.9 kPa and period 7.5 s. Find the wave height, energy density and wave power per metre of crest.

The dynamic (i.e. time-varying) part of the wave pressure is given by

$$p = \rho g A \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t)$$

and, hence, at the sea bed ($z = -h$):

$$5900 = \frac{1025 \times 9.81 \times A}{\cosh kh} \qquad (*)$$

To continue we need the wave property $kh$. Given

$h = 9$ m
$T = 7.5$ s

Then,

$$\omega = \frac{2\pi}{T} \quad = 0.8378 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 0.6440 = kh \tanh kh$$

The LHS is small, so iterate as

$$kh = \frac{1}{2}(kh + \frac{0.6440}{\tanh kh})$$

to get

$$kh = 0.8994$$

Returning to (*):

$$5900 = 7020 \times A$$

$$\Rightarrow \qquad A = 0.8405 \text{ m}$$

$$\Rightarrow \qquad H = 2A \quad = 1.681 \text{ m}$$

Then, for energy density:

$$E = \frac{1}{2}\rho g A^2 \quad \left(\text{or } \frac{1}{8}\rho g H^2\right) \quad = 3552 \text{ J m}^{-2}$$

For power density $P = Ec_g$, with $c_g = nc$:

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.8061$$

$$k = \frac{0.8994}{h} \quad = 0.09993 \text{ m}^{-1}$$

$$c = \frac{\omega}{k} \quad = 8.384 \text{ m s}^{-1}$$

$$c_g = nc \quad = 6.758 \text{ m s}^{-1}$$

so that power density is

$$P = Ec_g \quad = 3552 \times 6.758 \quad = 24000 \text{ W m}^{-1}$$

**Answer:** wave height = 1.68 m; energy density = 3.55 kJ $m^{-2}$; power density = 24 kW $m^{-1}$.

## Section 1.8

**Example.**
(a) Find the deep-water speed and wavelength of a wave of period 12 s.
(b) Find the speed and wavelength of a wave of period 12 s in water of depth 3 m. Compare with the shallow-water approximation.

(a) In deep water speed and wavelength come directly from the period.

Speed:

$$c = \frac{gT}{2\pi} \quad = 18.74 \text{ m s}^{-1}$$

Wavelength:

$$L = \frac{gT^2}{2\pi} \quad (\text{or just} \quad cT) \quad = 224.8 \text{ m}$$

**Answer:** speed = 18.7 m s$^{-1}$; wavelength = 225 m.

(b) Given:

$$h = 3 \text{ m}$$
$$T = 12 \text{ s}$$

Then,

$$\omega = \frac{2\pi}{T} \quad = 0.5236 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 0.08384 = kh \tanh kh$$

Iterate as either

$$kh = \frac{0.08384}{\tanh kh} \qquad \text{or (much better here)} \qquad kh = \frac{1}{2}(kh + \frac{0.08384}{\tanh kh})$$

to get

$$kh = 0.2937$$

$$k = \frac{0.2937}{h} \quad = 0.09790 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 64.18 \text{ m}$$

$$c = \frac{\omega}{k} \; \left(\text{or } \frac{L}{T}\right) \quad = 5.348 \text{ m s}^{-1}$$

By contrast, the "shallow-water" phase speed is

$$c = \sqrt{gh} \quad = 5.425 \text{ m s}^{-1}$$

and the wavelength is

$$L = cT \quad = 65.10 \text{ m}$$

As expected (since $kh < \pi/10$ so the conditions would be classified as "shallow-water") these are very close to those from the general formula.

**Answer:** wavelength = 64.2 m, phase speed = 5.35 m s$^{-1}$.

**Section 1.9**

**Example.**
An acoustic depth sounder indicates regular surface waves with apparent period 8 s in water of depth 12 m. Find the wavelength and absolute phase speed of the waves when there is:
(a) no mean current;
(b) a current of 3 m s$^{-1}$ in the same direction as the waves;
(c) a current of 3 m s$^{-1}$ in the opposite direction to the waves.

Given

$$h = 12 \text{ m}$$
$$T_a = 8 \text{ s}$$

Then, irrespective of the current speed:

$$\omega_a = \frac{2\pi}{T} \qquad = 0.7854 \text{ rad s}^{-1}$$

$T_a$ and $\omega_a$ are the absolute period and wave angular frequency in all cases.

Dispersion relation:

$$(\omega_a - kU)^2 = \omega_r^2 = gk \tanh kh$$

Iterate as

$$k = \frac{(\omega_a - kU)^2}{g \tanh kh}$$

i.e.

$$k = \frac{(0.7854 - kU)^2}{9.81 \tanh 12k} \quad \text{or (better here)} \quad k = \frac{1}{2}\left[k + \frac{(0.7854 - kU)^2}{9.81 \tanh 12k}\right]$$

A typical starting value for $k$ is 0.1 m$^{-1}$.

The phase speed in the *absolute* (i.e. fixed) frame is

$$c_a = \frac{L}{T_a}$$

(a) With $U = 0$, iterate

$$k = \frac{1}{2}\left[k + \frac{0.7854^2}{9.81 \tanh 12k}\right]$$

to get

$$k = \frac{0.9941}{h} \qquad = 0.08284 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \qquad = 75.85 \text{ m}$$

$$c = \frac{\omega_a}{k} \quad \left(\text{or } \frac{L}{T}\right) \quad = 9.481 \text{ m s}^{-1}$$

**Answer:** wavelength = 75.8 m;   phase speed = 9.48 m s$^{-1}$.

(b) With $U = +3 \text{ m s}^{-1}$, iterate

$$k = \frac{1}{2}\left[k + \frac{(0.7854 - 3k)^2}{9.81 \tanh 12k}\right]$$

to get

$$k = 0.06024 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 104.3 \text{ m}$$

$$c_a = \frac{\omega_a}{k} \quad = \frac{0.7854}{0.06024} \quad = 13.04 \text{ m s}^{-1}$$

**Answer:** wavelength = 104.3 m;   absolute phase speed = 13.0 m s$^{-1}$.

(c) With $U = -3 \text{ m s}^{-1}$, iterate

$$k = \frac{1}{2}\left[k + \frac{(0.7854 + 3k)^2}{9.81 \tanh 12k}\right]$$

to get (after a lot of iterations):

$$k = 0.1951 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 32.20 \text{ m}$$

$$c_a = \frac{\omega_a}{k} \quad = 4.026 \text{ m s}^{-1}$$

**Answer:** wavelength = 32.2 m;   absolute phase speed = 4.03 m s$^{-1}$.

**Section 2.1**

**Example.**

A straight coastline borders a uniformly-sloping sea bed. Regular waves are observed to cross the 8 m depth contour at an angle of 14° to the coastline-normal, with wavelength 45 m. Find:

(a) the wave period;
(b) the wavelength in deep water;
(c) the direction in deep water.

(a) In depth $h = 8$ m,

$$L = 45 \text{ m}$$

$$k = \frac{2\pi}{L} \quad = 0.1396 \text{ m}^{-1}$$

Dispersion relation

$$\omega^2 = gk \tanh kh$$

gives

$$\omega = 1.051 \text{ rad s}^{-1}$$

$$T = \frac{2\pi}{\omega} \quad = 5.978 \text{ s}$$

**Answer:** 5.98 s.

(b) Wavelength in deep water:

$$L_0 = \frac{gT^2}{2\pi} \quad = 55.80 \text{ m}$$

**Answer:** 55.8 m.

(c) The wavenumber in deep water is

$$k_0 = \frac{2\pi}{L_0} \quad = 0.1126 \text{ m}^{-1}$$

Snell's Law:

$$(k \sin\theta)_0 = (k \sin\theta)_{8\text{ m}}$$

$\Rightarrow \quad 0.1126 \sin\theta_0 = 0.1396 \sin 14°$

$\Rightarrow \quad \theta_0 = 17.45°$

**Answer:** 17.5º.

**Section 2.2**

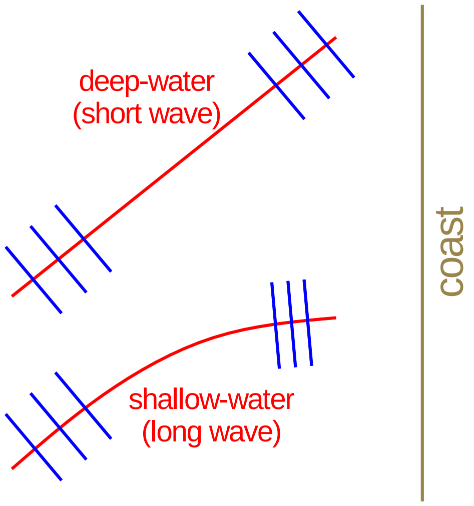
**Example (**Exam 2016, part)
Waves propagate towards a long straight coastline that has a very gradual bed slope normal to the coast. In water depth of 20 m, regular waves propagate at heading $\theta = 40°$ relative to the bed slope.

(a) Sketch the shape of a wave ray from the 20 m depth contour to the 5 m depth contour for a wave that is of deep-water type in both depths and, separately, for a wave that is of shallow water type in both depths. Calculations are not required.

(b) For a wave with period $T = 8$ s and height 1.2 m at 20 m depth, calculate the wave heading and wave height at the 5 m depth contour.

(a)
Deep-water waves (by definition) do not feel the effect of the bottom, so are unchanged on moving from 20 m to 5 m depth.

By contrast, shallow-water waves are bent toward the normal as they move into shallower water.




(b)

$$T = 8 \text{ s}$$

Hence

$$\omega = \frac{2\pi}{T} \qquad = 0.7854 \text{ rad s}^{-1}$$

The dispersion relation is

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

This may be iterated as either

$$kh = \frac{\omega^2 h/g}{\tanh kh} \qquad \text{or} \qquad kh = \frac{1}{2}(kh + \frac{\omega^2 h/g}{\tanh kh})$$

| | $h = 5$ m | $h = 20$ m |
|---|---|---|
| $\dfrac{\omega^2 h}{g}$ | 0.3144 | 1.258 |
| Iteration: | $kh = \dfrac{1}{2}\left(kh + \dfrac{0.3144}{\tanh kh}\right)$ | $kh = \dfrac{1.258}{\tanh kh}$ |
| $kh$ | 0.5918 | 1.416 |
| $k$ | 0.1184 m$^{-1}$ | 0.07080 m$^{-1}$ |
| $c = \dfrac{\omega}{k}$ | 6.633 m s$^{-1}$ | 11.09 m s$^{-1}$ |
| $n = \dfrac{1}{2}\left[1 + \dfrac{2kh}{\sinh 2kh}\right]$ | 0.8999 | 0.6674 |
| $\theta$ | ? | 40° |
| $H$ | ? | 1.2 m |

Refraction:

$$(k \sin\theta)_{5\text{ m}} = (k \sin\theta)_{20\text{ m}}$$

$$0.1184 \sin\theta = 0.07080 \sin 40°$$

Hence,

$$\theta = 22.60°$$

From the shoaling equation:

$$(H^2 nc \cos\theta)_{5\text{ m}} = (H^2 nc \cos\theta)_{20\text{ m}}$$

Hence:

$$H_{5\text{ m}} = H_{20\text{ m}} \sqrt{\frac{(nc\cos\theta)_{20\text{ m}}}{(nc\cos\theta)_{5\text{ m}}}} = 1.2 \times \sqrt{\frac{0.6674 \times 11.09 \times \cos 40°}{0.8999 \times 6.633 \times \cos 22.60°}} = 1.217\text{ m}$$

**Answer:** heading 22.6°;   wave height 1.22 m

**Section 2.3**

**Example.**
Waves propagate towards a long straight coastline that has a constant bed slope of 1 in 100. Consider the $x$-axis to be normal to the coastline and the $y$-axis parallel to the coastline. Waves propagate at an angle $\theta$ to the $x$-axis.

(a) A wave with period 7 s and height 1.2 m crosses the 36 m depth contour at angle $\theta = 22°$.
  (i) Determine the direction, height and power per metre width of wave crest at the 4 m depth contour.
  (ii) Explain how height changes between these depths.

(b) A wave with period 7 s and height 1.0 m crosses the 4 m depth contour at angle $\theta = 0°$. Determine the breaking wave height and breaking depth from their corresponding indices and identify the type of breaker expected.

(c) Further along the coast, waves propagate over the outflow of a river. In water depth of 14 m, measurements indicate a period of 7 s and depth-averaged flow velocity of 0.8 m s$^{-1}$ against the wave direction. Determine the wavelength.

(a)
(i)

$$T = 7 \text{ s}$$

Hence

$$\omega = \frac{2\pi}{T} \qquad = 0.8976 \text{ rad s}^{-1}$$

The dispersion relation is

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

This may be iterated as either

$$kh = \frac{\omega^2 h/g}{\tanh kh} \qquad \text{or} \qquad kh = \frac{1}{2}(kh + \frac{\omega^2 h/g}{\tanh kh})$$

| | $h = 4$ m | $h = 36$ m |
|---|---|---|
| $\frac{\omega^2 h}{g}$ | 0.3285 | 2.957 |
| Iteration: | $kh = \frac{1}{2}\left(kh + \frac{0.3285}{\tanh kh}\right)$ | $kh = \frac{2.957}{\tanh kh}$ |

| | | |
|---|---|---|
| $kh$ | 0.6065 | 2.973 |
| $k$ | 0.1516 m$^{-1}$ | 0.08258 m$^{-1}$ |
| $c = \dfrac{\omega}{k}$ | 5.921 m s$^{-1}$ | 10.87 m s$^{-1}$ |
| $n = \dfrac{1}{2}\left[1 + \dfrac{2kh}{\sinh 2kh}\right]$ | 0.8956 | 0.5156 |
| $\theta$ | ? | 22° |
| $H$ | ? | 1.2 m |

Refraction:

$$(k \sin\theta)_{4\text{ m}} = (k \sin\theta)_{36\text{ m}}$$

$$0.1516 \sin\theta = 0.08258 \sin 22°$$

Hence,

$$\theta = 11.77°$$

From the shoaling equation:

$$(H^2 nc \cos\theta)_{4\text{ m}} = (H^2 nc \cos\theta)_{36\text{ m}}$$

Hence:

$$H_{4\text{ m}} = H_{36\text{ m}} \sqrt{\frac{(nc\cos\theta)_{36\text{ m}}}{(nc\cos\theta)_{4\text{ m}}}} = 1.2 \times \sqrt{\frac{0.5156 \times 10.87 \times \cos 22°}{0.8956 \times 5.921 \times \cos 11.77°}} = 1.201\text{ m}$$

The power per metre of crest is

$$P = \frac{1}{8}\rho g H^2 (nc) = \frac{1}{8} \times 1025 \times 9.81 \times 1.201^2 \times (0.8956 \times 5.921) = 9614\text{ W m}^{-1}$$

**Answer:** heading 11.8°;   wave height 1.20 m;   power = 9.61 kW m$^{-1}$.

(ii) The height initially reduces (because of the group velocity) and then increases again.

(b) At 4 m depth we have

$$H = 1.0\text{ m}$$

$$\theta = 0°$$

and from the above:

$$n = 0.8956$$

$$c = 5.921 \text{ m s}^{-1}$$

In deep water,

$$n = \frac{1}{2}$$

$$c = \frac{gT}{2\pi} \quad = 10.93 \text{ m s}^{-1}$$

$$L_0 = \frac{gT^2}{2\pi} \quad = 76.50 \text{ m}$$

whilst the deep-water height can be deduced from the shoaling equation:

$$(H^2 nc)_0 = (H^2 nc)_{4\text{ m}}$$

whence

$$H_0 = H_{4\text{ m}} \sqrt{\frac{(nc)_{4\text{ m}}}{(nc)_0}} \quad = 1.0 \times \sqrt{\frac{0.8956 \times 5.921}{0.5 \times 10.93}} \quad = 0.9851 \text{ m}$$

Then the given breaking criterion ("breaker height index") gives

$$H_b = 0.56 H_0 \left(\frac{H_0}{L_0}\right)^{-1/5} \quad = 0.56 \times 0.9851 \times \left(\frac{0.9851}{76.50}\right)^{-1/5} \quad = 1.317 \text{ m}$$

From the formula sheet the breaker depth index is

$$\gamma_b \equiv \left(\frac{H}{h}\right)_b = b - a\frac{H_b}{gT^2}$$

where

$$a = 43.8(1 - \mathrm{e}^{-19m}) \quad = 43.8\left(1 - \mathrm{e}^{-19/100}\right) \quad = 7.579$$

$$b = \frac{1.56}{1 + \mathrm{e}^{-19.5m}} \quad = \frac{1.56}{1 + \mathrm{e}^{-19.5/100}} \quad = 0.8558$$

Hence,

$$\frac{1.317}{h_b} = 0.8558 - 7.579 \times \frac{1.317}{9.81 \times 7^2}$$

giving breaking depth

$$h_b = 1.577 \text{ m}$$

The surf-similarity parameter (Irribarren number) is

$$\xi_0 = \frac{m}{\sqrt{H_0/L_0}} \quad = \frac{0.01}{\sqrt{0.9851/76.50}} \quad = 0.0881$$

This is significantly less than 0.5, so we expect spilling breakers.

**Answer:** breaking height = 1.32 m; breaking depth = 1.58 m; spilling breakers.

(c) Given

$$h = 14 \text{ m}$$

$$T_a = 7 \text{ s}$$

$$U = -0.8 \text{ m s}^{-1}$$

Then

$$\omega_a = \frac{2\pi}{T_a} \quad = 0.8976 \text{ rad s}^{-1}$$

Dispersion relation:

$$(\omega_a - kU)^2 = \omega_r^2 \; = gk \tanh kh$$

Rearrange as

$$k = \frac{(\omega_a - kU)^2}{g \tanh kh}$$

or here, in m-s units:

$$k = \frac{(0.8976 + 0.8k)^2}{9.81 \tanh 14k}$$

to get

$$k = 0.1087 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 57.80 \text{ m}$$

**Answer:** 57.8 m.

**Example.** (Exam 2017, part)
Waves propagate towards a straight shoreline. The wave heading is equal to the angle formed between wave crests and the bed contours. The bed slope is less than 1 in 100. Waves are measured in 30 m depth and wave conditions at 6 m depth are required to inform design of nearshore structures.

Regular waves are measured with period of 7 s and height of 3 m.

(a) Determine the water depth in which waves with this period can be considered as deep-water waves.

(b) For 30 m depth, determine the breaking height by the Miche criterion and briefly describe this type of breaking wave.

(c) If the heading is zero degrees, calculate wave height in 6 m depth. State your assumptions.

(d) If the heading of the measured conditions is 30°, calculate the wave heading and height in 6 m depth. Hence calculate the change of wave power per unit width of wave crest (kW m$^{-1}$) between the two depths.

(a) Given

$$T = 7 \text{ s}$$

Then

$$\omega = \frac{2\pi}{T} \quad = 0.8976 \text{ rad s}^{-1}$$

Deep-water waves if

$$kh > \pi$$

From the dispersion relation

$$\omega^2 = gk \tanh kh$$

$$\frac{\omega^2 h}{g} = kh \tanh kh$$

Hence, one has deep-water waves if

$$\frac{\omega^2 h}{g} > \pi \tanh \pi$$

i.e. if

$$h > \pi \tanh \pi \times \frac{g}{\omega^2} \quad = 38.11$$

**Answer:** deep-water waves if $h > 38.1$ m.

(b) Need wave conditions for $h = 30$ m (not deep water, from part (a)).

$$\frac{\omega^2 h}{g} = 2.464$$

Rearranging the dispersion relation:

$$kh = \frac{2.464}{\tanh kh}$$

which iterates to give

$$kh = 2.498$$

$$k = \frac{2.498}{h} \quad = 0.08327 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 75.46 \text{ m}$$

Needed later in part (c):

$$c = \frac{\omega}{k} \quad = 10.78 \text{ m s}^{-1}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.5338$$

If the waves are to break at a depth of 30 m then, from the Miche criterion:

$$\frac{H_b}{L} = 0.14 \tanh kh$$

Hence,

$$H_b = L \times 0.14 \times \tanh kh \quad = 75.46 \times 0.14 \times \tanh 2.498 \quad = 10.42 \text{ m}$$

The surf-similarity parameter (Irribarren number) is

$$\xi_b = \frac{m}{\sqrt{H_b/L_0}}$$

where $L_0$ is the deep-water wavelength:

$$L_0 = \frac{gT^2}{2\pi} \quad = \frac{9.81 \times 7^2}{2\pi} \quad = 76.50 \text{ m}$$

Since the beach slope $m$ is less than 0.01, then

$$\xi_b < \frac{0.01}{\sqrt{10.42/76.50}} \quad = 0.0271$$

This is (much) less than 0.4; hence the breaking-wave type is “spilling breakers”.

**Answer:** 10.4 m;   spilling breakers.

(c) Need wave parameters for $h = 6$ m. As before:

$$\omega = 0.8976 \text{ rad s}^{-1}$$

$$\frac{\omega^2 h}{g} = 0.4928$$

Rearranging the dispersion relation for iteration:

$$kh = \frac{1}{2}\left(kh + \frac{0.4928}{\tanh kh}\right)$$

which iterates to give

$$kh = 0.7651$$

$$k = \frac{0.7651}{h} \quad = 0.1275 \text{ m}^{-1}$$

$$c = \frac{\omega}{k} \quad = 7.040 \text{ m s}^{-1}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.8476$$

Shoaling equation with no refraction:

$$(H^2 nc)_{6\text{ m}} = (H^2 nc)_{30\text{ m}}$$

Hence:

$$H_{6\text{ m}} = H_{30\text{ m}}\sqrt{\frac{(nc)_{30\text{ m}}}{(nc)_{6\text{ m}}}} = 3 \times \sqrt{\frac{0.5338 \times 10.78}{0.8476 \times 7.040}} = 2.946 \text{ m}$$

**Answer:** 2.95 m;   assumes negligible energy dissipation.

(d) Refraction:

$$(k \sin\theta)_{6\text{ m}} = (k \sin\theta)_{30\text{ m}}$$

$$0.1275 \sin\theta = 0.08327 \sin 30°$$

Hence,

$$\theta = 19.06°$$

From the shoaling equation:

$$(H^2 nc \cos\theta)_{6\text{ m}} = (H^2 nc \cos\theta)_{30\text{ m}}$$

Hence:

$$H_{6\text{ m}} = H_{30\text{ m}}\sqrt{\frac{(nc\cos\theta)_{30\text{ m}}}{(nc\cos\theta)_{6\text{ m}}}} = 3 \times \sqrt{\frac{0.5338 \times 10.78 \times \cos 30°}{0.8476 \times 7.040 \times \cos 19.06°}} = 2.820 \text{ m}$$

Hence the change of power is

$$
\begin{aligned}
\Delta P &= \left(\frac{1}{8}\rho g H^2 nc\right)_{30\ \mathrm{m}} - \left(\frac{1}{8}\rho g H^2 nc\right)_{6\ \mathrm{m}} \\
&= \frac{1}{8} \times 1025 \times 9.81 \times (3^2 \times 0.5338 \times 10.78 - 2.82^2 \times 0.8476 \times 7.040) \\
&= 5451\ \mathrm{W\ m^{-1}}
\end{aligned}
$$

**Answer:** power change = 5.45 kW per metre crest.

**Section 2.4**

**Example.**
A harbour is to be protected by an L-shaped breakwater as sketched. Determine the length $X$ of the outer arm necessary for the wave height at point P to be 0.3 m when incident waves have a height of 3 m and a period of 5 s.

The depth is everywhere uniform at 5 m. The diffraction diagram for the appropriate approach angle is shown. Neglect reflections within the harbour

We require

$$\frac{H}{H_{in}} = \frac{0.3}{3} \quad = 0.1$$

From the diffraction diagram the 0.1 contour is almost vertical, with horizontal distance from the point $4L$. Hence, the total arm length is

$$X = 4L + 90$$

So, we need the wavelength $L$ for the specified wave.

Given

$$h = 5 \text{ m}$$
$$T = 5 \text{ s}$$

Then,

$$\omega = \frac{2\pi}{T} \quad = 1.257 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 0.8053 = kh \tanh kh$$

Iterate as either

$$kh = \frac{0.8053}{\tanh kh} \qquad \text{or (better here)} \qquad kh = \frac{1}{2}(kh + \frac{0.8053}{\tanh kh})$$

to get

$$kh = 1.037$$

$$k = \frac{1.037}{h} \quad = 0.2074 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 30.30 \text{ m}$$

Finally, for the arm length:

$$X = 90 + 4 \times 30.3 \quad = 211.2 \text{ m}$$

**Answer:** arm length 211 m.

**Section 2.5**

**Example.**
Lake Baikal in Siberia contains about one fifth of the world's fresh-water resources. It is 636 km long, with an average depth of 744 m. Find the fundamental period for seiching.

In the fundamental mode, the length of the lake is half a wavelength. Hence the wavelength is

$$L = 2 \times 636000 \quad = 1272000 \text{ m}$$

The depth is considerably less than 1/20th of a wavelength, so these are shallow-water waves with speed

$$c = \sqrt{gh} \quad = \sqrt{9.81 \times 744} \quad = 85.43 \text{ m s}^{-1}$$

Then,

$$\text{period} = \frac{\text{wavelength}}{\text{speed}} \quad = \frac{L}{c} \quad = 14890 \text{ s}$$

or

$$\frac{14890}{3600} = 4.136 \text{ hrs}$$

**Answer:** seiching period = 4.1 hours.

**Section 3.2**

**Example:**
Near a pier, 400 consecutive wave heights are measured. Assume that the sea state is narrow-banded.

(a) How many waves are expected to exceed $2H_{\text{rms}}$?
(b) If the significant wave height is 2.5 m, what is $H_{\text{rms}}$?
(c) Estimate the wave height exceeded by 80 waves.
(d) Estimate the number of waves with a height between 1.0 and 3.0 m.

(a) For a narrow-banded (i.e. small-range-of-different-frequencies) sea state the Rayleigh distribution is appropriate; i.e.

$$P(\text{height} > H) = \mathrm{e}^{-(H/H_{\text{rms}})^2}$$

Hence,

$$P(\text{height} > 2H_{\text{rms}}) = \mathrm{e}^{-4} \quad = 0.01832$$

For 400 waves, the expected number exceeding this is

$$400 \times 0.01832 \quad = 7.328$$

**Answer:** expectation = 7.3.

(b) From the ratio given in the notes, with $H_s = H_{1/3}$:

$$H_{1/3} = 1.416\, H_{\text{rms}}$$

Hence,

$$H_{\text{rms}} = \frac{H_{1/3}}{1.416} \quad = \frac{2.5}{1.416} \quad = 1.766 \text{ m}$$

**Answer:** $H_{\text{rms}} = 1.77$ m

(c) 80 from 400 corresponds to a probability of 0.2. Hence,

$$P(\text{height} > H) = \mathrm{e}^{-(H/H_{\text{rms}})^2} \quad = 0.2$$

i.e.

$$\frac{H}{H_{\text{rms}}} = \sqrt{-\ln 0.2} \quad = 1.269$$

$$H = 1.269 \times 1.766 \quad = 2.241 \text{ m}$$

**Answer:** wave height = 2.24 m.

(d)

$$
\begin{aligned}
P(1.0 < \text{height} < 3.0) \quad &= P(\text{height} > 1.0) - P(\text{height} > 3.0) \\
&= \mathrm{e}^{-(1/H_{\text{rms}})^2} - \mathrm{e}^{-(3/H_{\text{rms}})^2} \\
&= 0.6699
\end{aligned}
$$

Hence, from 400 waves the number expected within this range is

$$400 \times 0.6699 = 268.0$$

**Answer:** 268 waves.

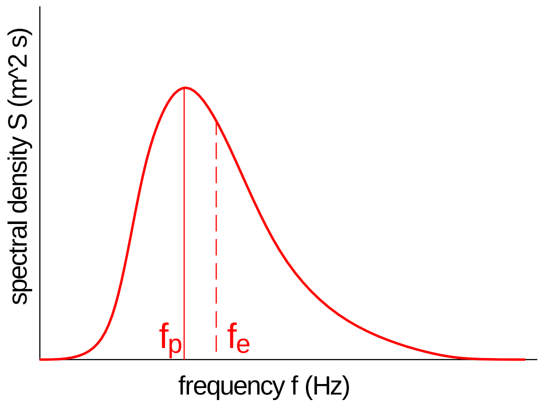
**Section 3.4**

**Example.**
An irregular wavefield at a deep-water location is characterised by peak period of 8.7 s and significant wave height of 1.5 m.

(a) Provide a sketch of a Bretschneider spectrum, labelling both axes with variables and units and indicating the frequencies corresponding to both the peak period and the energy period.
**Note:** Calculations are not needed for this part.

(b) Determine the power density (in kW m$^{-1}$) of a regular wave component with frequency 0.125 Hz that represents the frequency range 0.12 to 0.13 Hz of the irregular wave field.

(a) The energy period is slightly less than the peak period; consequently, the energy frequency $f_e$ is slightly higher than the peak frequency $f_p$.




(b) Given

$$T_p = 8.7 \text{ s}$$

then

$$f_p = \frac{1}{T_p} \quad = 0.1149 \text{ Hz}$$

For $f = 0.125$ Hz and $H_s = 1.5$ m the Bretschneider spectrum gives

$$S(f) = \frac{5}{16} H_s^2 \frac{f_p^4}{f^5} \exp\left(-\frac{5}{4}\frac{f_p^4}{f^4}\right) \quad = 1.645 \text{ m}^2\text{s}$$

The energy density is then

$$E = \rho g \times S(f)\Delta f \quad = 1025 \times 9.81 \times 1.645 \times 0.01 \quad = 165.4 \text{ J m}^{-2}$$

The period of this component is

$$T = \frac{1}{f} \quad = 8 \text{ s}$$

And hence, as a deep-water wave:

$$c = \frac{gT}{2\pi} \quad = 12.49 \text{ m s}^{-1}$$

$$n = \frac{1}{2}$$

Hence

$$c_g = nc \quad = 6.245 \text{ m s}^{-1}$$

The power density is then

$$P = Ec_g \quad = 165.4 \times 6.245 \quad = 1033 \text{ W m}^{-1}$$

**Answer:** power density = 1.03 kW m$^{-1}$.

**Example.**
Wave measurements are obtained from a stationary sensor located in deep water. The measured surface elevation of an irregular wave can be modelled as the sum of four regular wave components:

| Period (s) | 6 | 7 | 8 | 9 |
|---|---|---|---|---|
| Amplitude (m) | 0.8 | 1.2 | 0.8 | 0.4 |

(a) In the context of modelling an irregular wave, explain the meaning of the following terms:
(i) significant wave height;
(ii) significant wave period;
(iii) duration-limited.

(b) Obtain the total power conveyed by these deep-water wave components per metre width of wave crest if conditions were measured:
(i) with zero current;
(ii) with an opposing current of 1.0 m $s^{-1}$.

(a)
(i) "Significant wave height" is the average height of the highest 1/3 waves.

(ii) "Significant wave period" is the average period of the highest 1/3 waves.

(iii) "Duration-limited" describes conditions when the wind has not blown long enough for wave energy to propagate over the whole fetch; i.e. not long enough for a statistically-steady state to be obtained.

(b) In each case we have

$$P = Ec_g \quad = \frac{1}{2}\rho g A^2 \times \frac{1}{2} c_r$$

where $A$ is amplitude and the (relative) group velocity $c_g$ is (in deep water) half the phase velocity $c_r$. Thus, we need to find, for each component, the relative phase velocity.

(i) In no current and in deep water:

$$c = \frac{gT}{2\pi}$$

and hence, in kg-m-s units:

$$P = \frac{1}{2}\rho g A^2 \times \frac{1}{2} \times \frac{gT}{2\pi} \quad = 3925 A^2 T$$

From the given data this gives

$$P = [15072, 39564, 20096, 5652] \text{ W m}^{-1}$$

(ii) In current it is necessary to solve the (deep-water) dispersion equation

$$(\omega_a - kU_0)^2 = \omega_r^2 = gk$$

for each wavenumber $k$ and thence the relative phase velocity. Rearranging for iteration:

$$k = \frac{(\omega_a - kU_0)^2}{g}$$

or, here,

$$k = \frac{\left(\frac{2\pi}{T_a} + k\right)^2}{9.81}$$

and thence the phase speeds

$$c_r = \frac{\omega_r}{k} \quad = \frac{\frac{2\pi}{T_a} + k}{k}$$

to get the following table:

| $T_a$ (s) | 6 | 7 | 8 | 9 |
|---|---|---|---|---|
| $A$ (m) | 0.8 | 1.2 | 0.8 | 0.4 |
| $k$ ($\text{m}^{-1}$) | 0.1448 | 0.1018 | 0.07556 | 0.05833 |
| $c_r$ ($\text{m s}^{-1}$) | 8.232 | 9.817 | 11.39 | 12.97 |

Power density then follows as

$$P = \frac{1}{2}\rho g A^2 \times \frac{1}{2}c_r \quad = 2514A^2 c_r$$

giving the following values for the different components:

$$P = [13244, 35540, 18330, 5217] \text{ W m}^{-1}$$

**Section 3.5**

| |
|---|
| **Example.**<br>(a) Wind has blown at a consistent $U = 20 \text{ m s}^{-1}$ over a fetch $F = 100$ km for $t = 6$ hrs. Determine $H_s$ and $T_p$ using the JONSWAP curves.<br>(b) If the wind blows steadily for another 4 hours what are $H_s$ and $T_p$? |

(a)
Given

$$U = 20 \text{ ms}^{-1}$$

$$F = 100000 \text{ m}$$

$$t = 6 \times 3600 \quad = 21600 \text{ s}$$

The dimensionless fetch and duration are

$$\hat{F} \equiv \frac{gF}{U^2} \quad = 2453$$

$$\hat{t} = \frac{gt}{U} \quad = 10590$$

but the minimum (dimensionless) time required to establish fetch-limited waves is

$$\hat{t}_{\text{min}} = 68.8\hat{F}^{2/3} \quad = 12510$$

Thus, the given storm has not blown for sufficiently long for wave energy to propagate across the whole fetch (10590 < 12510 in dimensionless terms). Hence the waves are *duration-limited* (i.e. determined primarily by the time, not distance, that the wind has blown). Thus we need to use smaller *effective fetch* to determine the sea state:

$$\hat{F}_{\text{eff}} = \left(\frac{\hat{t}}{68.8}\right)^{3/2} \quad = 1910$$

Then,

$$\frac{gH_s}{U^2} \equiv \hat{H}_s \quad = 0.0016\hat{F}_{\text{eff}}^{1/2} \quad = 0.06993$$

$$\frac{gT_p}{U} \equiv \hat{T}_p \quad = 0.286\hat{F}_{\text{eff}}^{1/3} \quad = 3.548$$

These give dimensionless height and period

$$H_s = 0.06993 \times \frac{U^2}{g} \quad = 2.851 \text{ m}$$

$$T_p = 3.548 \times \frac{U}{g} \quad = 7.233 \text{ s}$$

**Answer:** wave height = 2.85 m; peak wave period = 7.23 s.

(b) If the duration is increased by 4 hours to

$$t = 10 \times 3600 \quad = 36000 \text{ s}$$

then

$$\hat{t} = \frac{gt}{U} \quad = 17660$$

This exceeds $\hat{t}_{min}$ (i.e. $t > t_{\text{min}}$) and hence we may use the actual non-dimensional fetch

$$\hat{F} = 2453$$

in the formulae for significant wave height and peak period. Hence,

$$\hat{H}_s = 0.0016\hat{F}^{1/2} \quad = 0.07924$$

$$\hat{T}_p = 0.286\hat{F}^{1/3} \quad = 3.857$$

These give dimensionless height and period

$$H_s = 0.07924 \times \frac{U^2}{g} \quad = 3.231 \text{ m}$$

$$T_p = 3.857 \times \frac{U}{g} \quad = 7.863 \text{ s}$$

**Answer:** wave height = 3.23 m;   peak wave period = 7.86 s.

**Section 4.4**

**Example.**
A regular wave of period 8 s and height 1.2 m is normally incident to a caisson type breakwater of 4 m breadth and located in water depth of 5 m. Using linear theory, determine the following:

(a) Caisson height required for 1 m freeboard above the peak wave elevation at the breakwater.

(b) Maximum wave-induced horizontal force per metre of wave crest.

(c) Maximum wave-induced overturning moment per metre of wave crest.

(a) Peak wave elevation of a reflected wave above the SWL is $2A$, i.e. $H$. Hence caisson height required is

$$\text{water depth + wave elevation + freeboard} = 5 + 1.2 + 1 \quad = 7.2 \text{ m}$$

**Answer:** caisson height = 7.2 m.

(b), (c) Wave properties:

$$h = 5 \text{ m}$$

$$T = 8 \text{ s}$$

$$\omega = \frac{2\pi}{T} \quad = 0.7854 \text{ rad s}^{-1}$$

The dispersion relation is

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \quad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \quad 0.3144 = kh \tanh kh$$

Since the LHS is small this may be iterated as

$$kh = \frac{1}{2}(kh + \frac{0.3144}{\tanh kh})$$

to give

$$kh = 0.5918$$

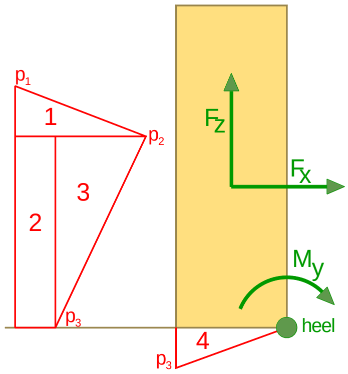
Approximate the wave-induced pressure distribution as shown in the figure.

Pressures:

$$p_1 = 0$$

$$p_2 = \rho g H \quad = 1025 \times 9.81 \times 1.2 \quad = 12070 \text{ Pa}$$

$$P_3 = \frac{\rho g H}{\cosh kh} \quad = \frac{p_2}{\cosh 0.5918} \quad = 10220 \text{ Pa}$$




Decompose the pressure forces on the breakwater as shown. Relevant dimensions are:

$$h = 5 \text{ m}$$

$$H = 1.2 \text{ m}$$

$$b = 4 \text{ m}$$

| Region | Force, $F_x$ or $F_z$ | Moment arm |
|---|---|---|
| 1 | $F_{x1} = \frac{1}{2} p_2 \times H \quad = 7242 \text{ N/m}$ | $z_1 = h + \frac{1}{3} H \quad = 5.4 \text{ m}$ |
| 2 | $F_{x2} = p_3 \times h \quad = 51100 \text{ N/m}$ | $z_2 = \frac{1}{2} h \quad = 2.5 \text{ m}$ |
| 3 | $F_{x3} = \frac{1}{2}(p_2 - p_3) \times h \quad = 4625 \text{ N/m}$ | $z_3 = \frac{2}{3} h \quad = 3.333 \text{ m}$ |
| 4 | $F_{z4} = \frac{1}{2} p_3 \times b \quad = 20440 \text{ N/m}$ | $x_4 = \frac{2}{3} b \quad = 2.667 \text{ m}$ |

Horizontal force:

$$F_{x1} + F_{x2} + F_{x3} = 62970 \text{ N /m}$$

The sum of all clockwise moments about the heel:

$$F_{x1} z_1 + F_{x2} z_2 + F_{x3} z_3 + F_{z4} x_4 = 236800 \text{ N m/m}$$

**Answer:** force = 63.0 kN /m;   overturning moment = 237 kN m /m.