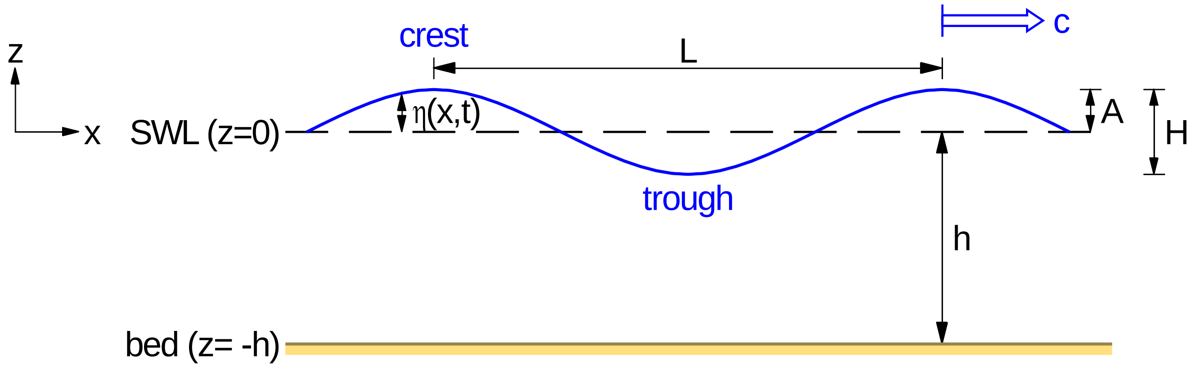
1.1 Main Wave Parameters



Consider a single-frequency ("monochromatic") *progressive* wave on still water:

$$\eta = A\cos(kx - \omega t)$$

We will develop a *linear wave theory* (or *Airy*[1] *wave theory*), based on the assumption that the wave amplitude A is small (compared with the depth h and wavelength L), and, hence, we may neglect second-order and higher products and powers of wave-related perturbations.

Despite the approximations embodied in the assumption of small amplitudes, linear-wave theory has proved remarkably successful at describing the behaviour of real waves, including wave transformations such as refraction, diffraction, shoaling, reflection and predicting the onset of breaking. An additional advantage of linearity is that real (multi-frequency, multi-direction) wave fields can be obtained by superposition of simple wave components.

Amplitude and Height

The *amplitude* A is the maximum displacement from *still water level* (SWL). The *wave height* H is the vertical distance between neighbouring crest and trough. For sinusoidal waves,

$$A = \frac{H}{2}, \qquad \text{or} \qquad H = 2A$$

For non-sinusoidal waves H is the more easily defined and measured quantity.

Wavenumber and Wavelength

k is the *wavenumber*. Since the wave goes through a single cycle when kx changes by 2π, the *wavelength* L is given by

$$L = \frac{2\pi}{k}$$

[1] After George Biddell Airy, former Astronomer Royal, who first developed the theory.

Frequency and Period

ω is the *wave angular frequency*. Since the wave goes through a single cycle in time when ωT changes by 2π, the *period* T is given by

$$T = \frac{2\pi}{\omega}$$

with actual *frequency* f (in cycles per second, or Hertz) given by

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Wave Speed (Phase Velocity, or "Celerity")

The wave profile above can also be written

$$\eta = A\cos[k(x - \mathrm{c}t)]$$

where

$$c = \frac{\omega}{k}$$

η is the same for $x - ct =$ constant, or $x = ct +$ constant, so c represents the velocity of travel of a pure sinusoidal wave (the speed at which its crest moves forward), also called the *phase velocity*, or *celerity*. It can be computed by any of

$$\begin{aligned} c &= \frac{\omega}{k} \\ &= \frac{L}{T} && \left(\frac{\text{wavelength}}{\text{period}}\right) \\ &= fL && (\text{frequency} \times \text{wavelength}) \end{aligned}$$

ω and k are not independent, but are related via a *dispersion relation*, saying how wavelength changes with frequency. Later we shall see that for a collection of waves or *wave packet* the natural velocity of energy transport is not the phase velocity c but the *group velocity*

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}$$

Note that c represents the speed of the wave *form*, not the individual fluid particles, whose velocity, as we shall see, is considerably smaller.

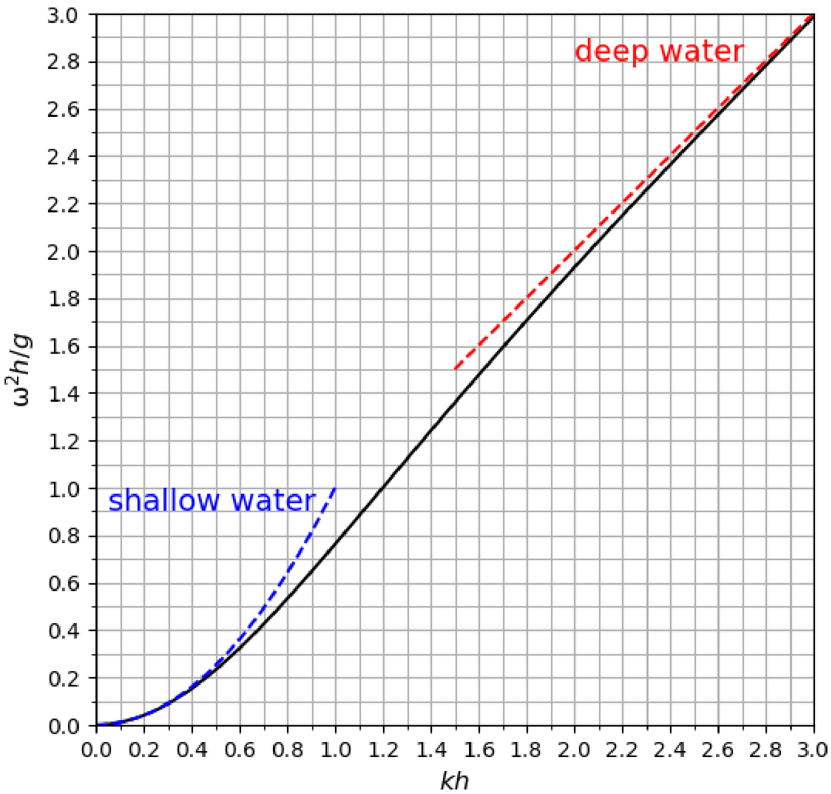
1.2 Dispersion Relationship

At a given depth, waves of different frequencies (hence of different wavelengths) travel at different speeds. This phenomenon is called *dispersion*.

For a given frequency, the wavelength, and hence wave speed, must change with depth.

The mathematical theory is given in APPENDICES A1–A3, which should be read here. This predicts the *dispersion relationship* (in still water) between frequency ω and wavenumber k (or, indirectly, between period T and wavelength L):

$$\omega^2 = gk \tanh kh \tag{1}$$



For the "shallow-water" and "deep-water" approximations, see later.

The phase speed is

$$c \equiv \frac{\omega}{k} \quad = \sqrt{\frac{g}{k} \tanh kh}$$

Thus, in general, *wave speed changes with wavelength* (or frequency).

For a given depth h:
phase speed c and period T increase as wavelength L increases; (*exercise*: prove it).
For a given period T:
phase speed c and wavelength L decrease as depth decreases; (*exercise*: prove it).

The latter is why waves undergo transformation (refraction, shoaling, breaking) as they move into shallow water.

Solving the Dispersion Relation

(i) To find period or frequency given the wavelength, simply find the wavenumber k $(= 2\pi/L)$ and substitute in equation (1) for ω. The period is given by $T = 2\pi/\omega$.

(ii) To find wavelength given the period or frequency, it is usually convenient to first write equation (1) in non-dimensional form:

$$\frac{\omega^2 h}{g} = kh \tanh kh$$

This is of the form

$$Y = X \tanh X$$

and may be solved graphically or numerically.

Methods of Solving $Y = X \tanh X$

By hand:

- **Repeated trial**
  Always works, but can take a while.

- **Iteration**
  For large Y (say $Y > 1$) the naive iterative formula
  $$X = \frac{Y}{\tanh X}$$
  works fine. However, for small Y, it takes a vast number of iterations. In the latter case a more robust iterative formula simply averages this with the current value:
  $$X = \frac{1}{2}\left(X + \frac{Y}{\tanh X}\right)$$
  I suggest that you use the first if $Y > 1$ and the second if $Y \leq 1$.

By computer:

- The **Newton-Raphson method** works extremely well. (*Exercise*: use any programming language of your choice to find the wavelength for arbitrary period and depth.)

For a first guess one could take anything positive for X, e.g. 1, but iterative techniques can often be improved by either reading an approximate solution from the graph or observing *shallow-water* ($X \approx \sqrt{Y}$) and *deep-water* ($X \approx Y$) limits – see later.

Example.
Find, in still water of depth 15 m:
(a) the period of a wave with wavelength 45 m;
(b) the wavelength of a wave with period 8 s.
In each case write down the phase speed (celerity).

1.3 Wave Velocity and Pressure

The velocity and pressure fields for a harmonic surface displacement:

$$\eta = A\cos(kx-\omega t)$$

are also derived in APPENDICES A1–A3. They are conveniently summarised by a *velocity potential* ϕ:

$$\phi = \frac{Ag}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx-\omega t)$$

The horizontal and vertical velocity components are:

$$u \equiv \frac{\partial\phi}{\partial x} \qquad = \frac{Agk}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx-\omega t)$$

$$w \equiv \frac{\partial\phi}{\partial z} \qquad = \frac{Agk}{\omega}\frac{\sinh k(h+z)}{\cosh kh}\sin(kx-\omega t)$$

The horizontal velocity component u is in phase with the surface disturbance: it is largest underneath a wave crest. The vertical component w is 90º out of phase with the surface disturbance: it is zero underneath a wave crest. Since $\sinh X < \cosh X$, the vertical component is always smaller than the horizontal one, the more so near the bed.

The pressure is

$$p = -\rho g z - \rho\frac{\partial\phi}{\partial t} \qquad = \underbrace{-\rho g z}_{\text{hydrostatic}} + \underbrace{\rho g A\frac{\cosh k(h+z)}{\cosh kh}\cos(kx-\omega t)}_{\text{hydrodynamic (i.e. wave)}}$$

The pressure field consists of two parts:

- a *hydrostatic* pressure $-\rho g z$, which is always present (below the SWL), regardless of whether waves exist;
- a wave-induced *hydrodynamic* pressure $-\rho\partial\phi/\partial t$, which exists only if waves are present. At a fixed height this varies (as might be anticipated from the amount of water above) sinusoidally from maximum positive beneath a wave crest to maximum negative beneath a wave trough.

Example.
A pressure sensor is located 0.6 m above the sea bed in a water depth $h = 12$ m. The pressure fluctuates with period 15 s. A maximum gauge pressure of 124 kPa is recorded.

(a) What is the wave height?

(b) What are the maximum horizontal and vertical velocities at the surface?

1.4 Wave Energy

Wave energy is of two forms: kinetic and potential. The amount of each (per unit horizontal area) can be found by integrating over the water column and averaging over a period and wavelength. This is done in APPENDIX A4. The result is (with the small-amplitude wave assumption):

$$\overline{\mathrm{KE}} = \overline{\int_{z=-h}^{\eta} \frac{1}{2}\rho(u^2 + w^2)\,\mathrm{d}z} \qquad = \frac{1}{4}\rho g A^2$$

$$\overline{\mathrm{PE}} = \overline{\int_{z=-h}^{\eta} \rho g z\,\mathrm{d}z} \qquad = \frac{1}{4}\rho g A^2 + \text{constant}$$

It is seen that (under linear wave theory) the average wave-related kinetic and potential energies are equal. The total energy (per unit horizontal area) is the sum of these:

$$E = \frac{1}{2}\rho g A^2 \qquad \text{or, in terms of wave height,} \qquad E = \frac{1}{8}\rho g H^2$$

IMPORTANT: *wave energy varies as the* ***square*** *of the wave height.*

1.5 Group Velocity

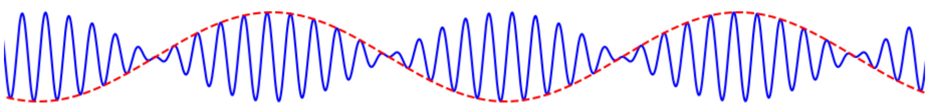
Real wave fields do not consist of individual harmonic components, but *wave packets* comprised of a combination of waves of different frequency.

Consider first two superposed waves of similar amplitude a but different frequencies $\omega \pm \Delta\omega$ and corresponding wavenumbers $k \pm \Delta k$:

$$\eta = \underbrace{a\cos[(k + \Delta k)x - (\omega + \Delta\omega)t]}_{\text{component 1}} \quad + \quad \underbrace{a\cos[(k - \Delta k)x - (\omega - \Delta\omega)t]}_{\text{component 2}}$$

Using the trigonometric formula $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, we find

$$\eta = 2a\cos(kx - \omega t)\cos(\Delta k.x - \Delta\omega.t)$$



This represents (see the figure) an underlying fast-varying wave motion $\cos(kx - \omega t)$ with the phase velocity $c = \omega/k$, but modulated by an envelope with *time-varying amplitude*:

$$A(t) = 2a\cos(\Delta k\,x - \Delta\omega\,t)$$

This amplitude function represents a wave with smaller wavenumber and frequency (i.e. longer wavelength and period), travelling at speed

$$\frac{\Delta\omega}{\Delta k}$$

In general, real *wave packets* are formed from not just two but a continuous spread of wavenumbers. It may be shown mathematically[2] that, for a wave packet strongly peaked around one wavenumber k, the packet as a whole travels with the *group velocity*

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}$$

For regular waves, using the dispersion relation $\omega^2 = gk \tanh kh$ (see APPENDIX A5):

$$c_g = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]\frac{\omega}{k}$$

or

$$c_g = nc$$

where

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right], \qquad c = \frac{\omega}{k}$$

n is the ratio of group velocity to phase velocity. It is always between ½ and 1; the group velocity is always less than the phase velocity.

1.6 Energy Transfer (Wave Power)

Since wave energy depends on the square of the amplitude, and the amplitude envelope travels at the group velocity, the latter also represents the velocity at which energy is transferred.

- *Phase velocity*, c, represents the speed at which the *wave form propagates*.
- *Group velocity*, c_g, represents the speed at which the *wave transmits energy*.

The rate at which wave energy is transferred is called the *energy flux* or *wave power* P and is usually expressed as the (average) wave energy passing a location per unit length of wave crest per unit time. Heuristically it may be deduced by noting that, if wave energy does travel at velocity c_g, then the energy in area $c_g \Delta t$ crosses such a line in time Δt, so $P\Delta t = E \times c_g \Delta t$, or

$$P = Ec_g$$

where $E = \frac{1}{2}\rho g A^2$ is the wave energy per unit area of surface and c_g is the group velocity.

A more formal derivation (see APPENDIX A6) is to integrate the rate of working by pressure forces (i.e. pressure × area × velocity), over the depth of the water column and then average in time.

Example.
A sea-bed pressure transducer in 9 m of water records a sinusoidal signal with amplitude 5.9 kPa and period 7.5 s. Find the wave height, energy density and wave power per metre of crest.

[2] Google "Method of stationary phase".

1.7 Particle Motion

The velocity field is

$$u = \frac{Agk}{\omega} \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t)$$
$$w = \frac{Agk}{\omega} \frac{\sinh k(h+z)}{\cosh kh} \sin(kx - \omega t)$$

Rearrange the dispersion relationship $\omega^2 = gk \tanh kh$ to get

$$\frac{gk}{\omega \cosh kh} = \frac{\omega}{\sinh kh}$$

the equations governing (small-amplitude) particle paths (X, Z) about (X_0, Z_0) in the wave field can be written

$$\frac{\mathrm{d}X}{\mathrm{d}t} = u \quad = a\omega \cos(kX_0 - \omega t)$$
$$\frac{\mathrm{d}Z}{\mathrm{d}t} = w \quad = b\omega \sin(kX_0 - \omega t)$$

where

$$a = A \frac{\cosh k(h + Z_0)}{\sinh kh}, \qquad b = A \frac{\sinh k(h + Z_0)}{\sinh kh}$$

The X and Z equations can then be integrated with respect to time to give

$$X = X_0 - a \sin(kX_0 - \omega t)$$
$$Z = Z_0 + b \cos(kX_0 - \omega t)$$

Since $\sin^2 \theta + \cos^2 \theta = 1$,

$$\frac{(X - X_0)^2}{a^2} + \frac{(Z - Z_0)^2}{b^2} = 1$$

The particle trajectory is an ellipse, centre (X_0, Z_0) and semi-axes a and b.

1.8 Shallow-Water and Deep-Water Behaviour

1.8.1 Limiting Behaviour

Consider the dispersion relation:

$$\omega^2 = gk \tanh kh$$

and the asymptotic behaviour of the tanh function (Appendix A1):

$$\tanh kh \sim kh \qquad (\text{as } kh \to 0)$$

$$\tanh kh \to 1 \qquad (\text{as } kh \to \infty)$$

<u>Shallow Water (or Long Waves)</u>

$$kh \ll 1$$

$$\omega^2 \approx k^2 gh$$

$$c = c_g = \sqrt{gh}$$

Thus, in shallow water, phase velocity and group velocity are equal and independent of wavelength ... *waves on shallow water are non-dispersive*.

<u>Deep Water (or Short Waves)</u>

$$kh \gg 1$$

$$\omega^2 \approx gk$$

$$L = \frac{gT^2}{2\pi}$$

$$c = \frac{L}{T} = \frac{gT}{2\pi}, \qquad n = \frac{1}{2}, \qquad c_g = \frac{1}{2}c$$

For deep-water waves, phase velocity is dependent on wavelength and the group velocity is half the phase velocity ... *deep-water waves are dispersive*.

1.8.2 Nominal Limits

What constitutes "shallow" or "deep" water (or, equivalently, "long" or "short" waves) is rather subjective. The following is common (see the dispersion-relation graph):

Shallow water (or long waves): $kh < \frac{\pi}{10}$ or $h < \frac{1}{20}L$

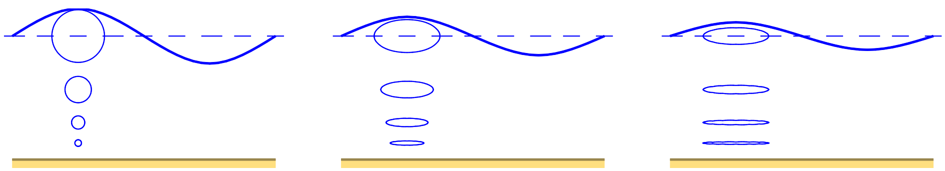
Deep water (or short waves): $kh > \pi$ or $h > \frac{1}{2}L$

Between these limits we refer to *intermediate depths*.

1.8.3 Particle Motions

As shown before, the trajectories of particles are ellipses with horizontal and vertical semiaxes

$$a = A\frac{\cosh k(h+Z_0)}{\sinh kh}, \qquad b = A\frac{\sinh k(h+Z_0)}{\sinh kh}$$



For *deep-water* waves ($kh \gg 1$),

$$a = b \approx A\mathrm{e}^{-k|Z_0|}$$

and the trajectories of particles are circles whose radius diminishes exponentially with depth over the order of half a wavelength.

For *shallow-water* waves ($kh \ll 1$),

$$a \approx \frac{A}{kh}, \qquad \frac{b}{a} \ll 1$$

the trajectories of particles are highly flattened ellipses; the horizontal excursion of water particles is similar at all depths and much greater than the vertical excursion.

1.8.4 Pressure

The pressure is given, in general, by

$$p = -\rho g z - \rho\frac{\partial\phi}{\partial t} \qquad = \underbrace{-\rho g z}_{\text{hydrostatic}} + \underbrace{\rho g \eta \frac{\cosh k(h+z)}{\cosh kh}}_{\text{hydrodynamic}}$$

For *deep-water* waves ($kh \gg 1$),

$$p \approx -\rho g z + \rho g \eta \mathrm{e}^{-k|z|}$$

and the hydrodynamic pressure decays exponentially with depth below the surface over the order of half a wavelength.

For *shallow-water* waves ($kh \ll 1$),

$$p \approx \rho g(\eta - z)$$

i.e. *shallow-water waves are hydrostatic*. This is because vertical accelerations are much smaller than g. This is the limit we have seen earlier in the Open-Channel Flow section.

1.8.5 Summary of Shallow-Water/Deep-Water Approximations

In **shallow water**:

- The phase velocity and group velocity are equal and independent of wavelength ... *waves on shallow water are non-dispersive*.

$$c = \sqrt{gh}$$

- Pressure is hydrostatic.
- Velocity is predominantly horizontal and almost independent of depth; particle paths are highly-flattened ellipses.

In **deep water**:

- The phase velocity is dependent on wavelength and the group velocity is half the phase velocity ... *deep-water waves are dispersive*.
- Phase velocity is independent of depth, as velocity and pressure perturbations do not reach the bed.
- Wavelength, speed and period are connected by

$$L = \frac{gT^2}{2\pi}\ , \qquad c = \frac{gT}{2\pi}$$

- Particle motions are circular and diminish in size over distance of order half a wavelength.

These mean that for any body of water deeper than about half a wavelength the bed will play no influence on wave motions.

Example.

(a) Find the deep-water speed and wavelength of a wave of period 12 s.

(b) Find the speed and wavelength of a wave of period 12 s in water of depth 3 m. Compare with the shallow-water approximation.

1.9 Waves on Currents

Until now we have considered waves on *still water*. In many cases waves at sea coexist with a background steady current U. The formulae then continue to hold in a frame of reference moving with the current; i.e. x is replaced by a value (subscript *r*):

$$x_r = x - Ut$$

In particular,

$$\begin{aligned}\eta &= A\cos(kx_r - \omega_r t) - \\ &= A\cos[kx - (\omega_r + kU)t] - \\ &= A\cos[k\{x - (\frac{\omega_r}{k} + U)t\}]\end{aligned}$$

Hence, in an *absolute frame* (subscript *a*) which is fixed; i.e. not moving with the current:

- wavenumber k and wavelength L are unchanged; (if you took a photograph from above, the fact that the waves are advected with the current doesn't change their length);
- the current is simply added to the wave speed (which you could probably have guessed):

  $$c_a = c_r + U$$

- the perceived frequency ω and period T are changed:

  $$\omega_a = \omega_r + kU$$

  e.g. if waves and current are in the same direction, then the absolute frequency ω_a is larger, and the absolute period T_a is smaller than their "relative" counterparts (those moving with the current). This is because the current shortens the time between successive wave crests passing a point.

Note, however, that it is the *relative* frequency

$$\omega_r = \omega_a - kU$$

that appears in the dispersion formula, and hence:

$$(\omega_a - kU)^2 = \omega_r^2 = gk\tanh kh$$

The change of apparent frequency in a fixed reference frame due to the motion of the source or transmitting medium is called a *Doppler shift*.

Note: be careful about absolute velocities (phase or group):

$$c_a = \frac{L}{T_a} \quad = \frac{\omega_r}{k} + U$$

To use the second form you would need to find $\omega_r = \omega_a - kU$ once you have solved for k.

Example.
An acoustic depth sounder indicates regular surface waves with apparent period 8 s in water of depth 12 m. Find the wavelength and absolute phase speed of the waves when there is:

(a) no mean current;
(b) a current of 3 m s^{-1} in the same direction as the waves;
(c) a current of 3 m s^{-1} in the opposite direction to the waves.