Q1.
Given:

$h = 20$ m
$A = 4$ m $\quad (H = 8$ m$)$
$T = 13$ s

Then,

$$\omega = \frac{2\pi}{T} \quad = 0.4833 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 0.4762 = kh \tanh kh$$

The LHS is small, so iterate as

$$kh = \frac{1}{2}(kh + \frac{0.4762}{\tanh kh})$$

to get

$$kh = 0.7499$$

$$k = \frac{0.7499}{h} \quad = 0.03750 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 167.6 \text{ m}$$

$$c = \frac{\omega}{k} \ \left(\text{or } \frac{L}{T}\right) \quad = 12.89 \text{ m s}^{-1}$$

Answer: wavelength 168 m, phase speed 12.9 m s^{-1}.

(b) By formula:

$$u = \frac{gkA}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx - \omega t)$$

$$a_x = \frac{\partial u}{\partial t} (+ \text{ neglected non-linear terms}) \quad = gkA\frac{\cosh k(h+z)}{\cosh kh}\sin(kx - \omega t)$$

Hence

$$a_{x,\max} = 1.137 \cosh k(h+z)$$

Then:

surface ($z = 0$): $\quad a_{x,\max} = 1.137 \cosh kh \quad = 1.472 \text{ m s}^{-2}$
mid-depth ($z = -h/2$): $\quad a_{x,\max} = 1.137 \cosh(kh/2) \quad = 1.218 \text{ m s}^{-2}$

bottom: ($z = -h$): $a_{x,\max} = 1.137 \text{ m s}^{-2}$

Answer: (1.47, 1.22, 1.14) m s^{-2}.

Q2.

Waves A and B (no current)

In the absence of current the dispersion relation is

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

This may be iterated as either

$$kh = \frac{\omega^2 h/g}{\tanh kh} \qquad \text{or} \qquad kh = \frac{1}{2}(kh + \frac{\omega^2 h/g}{\tanh kh})$$

	Wave A	**Wave B**
h	20 m	3 m
T	10 s	12 s
$\omega = \frac{2\pi}{T}$	0.6283 rad s^{-1}	0.5236 rad s^{-1}
$\frac{\omega^2 h}{g}$	0.8048	0.08384
Iteration:	$kh = \frac{1}{2}(kh + \frac{0.8048}{\tanh kh})$	$kh = \frac{1}{2}(kh + \frac{0.08384}{\tanh kh})$
kh	1.036	0.2937
k	0.0518 m^{-1}	0.0979 m^{-1}
$c = \frac{\omega}{k}$	12.13 m s^{-1}	5.348
Type:	Intermediate ($\pi/10 < kh < \pi$)	Shallow ($kh < \pi/10$)

Wave C (with current)

$$T_a = 6 \text{ s}$$

$$h = 24 \text{ m}$$

$$U = -0.8 \text{ m s}^{-1}$$

Then

$$\omega_a = \frac{2\pi}{T_a} = 1.047 \text{ rad s}^{-1}$$

With current, the dispersion relation is

$$(\omega_a - kU_0)^2 = \omega_r^2 = gk \tanh kh$$

Rearrange for iteration as

$$k = \frac{(\omega_a - kU_0)^2}{g \tanh kh}$$

i.e.

$$k = \frac{(1.047 + 0.8k)^2}{9.81 \tanh(24k)}$$

Solve:

$$k = 0.1367 \text{ m}^{-1}$$

$$kh = 3.2808$$

The speed relative to the fixed-position sensor is the *absolute* speed:

$$c_a = \frac{\omega_a}{k} \quad = \frac{1.047}{0.1367} \quad = 7.659 \text{ m s}^{-1}$$

Since $kh > \pi$ this is a *deep-water* wave.

Answer: A: intermediate, 12.1 m s^{-1};   B: shallow, 5.348 m s^{-1};   C: deep, 7.66 m s^{-1}.

Q3.
Given:

$$h = 15 \text{ m}$$
$$f = 0.1 \text{ Hz}$$
$$p_{\text{range}} = 9500 \text{ N m}^{-2}$$

Then,

$$\omega = \frac{2\pi}{T} \quad = 2\pi f \quad = 0.6283 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 0.6036 = kh \tanh kh$$

The LHS is small, so iterate as

$$kh = \frac{1}{2}(kh + \frac{0.6036}{\tanh kh})$$

to get

$$kh = 0.8642$$

$$k = \frac{0.8642}{h} \quad = 0.05761 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 109.1 \text{ m}$$

By formula, the wave part of the pressure is

$$p = \rho g A \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t)$$

and hence the range (max – min) at the seabed sensor ($z = -h$) is

$$p_{\text{range}} = \frac{2\rho g A}{\cosh kh} \quad = \frac{\rho g H}{\cosh kh}$$

Hence,

$$H = p_{\text{range}} \times \frac{\cosh kh}{\rho g} \quad = 9500 \times \frac{\cosh 0.8642}{1025 \times 9.81} \quad = 1.32 \text{ m}$$

Answer: 1.32 m.

Q4.

On the seabed ($z = -h$) the gauge pressure distribution is

$$p = \rho g h + \frac{\rho g A}{\cosh kh}\cos(kx - \omega_a t)$$

The average and the fluctuation of this will give the water depth and wave amplitude, respectively:

$$\bar{p} = \rho g h\,, \qquad \Delta p = \frac{\rho g A}{\cosh kh}$$

i.e.

$$h = \frac{\bar{p}}{\rho g}\,, \qquad A = \frac{\Delta p}{\rho g}\cosh kh$$

Here,

$$\bar{p} = \frac{98.8 + 122.6}{2} \times 1000 \quad = 110400 \text{ Pa}$$

$$\Delta p = \frac{122.6 - 98.8}{2} \times 1000 \quad = 11900 \text{ Pa}$$

Hence,

$$h = \frac{110400}{1025 \times 9.81} \quad = 10.98 \text{ m}$$

For the amplitude, we require also kh, which must be determined, via the dispersion relation, from the period. From, e.g., the difference between the peaks of 5 cycles,

$$T = \frac{49.2 - 6.8}{5} \quad = 8.48 \text{ s}$$

$$\omega_a = \frac{2\pi}{T} \quad = 0.7409 \text{ rad s}^{-1}$$

<u>(a) No current.</u>

The dispersion relation rearranges as

$$\frac{\omega_a^2 h}{g} = kh \tanh kh$$

$$0.6144 = kh \tanh kh$$

Rearrange for iteration. Since the LHS < 1 the most effective form is

$$kh = \frac{1}{2}(kh + \frac{0.6144}{\tanh kh})$$

Iteration from, e.g., $kh = 1$, gives

$$kh = 0.8737$$

Then the amplitude is

$$A = \frac{\Delta p}{\rho g} \cosh kh \quad = \frac{11900}{1025 \times 9.81} \times \cosh 0.8737 \quad = 1.665 \text{ m}$$

corresponding to a wave height

$$H = 2A \quad = 3.33 \text{ m}$$

Finally, for the wavelength,

$$k = \frac{kh}{h} \quad = \frac{0.8737}{10.98} \quad = 0.07957 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 78.96 \text{ m}$$

Answer: water depth = 11.0 m;   wave height = 3.33 m;   wavelength = 79.0 m.

<u>(b) With current $U = 2 \text{ m s}^{-1}$</u>

The dispersion relation is

$$(\omega_a - kU)^2 = gk \tanh kh$$

Rearrange as

$$k = \frac{(\omega_a - kU)^2}{g \tanh kh}$$

Here,

$$k = \frac{(0.7409 - 2k)^2}{9.81 \tanh(10.98k)}$$

This does not converge very easily. An alternative using under-relaxation is

$$k = \frac{1}{2}\left[k + \frac{(0.7409 - 2k)^2}{9.81 \tanh(10.98k)}\right]$$

Iteration from, e.g., $k = 0.1 \text{ m}^{-1}$, gives

$$k = 0.06362 \text{ m}^{-1}$$

and, with $h = 10.98$ m),

$$kh = 0.6985$$

Then the amplitude is

$$A = \frac{\Delta p}{\rho g} \cosh kh \quad = \frac{11900}{1025 \times 9.81} \times \cosh 0.6985 \quad = 1.484 \text{ m}$$

corresponding to a wave height

$$H = 2A \quad = 2.968 \text{ m}$$

Finally, for the wavelength,

$$L = \frac{2\pi}{k} \quad = 98.76 \text{ m}$$

Answer: water depth = 11.0 m;   wave height = 2.97 m;   wavelength = 98.8 m.

Q5.
There are three depths to be considered:

deep / transducer (22 m) / shallow (8 m)

The shoreward rate of energy transfer is constant; i.e.

$$(H^2nc)_{\text{deep}} = (H^2nc)_{\text{transducer}} = (H^2nc)_{\text{shallow}}$$

We can find n and c from the dispersion relation at all locations, and the height H from the transducer pressure measurements.

Given:

$$T = 12 \text{ s}$$

Then

$$\omega = \frac{2\pi}{T} \quad = 0.5236 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

Solve for kh, and thence k by iterating either

$$kh = \frac{\omega^2 h/g}{\tanh kh} \qquad \text{or (better here)} \qquad kh = \frac{1}{2}(kh + \frac{\omega^2 h/g}{\tanh kh})$$

together with

$$c = \frac{\omega}{k} \, , \qquad n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]$$

In shallow water ($h = 8$ m), $\omega^2 h/g = 0.2236$, and hence:

$$kh = 0.4912 \, , \quad k = 0.06140 \text{ m}^{-1}, \quad c = 8.528 \text{ m s}^{-1} \, , \quad n = 0.9278$$

At the transducer ($h = 22$ m), $\omega^2 h/g = 0.6148$, and hence:

$$kh = 0.8740 \, , \quad k = 0.03973 \text{ m}^{-1}, \quad c = 13.18 \text{ m s}^{-1} \, , \quad n = 0.8139$$

In deep water, $\tanh kh \to 1$ and so

$$\omega^2 = gk$$

whence

$$k = \frac{\omega^2}{g} = 0.02795$$

The phase speed is

$$c = \frac{\omega}{k} \quad = \frac{g}{\omega} \quad = 18.74 \text{ m s}^{-1}$$

and, in deep water,

$$n = \frac{1}{2}$$

At the pressure transducer we have

$$\rho g A \frac{\cosh k(h+z)}{\cosh kh} = 10000$$

with $\rho = 1025$ kg m^{-3} (seawater) and $z = -20$ m in water of depth $h = 22$ m. Hence,

$$A = \frac{10000 \times \cosh 0.8740}{1025 \times 9.81 \times \cosh[0.03973 \times 2]} \qquad = 1.395 \text{ m}$$

and hence

$$H_{\text{transducer}} = 2A \qquad = 2.790 \text{ m}$$

Then, from the shoaling equation:

$$H_{\text{deep}} = H_{\text{transducer}} \sqrt{\frac{(nc)_{\text{transducer}}}{(nc)_{\text{deep}}}} \; = 2.790 \times \sqrt{\frac{0.8139 \times 13.18}{0.5 \times 18.74}} \; = 2.985 \text{ m}$$

$$H_{\text{shallow}} = H_{\text{transducer}} \sqrt{\frac{(nc)_{\text{transducer}}}{(nc)_{\text{shallow}}}} \; = 2.790 \times \sqrt{\frac{0.8139 \times 13.18}{0.9278 \times 8.528}} \; = 3.249 \text{ m}$$

Answer: wave heights (a) nearshore: 3.25 m; (b) deep water: 2.99 m.

Q6.
Given:

$$h = 1 \text{ m}$$

$$a = 0.1 \text{ m}$$

$$b = 0.05 \text{ m}$$

First deduce kh, and hence the wavenumber, from the ratio of the semi-axes of the particle orbits:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = u \approx \frac{Agk}{\omega}\frac{\cosh k(h+z_0)}{\cosh kh}\cos(kx_0 - \omega t)$$

$$\Rightarrow \qquad X = x_0 - \frac{Agk}{\omega^2}\frac{\cosh k(h+z_0)}{\cosh kh}\sin(kx_0 - \omega t)$$

$$\Rightarrow \qquad X = x_0 - A\frac{\cosh k(h+z_0)}{\sinh kh}\sin(kx_0 - \omega t)$$

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = w \approx \frac{Agk}{\omega}\frac{\sinh k(h+z_0)}{\cosh kh}\sin(kx_0 - \omega t)$$

$$\Rightarrow \qquad Z = z_0 + \frac{Agk}{\omega^2}\frac{\sinh k(h+z_0)}{\cosh kh}\cos(kx_0 - \omega t)$$

$$\Rightarrow \qquad Z = z_0 + A\frac{\sinh k(h+z_0)}{\sinh kh}\cos(kx_0 - \omega t)$$

Where, in both directions we have used the dispersion relation $\omega^2 = gk\tanh kh$ to simplify.

Hence,

$$a = A\frac{\cosh(kh/2)}{\sinh kh} \qquad = 0.1$$

$$b = A\frac{\sinh(kh/2)}{\sinh kh} \qquad = 0.05$$

Then

$$\frac{b}{a} = \tanh(kh/2) \qquad = 0.5$$

$$\Rightarrow \qquad kh = 2\tanh^{-1}(0.5) \qquad = 1.099$$

$$\Rightarrow \qquad k = \frac{1.099}{1} \qquad = 1.099 \text{ m}^{-1}$$

Then wave height can be deduced from, e.g., the expression for a:

$$A = a\frac{\sinh kh}{\cosh(kh/2)} \qquad = 0.1 \times \frac{\sinh 1.099}{\cosh(1.099/2)} \qquad = 0.1155$$

$$H = 2A \quad = 0.2310 \text{ m}$$

From the dispersion relation:

$$\omega^2 = gk \tanh kh \quad \Rightarrow \quad \omega = 2.937 \text{ rad s}^{-1}$$

Finally,

$$T = \frac{2\pi}{\omega} \quad = 2.139 \text{ s}$$

$$L = \frac{2\pi}{k} \quad = 5.717 \text{ m}$$

Answer: height = 0.0268 m;   period = 2.14 s;   wavelength = 5.72 m.

Q7.
Given

$$h = 20 \text{ m}$$

$$A = 1 \text{ m} \qquad (H = 2 \text{ m})$$

(a) Here,

$$U = +1 \text{ m s}^{-1}$$

$$T_a = 3 \text{ s}$$

$$\omega_a = \frac{2\pi}{T_a} \quad = 2.094 \text{ rad s}^{-1}$$

Dispersion relation:

$$(\omega_a - kU)^2 = \omega_r^2 = gk \tanh kh$$

Iterate as

$$k = \frac{(\omega_a - kU)^2}{g \tanh kh}$$

i.e.

$$k = \frac{(2.094 - k)^2}{9.81 \tanh 20k}$$

to get

$$k = 0.3206$$

$$L = \frac{2\pi}{k} \quad = 19.60 \text{ m}$$

The maximum particle velocity occurs at the surface and is the *wave-relative* maximum velocity plus the current, i.e. from the wave-induced particle velocity

$$u_r = \frac{Agk}{\omega_r} \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega_r t)$$

we have

$$\omega_r = \omega_a - kU \quad = 2.094 - 0.3206 \times 1 \quad = 1.773 \text{ rad s}^{-1}$$

As the wave is travelling in the same direction as the current:

$$|u|_{\max} = \frac{Agk}{\omega_r} + U \quad = \frac{1 \times 9.81 \times 0.3206}{1.773} + 1 \quad = 2.774 \text{ m s}^{-1}$$

Answer: wavelength 19.6 m; maximum particle speed 2.77 m s^{-1}.

(b) Now,

$$U = -1 \text{ m s}^{-1}$$

$$T_a = 7 \text{ s}$$

$$\omega_a = \frac{2\pi}{T_a} \quad = 0.8976 \text{ rad s}^{-1}$$

Dispersion relation is rearranged for iteration as above:

$$k = \frac{(0.8976 + k)^2}{9.81 \tanh 20k}$$

to get

$$k = 0.1056$$

$$L = \frac{2\pi}{k} \quad = 59.50 \text{ m}$$

Wave-relative frequency:

$$\omega_r = \omega_a - kU \quad = 0.8976 + 0.1056 \times 1 \quad = 1.003 \text{ rad s}^{-1}$$

This time, as the current is opposing, the maximum speed is the magnitude of the backward velocity:

$$|u|_{\max} = \frac{Agk}{\omega_r} + |U| \quad = \frac{1 \times 9.81 \times 0.1056}{1.003} + 1 \quad = 2.033 \text{ m s}^{-1}$$

Answer: wavelength 59.5 m; maximum particle speed 2.03 m s^{-1}.

Q8.
As the waves move from deep to shallow water they refract (change angle θ) according to

$$k_0 \sin\theta_0 = k_1 \sin\theta_1$$

and undergo shoaling (change height, H) according to

$$(H^2 nc\cos\theta)_0 = (H^2 nc\cos\theta)_1$$

where subscript 0 indicates deep and 1 indicates the measuring station. Period is unchanged.

Given:

$$T = 5.5 \text{ s}$$

then

$$\omega = \frac{2\pi}{T} \quad = 1.142 \text{ rad s}^{-1}$$

In deep water,

$$k_0 = \frac{\omega^2}{g} \quad = 0.1329$$

$$n_0 = \frac{1}{2}$$

$$c_0 = \frac{gT}{2\pi} \quad (\text{or} \frac{\omega}{k}) \ = 8.587 \text{ m s}^{-1}$$

In the measured depth $h = 6$ m the wave height $H_1 = 0.8$ m. The dispersion relation is

$$\omega^2 = gk\tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh\tanh kh$$

$$\Rightarrow \qquad 0.7977 = kh\tanh kh$$

Iterate as

$$kh = \frac{0.7977}{\tanh kh} \qquad \text{or (better here)} \qquad kh = \frac{1}{2}(kh + \frac{0.7977}{\tanh kh})$$

to get (adding subscript 1):

$$k_1 h = 1.030$$

$$k_1 = \frac{1.030}{h} \quad = 0.1717 \text{ m}^{-1}$$

$$c_1 = \frac{\omega}{k_1} \quad = 6.651 \text{ m s}^{-1}$$

$$n_1 = \frac{1}{2}\left[1 + \frac{2k_1 h}{\sinh 2k_1 h}\right] \quad = 0.7669$$

<u>Refraction</u>

$$k_0 \sin\theta_0 = k_1 \sin\theta_1$$

Hence:

$$\sin\theta_0 = \frac{k_1}{k_0}\sin\theta_1 \quad = \frac{0.1717}{0.1329}\sin 47° \quad = 0.9449$$

$$\theta_0 = 70.89°$$

Shoaling

$$(H^2 nc\cos\theta)_0 = (H^2 nc\cos\theta)_1$$

Hence:

$$H_0 = H_1\sqrt{\frac{(nc\cos\theta)_1}{(nc\cos\theta)_0}} \quad = 0.8 \times \sqrt{\frac{0.7669 \times 6.651 \times \cos 47°}{0.5 \times 8.587 \times \cos 70.89°}} \quad = 1.259\ \mathrm{m}$$

Answer: deep-water wave height = 1.26 m;   angle = 70.9°.

Q9.
(a)
<u>Deep water</u>

$$L_0 = 300 \text{ m}$$

$$H_0 = 2 \text{ m}$$

From the wavelength,

$$k_0 = \frac{2\pi}{L_0} \quad = 0.02094 \text{ m}^{-1}$$

From the dispersion relation with $\tanh kh = 1$:

$$\omega^2 = g k_0$$

whence

$$\omega = \sqrt{g k_0} \quad = 0.4532 \text{ rad s}^{-1}$$

This stays the same as we move into shallower water.

$$c_0 = \frac{\omega}{k_0} \quad = 21.64 \text{ m s}^{-1}$$

$$n_0 = 0.5$$

<u>Depth $h = 30$ m</u>

The dispersion relation is

$$\omega^2 = g k \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 0.6281 = kh \tanh kh$$

Iterate as

$$kh = \frac{0.6281}{\tanh kh} \qquad \text{or (better here)} \qquad kh = \frac{1}{2}(kh + \frac{0.6281}{\tanh kh})$$

to get (adding subscript 1):

$$kh = 0.8856$$

$$k = \frac{0.8856}{h} \quad = 0.02952 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 212.8 \text{ m}$$

$$c = \frac{\omega}{k} \quad = 15.35 \text{ m s}^{-1}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.8103$$

$$c_g = nc \quad = 12.44 \text{ m s}^{-1}$$

From the shoaling relationship

$$(H^2 nc)_0 = H^2 nc$$

Hence,

$$H = H_0 \sqrt{\frac{(nc)_0}{nc}} \quad = 2 \times \sqrt{\frac{0.5 \times 21.64}{0.8103 \times 15.35}} \quad = 1.865 \text{ m}$$

Answer: wavelength = 213 m;   height = 1.87 m;   group velocity = 12.4 m s^{-1}.

(b)

$$E = \frac{1}{2}\rho g A^2 \quad = \frac{1}{8}\rho g H^2 \quad = \frac{1}{8} \times 1025 \times 9.81 \times 1.865^2 \quad = 4372 \text{ J m}^{-2}$$

Answer: energy = 4370 J m^{-2}.

(c) If the wave crests were obliquely oriented then the wavelength and group velocity would not change (because they are fixed by period and depth). However, the direction would change (by refraction) and the height would change (due to shoaling).

Refraction:

$$k_0 \sin\theta_0 = k \sin\theta$$

Hence:

$$\sin\theta = \frac{k_0}{k}\sin\theta_0 \quad = \frac{0.02094}{0.02952}\sin 60° \quad = 0.6143$$

$$\theta_0 = 37.90°$$

Shoaling:

$$(H^2 nc\cos\theta)_0 = H^2 nc\cos\theta$$

Hence:

$$H = H_0 \sqrt{\frac{(nc\cos\theta)_0}{nc\cos\theta}} \quad = 2 \times \sqrt{\frac{0.5 \times 21.64 \times \cos 60°}{0.8103 \times 15.35 \times \cos 37.90°}} \quad = 1.485 \text{ m}$$

Answer: wavelength and group velocity unchanged;   height = 1.48 m.

Q10.
(a) Let subscript 0 denote deep-water conditions and the absence of a subscript denote inshore conditions (at the 9 m depth contour).

Wavenumber (at 9 m depth):

$$k = \frac{2\pi}{L} \quad = \frac{2\pi}{55} \quad = 0.1142\ \text{m}^{-1}$$

From the dispersion relation:

$$\omega^2 = gk \tanh kh \quad = 9.81 \times 0.1142 \tanh(0.1142 \times 9) \quad = 0.8660\ (\text{rad s}^{-1})^2$$

Hence, wave angular frequency

$$\omega = \sqrt{0.8660} \quad = 0.9306\ \text{rad s}^{-1}$$

and period

$$T = \frac{2\pi}{\omega} \quad = \frac{2\pi}{0.9306} \quad = 6.752\ \text{s}$$

Answer: 6.75 s.

(b) The wave period and angular frequency do not change with depth. In deep water $kh \to \infty$ and $\tanh kh \to 1$, so the deep-water wavenumber is given by

$$\omega^2 = gk_0$$

Hence

$$k_0 = \frac{0.8660}{9.81} \quad = 0.08828\ \text{m}^{-1}$$

and the deep-water wavelength is

$$L_0 = \frac{2\pi}{k_0} \quad = 71.17\ \text{m}$$

(Note: one could also use $L_0 = \frac{gT^2}{2\pi}$ directly for this part if preferred. However, k_0 is needed in the next part, so it is useful to calculate it here.)

Answer: 71.2 m.

(c) Refraction. By Snell's law:

$$k_0 \sin\theta_0 = k \sin\theta$$

Hence,

$$\sin\theta_0 = \frac{k}{k_0}\sin\theta \quad = \frac{0.1142}{0.08828}\sin 25° \quad = 0.5467$$

$$\theta_0 = 33.14°$$

Answer: 33.1°.

(d) Shoaling:

$$(H^2 nc\cos\theta)_0 = (H^2 nc\cos\theta)$$

Here, in deep water:

$$n_0 = \frac{1}{2}$$

$$c_0 = \frac{L_0}{T} \quad = \frac{71.17}{6.752} \quad = 10.54 \text{ m s}^{-1}$$

whilst at the inshore depth:

$$kh = 0.1142 \times 9 \quad = 1.028$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.7675$$

$$c = \frac{L}{T} \quad = \frac{55}{6.752} \quad = 8.146 \text{ m s}^{-1}$$

Hence:

$$H_0 = H\sqrt{\frac{(nc\cos\theta)}{(nc\cos\theta)_0}} \quad = 1.8 \times \sqrt{\frac{0.7675 \times 8.146 \times \cos 25°}{0.5 \times 10.54 \times \cos 33.14°}} \quad = 2.040 \text{ m}$$

Answer: 2.04 m.

Q11.
(a)

$$T = 8 \text{ s}$$

$$\omega = \frac{2\pi}{T} \quad = 0.7854 \text{ rad s}^{-1}$$

Shoaling from 10 m depth to 3 m depth. Need wave properties at these two depths.

The dispersion relation is

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

This may be iterated as either

$$kh = \frac{\omega^2 h/g}{\tanh kh} \qquad \text{or} \qquad kh = \frac{1}{2}(kh + \frac{\omega^2 h/g}{\tanh kh})$$

	$h = 3$ m	$h = 10$ m
$\dfrac{\omega^2 h}{g}$	0.1886	0.6288
Iteration:	$kh = \frac{1}{2}(kh + \frac{0.1886}{\tanh kh})$	$kh = \frac{1}{2}(kh + \frac{0.6288}{\tanh kh})$
kh	0.4484	0.8862
k	0.1495 m^{-1}	0.08862 m^{-1}
$c = \dfrac{\omega}{k}$	5.254 m s^{-1}	8.863 m s^{-1}
$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]$	0.9388	0.8101
H	?	1 m

From the shoaling equation:

$$(H^2 nc)_{3 \text{ m}} = (H^2 nc)_{10 \text{ m}}$$

Hence:

$$H_{3\text{m}} = H_{10\text{m}} \sqrt{\frac{(nc)_{10 \text{ m}}}{(nc)_{3 \text{ m}}}} \quad = 1 \times \sqrt{\frac{0.8101 \times 8.863}{0.9388 \times 5.254}} \quad = 1.207 \text{ m}$$

Answer: 1.21 m.

(b) The wave continues to shoal:

$$(H^2nc)_b = (H^2nc)_{3\text{ m}} \quad = 7.186$$

(in m-s units). Assuming that it breaks as a shallow-water wave, then

$$n_b = 1$$

$$c_b = \sqrt{gh_b}$$

and we are given, from the breaker depth index:

$$H_b = 0.78h_b$$

Substituting in the shoaling equation,

$$(0.78h_b)^2\sqrt{gh_b} \quad = 7.186$$

$$\Rightarrow \qquad 1.906h_b^{5/2} = 7.186$$

giving

$$h_b = 1.700\text{ m}$$

$$H_b = 0.78h_b \quad = 1.326\text{ m}$$

Answer: breaking wave height 1.33 m in water depth 1.70 m.

Q12.
Narrow-band spectrum means that a Rayleigh distribution is appropriate, for which

$$P(\text{wave height} > H) = \exp\left[-\left(\frac{H}{H_{\text{rms}}}\right)^2\right]$$

Central frequency of 0.2 Hz corresponds to a period of 5 seconds; i.e. 12 waves per minute. In 8 minutes there are (on average) $8 \times 12 = 96$ waves, so the question data says basically that

$$P(\text{wave height} > 2\text{ m}) = \frac{1}{96}$$

Comparing with the Rayleigh distribution with $H = 2$ m:

$$\exp\left[-\left(\frac{2}{H_{\text{rms}}}\right)^2\right] = \frac{1}{96}$$

$$\Rightarrow \qquad \exp\left[\left(\frac{2}{H_{\text{rms}}}\right)^2\right] = 96$$

$$\Rightarrow \qquad \frac{4}{H_{\text{rms}}^2} = \ln 96$$

$$\Rightarrow \qquad H_{\text{rms}} = \sqrt{\frac{4}{\ln 96}} \quad = 0.9361\text{ m}$$

(a)

$$P(\text{wave height} > 3\text{ m}) = \exp\left[-\left(\frac{3}{H_{\text{rms}}}\right)^2\right] \quad = 3.463 \times 10^{-5} \quad = \frac{1}{28877}$$

This corresponds to once in every 28877 waves, or, at 12 waves per minute,

$$\frac{28877}{12} = 2406\text{ min} \quad = 40.1\text{ hours}$$

Answer: about once every 40 hours.

(b) The median wave height, H_{med}, is such that

$$P(\text{wave height} > H_{\text{med}}) = \frac{1}{2}$$

Hence

$$\exp\left[-\left(\frac{H_{\text{med}}}{H_{\text{rms}}}\right)^2\right] = \frac{1}{2}$$

$$\Rightarrow \qquad \exp\left[\left(\frac{H_{\text{med}}}{H_{\text{rms}}}\right)^2\right] = 2$$

$$\Rightarrow \qquad \left(\frac{H_{\text{med}}}{H_{\text{rms}}}\right)^2 = \ln 2$$

$$\Rightarrow \qquad H_{\text{med}} = H_{\text{rms}}\sqrt{\ln 2} \quad = 0.9361\sqrt{\ln 2} \quad = 0.7794 \text{ m}$$

Answer: 0.779 m.

Q13.
(a) Narrow-banded, so a Rayleigh distribution is appropriate. Then

$$H_{\mathrm{rms}} = \frac{H_s}{1.416} \quad = \frac{2}{1.416} \quad = 1.412 \text{ m}$$

Answer: 1.41 m.

(b)

$$H_{1/10} = 1.800\, H_{\mathrm{rms}} \quad = 1.800 \times 1.412 \quad = 2.542 \text{ m}$$

Answer: 2.54 m.

(c)

$$\begin{aligned} P(4 \text{ m} < \text{height} < 5 \text{ m}) \quad &= P(\text{height} > 4) - P(\text{height} > 5) \\ &= \exp\left[-\left(\frac{4}{1.412}\right)^2\right] - \exp\left[-\left(\frac{5}{1.412}\right)^2\right] \\ &= 3.235 \times 10^{-4} \end{aligned}$$

or about once in every 3090 waves.

Answer: $3.24{\times}10^{-4}$.

Q14.

The Bretschneider spectrum is

$$S(f) = \frac{5}{16} H_s^2 \frac{f_p^4}{f^5} \exp\left(-\frac{5}{4}\frac{f_p^4}{f^4}\right)$$

Here we have

$$H_s = 2.5 \text{ m}$$

$$f_p = \frac{1}{T_p} \quad = \frac{1}{6} \text{ Hz}$$

Amplitudes

For a single component the energy (per unit weight) is

$$\frac{1}{2} a^2 = S(f)\Delta f$$

Hence,

$$a_i = \sqrt{2S(f)\Delta f}$$

In this instance, $\Delta f = 0.25 f_p$, so that, numerically,

$$a_i = \sqrt{\frac{0.9766}{(f_i/f_p)^5} \exp\left[-\frac{1.25}{(f_i/f_p)^4}\right]}$$

These are computed (using Excel) in a column of the table below.

Wavenumbers

For deep-water waves,

$$\omega^2 = gk$$

or, since $\omega = 2\pi f$

$$k_i = \frac{4\pi^2 f_i^2}{g}$$

Numerically here:

$$k_i = 0.1118 \left(\frac{f_i}{f_p}\right)^2$$

These are computed (using Excel) in a column of the table below.

Velocity

The maximum particle velocity for one component is

$$u_i = \frac{gk_i a_i}{\omega_i} \frac{\cosh k_i(h+z)}{\cosh k_i h}$$

or, since the water is deep and hence $\cosh X \sim \frac{1}{2}\mathrm{e}^X$:

$$u_i = \frac{gk_i a_i}{2\pi f_i} \exp k_i z$$

Numerically here, with $z = -5$ m:

$$u_i = \frac{9.368 k_i a_i}{f_i/f_p} \exp(-5k_i)$$

These are computed (using Excel) in a column of the table below.

f_i/f_p	a_i	k_i	u_i
0.75	0.281410	0.062888	0.161410
1	0.528962	0.111800	0.316769
1.25	0.437929	0.174688	0.239372
1.5	0.316967	0.251550	0.141566
1.75	0.228204	0.342388	0.075503
2	0.168004	0.447200	0.037614
Sum:	1.961475		0.972235

From the table, with all components instantaneously in phase

$$\eta_{\max} = \sum a_i \quad = 1.961 \text{ m}$$

$$u_{\max} = \sum u_i \quad = 0.9722 \text{ m s}^{-1}$$

Answers: (a), (b): $[a_i]$ and $[k_i]$ given in the table; (c) $\eta_{\max} = 1.96$ m, $u_{\max} = 0.972$ m s^{-1}.

Q15.
Given

$$T_p = 9.1 \text{ s}$$

then

$$f_p = \frac{1}{T_p} \quad = 0.1099 \text{ Hz}$$

The middle frequency of the given range is $f = 0.155$ Hz. With $H_s = 2.1$ m the Bretschneider spectrum gives

$$S(f) = \frac{5}{16} H_s^2 \frac{f_p^4}{f^5} \exp\left(-\frac{5}{4}\frac{f_p^4}{f^4}\right) \quad = 1.638 \text{ m}^2 \text{ s}$$

The energy density is then

$$E = \rho g \times S(f)\Delta f \quad = 1025 \times 9.81 \times 1.638 \times 0.01 \quad = 164.7 \text{ J m}^{-2}$$

The period of this component is

$$T = \frac{1}{f} \quad = 6.452 \text{ s}$$

And hence, as a deep-water wave:

$$c = \frac{gT}{2\pi} \quad = 10.07 \text{ m s}^{-1}$$

$$n = \frac{1}{2}$$

Hence,

$$c_g = nc \quad = 5.035 \text{ m s}^{-1}$$

The power density is then

$$P = Ec_g \quad = 164.7 \times 5.035 \quad = 829.3 \text{ W m}^{-1}$$

Answer: 0.829 kW m^{-1}.

Q16.
(a) Waves are duration-limited if the wind has not blown for sufficient time for wave energy to propagate across the entire fetch. Otherwise they are fetch-limited, and the precise duration of the storm does not affect wave parameters.

(b) Given

$$U = 13.5 \text{ m s}^{-1}$$

$$F = 64000 \text{ m}$$

$$t = 3 \times 60 \times 60 \quad = 10800 \text{ s}$$

then the relevant non-dimensional fetch is

$$\hat{F} \equiv \frac{gF}{U^2} \quad = 3445$$

The *minimum* non-dimensional time required for fetch-limited waves is found from

$$\hat{t}_{\min} \equiv \left(\frac{gt}{U}\right)_{\min} \quad = 68.8\hat{F}^{2/3} \quad = 15693$$

but the *actual* non-dimensional time for which the wind has blown is

$$\hat{t} \equiv \frac{gt}{U} \quad = 7848$$

which is less. Hence, the waves are *duration-limited* and in the predictive curves we must use an effective fetch F_{eff} given by

$$68.8\hat{F}_{\text{eff}}^{2/3} \quad = 7848$$

whence

$$\hat{F}_{\text{eff}} = 1218$$

Then

$$\frac{gH_s}{U^2} \equiv \hat{H}_s \quad = 0.0016\hat{F}_{\text{eff}}^{1/2} \quad = 0.05584$$

$$\frac{gT_p}{U} \equiv \hat{T}_p \quad = 0.286\hat{F}_{\text{eff}}^{1/3} \quad = 3.054$$

from which the corresponding significant wave height and peak period are

$$H_s = 1.037 \text{ m}$$

$$T_p = 4.203 \text{ s}$$

The significant wave period is estimated as

$$T_s = 0.945T_p \quad = 3.972 \text{ s}$$

Answer: duration-limited; significant wave height = 1.04 m and period 3.97 s.

Q17.
<u>Wave Properties</u>

Given:

$h = 6$ m
$T = 6$ s

Then,

$$\omega = \frac{2\pi}{T} \quad = 1.047 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 0.6705 = kh \tanh kh$$

Iterate as

$$kh = \frac{0.6705}{\tanh kh} \qquad \text{or (better here)} \qquad kh = \frac{1}{2}(kh + \frac{0.6705}{\tanh kh})$$

to get

$$kh = 0.9223$$

$$k = \frac{0.9223}{h} \quad = 0.1537 \text{ m}^{-1}$$

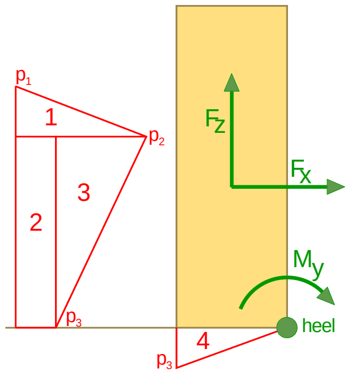
<u>Forces and Moments</u>

The height of the fully reflected wave is $2H$, where H is the height of the incident wave. Thus the maximum crest height above SWL is $H = 1.5$ m. Since the SWL depth is 6 m, the maximum crest height does not overtop the caisson.

The modelled pressure distribution is as shown, consisting of

hydrostatic (from $p_1 = 0$ at the crest to $p_2 = \rho g H$ at the SWL);
linear (from p_2 at the SWL to the wave value p_3 at the base);
linear underneath (from p_3 at the front of the caisson to 0 at the rear).

To facilitate the computation of moments, the linear distribution on the front face can be decomposed into a constant distribution (p_3) and a triangular distribution with top $p_2 - p_3$.



The relevant pressures are:

$$p_1 = 0$$

$$p_2 = \rho g H \quad = 1025 \times 9.81 \times 1.5 \quad = 15080 \text{ Pa}$$

$$p_3 = \frac{\rho g H}{\cosh kh} \quad = \frac{1025 \times 9.81 \times 1.5}{\cosh(0.1537 \times 6)} \quad = 10360 \text{ Pa}$$

This yields the equivalent forces and points of application shown in the diagram below. Sums of forces and moments (*per metre width*) are given in the table.

Region	Force, F_x or F_z	Clockwise turning moment about heel
1	$F_{x1} = \frac{1}{2} p_2 \times H \quad = 11310 \text{ N/m}$	$F_{x1} \times \left(h + \frac{1}{3} H\right) \quad = 73520 \text{ N m/m}$
2	$F_{x2} = p_3 \times h \quad = 62160 \text{ N/m}$	$F_{x2} \times \frac{1}{2} h \quad = 186480 \text{ N m/m}$
3	$F_{x3} = \frac{1}{2}(p_2 - p_3) \times h \quad = 14160 \text{ N/m}$	$F_{x3} \times \frac{2}{3} h \quad = 56640 \text{ N m/m}$
4	$F_{z4} = \frac{1}{2} p_3 \times w = 20720 \text{ N /m}$	$F_{z4} \times \frac{2}{3} w \quad = 55250 \text{ N m/m}$

The sum of the *horizontal* forces

$$F_{x1} + F_{x2} + F_{x3} = 87630 \text{ N/m}$$

The sum of all clockwise moments

371900 N m / m

Answer: per metre, horizontal force = 87.6 kN;   overturning moment = 372 kN m.

Q18.
Period:

$$T = \frac{1}{f} \quad = \frac{1}{0.1} \quad = 10 \text{ s}$$

The remaining quantities require solution of the dispersion relationship.

$$\omega = \frac{2\pi}{T} \quad = 0.6283 \text{ rad s}^{-1}$$

Rearrange the dispersion relation $\omega^2 = gk \tanh kh$ to give

$$\frac{\omega^2 h}{g} = kh \tanh kh$$

Here, with $h = 18$ m,

$$0.7243 = kh \tanh kh$$

Since the LHS < 1, rearrange for iteration as

$$kh = \frac{1}{2}\left[kh + \frac{0.7243}{\tanh kh}\right]$$

Iteration from, e.g., $kh = 1$ produces

$$kh = 0.9683$$

$$k = \frac{0.9683}{h} \quad = 0.05379 \text{ m}^{-1}$$

The wavelength is then

$$L = \frac{2\pi}{k} \quad = 116.8 \text{ m}$$

and the phase speed and ratio of group to phase velocities are

$$c = \frac{\omega}{k} \text{ (or } \frac{L}{T}) \quad = 11.68 \text{ m s}^{-1}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.7852$$

For the wave power,

$$\text{power (per m)} = \frac{1}{8}\rho g H^2 nc \quad = \frac{1}{8} \times 1025 \times 9.81 \times 2.1^2 \times 0.7852 \times 11.68$$
$$= 50840 \text{ W m}^{-1}$$

Answer: period = 10 s; wavelength = 117 m; phase speed = 11.68 m s^{-1};
power = 50.8 kW m^{-1}.

(b) First need new values of k, n and c in 6 m depth.

$$\frac{\omega^2 h}{g} = kh \tanh kh$$

Here, with $h = 6$ m,

$$0.2414 = kh \tanh kh$$

Since the LHS < 1, rearrange for iteration as

$$kh = \frac{1}{2}\left[kh + \frac{0.2414}{\tanh kh}\right]$$

Iteration from, e.g., $kh = 1$ produces

$$kh = 0.5120$$

$$k = \frac{0.5120}{h} \quad = 0.08533 \text{ m}^{-1}$$

$$c = \frac{\omega}{k} \quad = 7.363 \text{ m s}^{-1}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.9222$$

For direction, use Snell's Law:

$$(k \sin\theta)_{18\text{ m}} = (k\sin\theta)_{6\text{ m}}$$

whence

$$(\sin\theta)_{6\text{ m}} = \frac{(k\sin\theta)_{18\text{ m}}}{k_{6\text{ m}}} \quad = \frac{0.05379 \times \sin 25°}{0.08533} \quad = 0.2664$$

and hence the wave direction in 6 m depth is 15.45°.

For wave height, use the shoaling equation:

$$(H^2 nc\cos\theta)_{6\text{ m}} = (H^2 nc\cos\theta)_{18\text{ m}}$$

Hence,

$$H_{6\text{ m}} = H_{18\text{ m}}\sqrt{\frac{(nc\cos\theta)_{18\text{ m}}}{(nc\cos\theta)_{6\text{ m}}}} \quad = 2.1\sqrt{\frac{0.7852 \times 11.68 \times \cos 25°}{0.9222 \times 7.363 \times \cos 15.45°}} \quad = 2.367 \text{ m}$$

Answer: wave height = 2.37 m;   wave direction = 15.5°.

(c) The Miche breaking criterion gives

$$\left(\frac{H}{L}\right)_b = 0.14\tanh(kh)_b$$

If we are to assume shallow-water behaviour, then $\tanh kh \sim kh$ and so

$$\left(\frac{Hk}{2\pi}\right)_b = 0.14(kh)_b$$

whence

$$\left(\frac{H}{h}\right)_b = 2\pi \times 0.14 \quad = 0.8796$$

or

$$H_b = 0.8796 h_b \qquad (*)$$

This is used to eliminate wave height on the LHS of the shoaling equation

$$(H^2 nc \cos\theta)_b = (H^2 nc \cos\theta)_{6\text{ m}}$$

For refraction, Snell's Law in the form $\sin\theta / c =$ constant, together with the shallow-water phase speed at breaking, gives

$$\left(\frac{\sin\theta}{\sqrt{gh}}\right)_b = \left(\frac{\sin\theta}{c}\right)_{6\text{ m}} \quad = \frac{\sin 15.45°}{7.363} \quad = 0.03618$$

whence

$$\sin\theta_b = 0.1133\sqrt{h_b}$$

$$\cos\theta_b = \sqrt{1 - \sin^2\theta_b} \quad = \sqrt{1 - 0.01284 h_b}$$

With the shallow water approximations $n = 1$, $c = \sqrt{gh}$, and the relation (*), the shoaling equation becomes

$$(0.8796 h_b)^2 \sqrt{g h_b} \sqrt{1 - 0.01284 h_b} = 36.67$$

or

$$2.423 h_b^{5/2} (1 - 0.01284 h_b)^{1/2} = 36.67$$

This rearranges for iteration:

$$h_b = 2.965(1 - 0.01284 h_b)^{-1/5}$$

to give

$$h_b = 2.988 \text{ m}$$

Answer: 2.99 m.

Q19.
(a)

$$U = 35 \text{ m s}^{-1}$$

$$F = 150000 \text{ m}$$

From these,

$$\hat{F} = \frac{gF}{U^2} \quad = 1201$$

From the JONSWAP curves,

$$\hat{t}_{\min} = 68.8\hat{F}^{2/3} \quad = 7774$$

(i) For $t = 4$ hr $= 14400$ s,

$$\hat{t} = \frac{gt}{U} \quad = 4036$$

This is less than 7774. We conclude that there is insufficient time for wave energy to have propagated right across the fetch; the waves are duration-limited. Hence, instead of $\hat{F}$, we use an effective dimensionless fetch $\hat{F}_{\text{eff}}$ such that

$$68.8\hat{F}_{\text{eff}}^{2/3} = 4036$$

$$\hat{F}_{\text{eff}} = 449.3$$

Then

$$\hat{H}_s = 0.0016\hat{F}_{\text{eff}}^{1/2} \quad = 0.03391$$

$$\hat{T}_p = 0.2857\hat{F}_{\text{eff}}^{1/3} \quad = 2.188$$

The dimensional significant wave height and peak wave period are

$$H_s = \frac{\hat{H}_s U^2}{g} \quad = 4.234 \text{ m}$$

$$T_p = \frac{\hat{T}_p U}{g} \quad = 7.806 \text{ s}$$

Answer: significant wave height = 4.23 m;   peak period = 7.81 s.

(ii) For $t = 8$ hr $= 28800$ s,

$$\hat{t} = \frac{gt}{U} \quad = 8072$$

This is greater than 7774, so the wave statistics are fetch-limited and we can use $\hat{F} = 1201$. Then

$$\hat{H}_s = 0.0016\hat{F}^{1/2} \quad = 0.05545$$

$$\hat{T}_p = 0.2857\hat{F}^{1/3} \quad = 3.037$$

The dimensional significant height and peak wave period are

$$H_s = \frac{\hat{H}_s U^2}{g} \quad = 6.924 \text{ m}$$

$$T_p = \frac{\hat{T}_p U}{g} \quad = 10.84 \text{ s}$$

Answer: significant wave height = 6.92 m;   period = 10.8 s.

(b) (i) The wave spectrum is narrow-banded, so it is appropriate to use the Rayleigh probability distribution for wave heights.

In deep water H_{rms} is known: $H_{\text{rms}} = 1.8$ m. For a single wave:

$$P(H > 3 \text{ m}) = \exp[-(3/1.8)^2] \quad = 0.06218$$

This is once in every

$$\frac{1}{p} = 16.08 \text{ waves}$$

or, with wave period 9 s, once every 144.7 s.

Answer: every 145 s.

(ii) In this part the wave height is $H = 3$ m at depth 10 m, but H_{rms} is given in deep water. So we either need to transform H to deep water or H_{rms} to the 10 m depth. We'll do the former.

Use the shoaling equation (for normal incidence):

$$(H^2 nc)_0 = (H^2 nc)_{10 \text{ m}}$$

First find c and n in 10 m depth:

$$T = 9 \text{ s}$$

$$\omega = \frac{2\pi}{T} \quad = 0.6981 \text{ rad s}^{-1}$$

Rearrange the dispersion relation $\omega^2 = gk \tanh kh$ to give

$$\frac{\omega^2 h}{g} = kh \tanh kh$$

Here, with $h = 10$ m,

$$0.4968 = kh \tanh kh$$

Since the LHS < 1, rearrange for iteration as

$$kh = \frac{1}{2}\left[kh + \frac{0.4968}{\tanh kh}\right]$$

Iteration from, e.g., $kh = 1$ produces

$$kh = 0.7688$$

Then,

$$k = \frac{0.7688}{h} \quad = 0.07688 \text{ m}^{-1}$$

$$c = \frac{\omega}{k} \quad = 9.080 \text{ m s}^{-1}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.8464$$

In deep water:

$$c_0 = \frac{gT}{2\pi} \quad = 14.05 \text{ m s}^{-1}$$

$$n_0 = \frac{1}{2}$$

Hence, when the inshore wave height $H_{10} = 3$ m, the deep-water wave height, from the shoaling equation, is

$$H_0 = H_{10}\sqrt{\frac{(nc)_{10}}{(nc)_0}} \quad = 3 \times \sqrt{\frac{0.8464 \times 9.080}{0.5 \times 14.05}} \quad = 3.138 \text{ m}$$

(If we had transformed H_{rms} to the 10 m depth instead we would have got 1.721 m.)

Now use the Rayleigh distribution to determine wave probabilities. For a single wave:

$$P(H_{10} > 3 \text{ m}) = P(H_0 > 3.138 \text{ m}) \quad = \exp[-(3.138/1.8)^2] \quad = 0.04787$$

This is once in every

$$\frac{1}{p} = 20.89 \text{ waves}$$

or, with wave period 9 s, once every 188.0 s.

Answer: every 188 s.

Q20.

(a)

(i) *Refraction* – change of *direction* as oblique waves move into shallower water;

(ii) *Diffraction* – *spreading* of waves into a region of shadow;

(iii) S*hoaling* – change of *height* as waves move into shallower water.

(b)

$$\omega = \frac{2\pi}{T} \quad = \frac{2\pi}{12} \quad = 0.5236 \text{ rad s}^{-1}$$

Deep water

Dispersion relation

$$\omega^2 = gk_0$$

Hence:

$$k_0 = \frac{\omega^2}{g} \quad = 0.02795 \text{ m}^{-1}$$

$$L_0 = \frac{2\pi}{k_0} \quad = 224.8 \text{ m}$$

$$c_0 = \frac{\omega}{k_0} \ \left(\text{or } \frac{L_0}{T}\right) \quad = 18.73 \text{ m s}^{-1}$$

$$n_0 = \frac{1}{2}$$

Shallow water

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 0.1397 = kh \tanh kh$$

Iterate as either

$$kh = \frac{0.1397}{\tanh kh} \qquad \text{or (better here)} \qquad kh = \frac{1}{2}(kh + \frac{0.1397}{\tanh kh})$$

to get

$$kh = 0.3827$$

$$k = \frac{0.3827}{h} \quad = 0.07654 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 82.09 \text{ m}$$

$$c = \frac{\omega}{k} \quad \left(\text{or } \frac{L}{T}\right) \quad = 6.841 \text{ m s}^{-1}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.9543$$

Refraction

$$k \sin\theta = k_0 \sin\theta_0$$

Hence,

$$\sin\theta = \frac{k_0 \sin\theta_0}{k} \quad = \frac{0.02795 \times \sin 20°}{0.07654} \quad = 0.1249$$

$$\theta = 7.175°$$

Shoaling

$$(H^2 nc\cos\theta)_{5\text{ m}} = (H^2 nc\cos\theta)_0$$

Hence:

$$H_{5\text{ m}} = H_0\sqrt{\frac{(nc\cos\theta)_0}{(nc\cos\theta)_{5\text{ m}}}} \quad = 3 \times \sqrt{\frac{0.5 \times 18.73 \times \cos 20°}{0.9543 \times 6.841 \times \cos 7.175°}} \quad = 3.497 \text{ m}$$

Answer: wavelength = 82.1 m; direction = 7.18°; height = 3.50 m.

(c) The breaker height index is

$$\frac{H_b}{H_0} = 0.56\left(\frac{H_0}{L_0}\right)^{-1/5} \quad = 0.56 \times \left(\frac{3}{224.8}\right)^{-1/5} \quad = 1.328$$

Hence,

$$H_b = 1.328\, H_0 \quad = 3.984 \text{ m}$$

The breaker depth index is

$$\gamma_b \equiv \left(\frac{H}{h}\right)_b = b - a\frac{H_b}{gT^2}$$

where, with a beach slope $m = 1/20 = 0.05$:

$$a = 43.8(1 - \mathrm{e}^{-19m}) \quad = 26.86$$

$$b = \frac{1.56}{1 + \mathrm{e}^{-19.5m}} \quad = 1.133$$

Hence,

$$\gamma_b = 1.133 - 26.86 \times \frac{3.984}{9.81 \times 12^2} \quad = 1.058$$

giving:

$$h_b = \frac{H_b}{\gamma_b} \quad = 3.766 \text{ m}$$

Answer: breaker height = 3.98 m;   breaking depth = 3.77 m.

Q21.
(a)
(i) "Narrow-banded sea state" – narrow range of frequencies.

(ii) "Significant wave height" – either:
average height of the highest one-third of waves
or, from the wave spectrum,

$$H_{m0} = 4\sqrt{\overline{\eta^2}} \quad = 4m_0$$

(iii) "Energy spectrum": distribution of wave energy with frequency; specifically,

$$S(f)\,\mathrm{d}f$$

is the wave energy (divided by ρg) in the small interval $(f, f + \mathrm{d}f)$.

(iv) "Duration-limited" – condition of the sea state when the storm has blown for insufficient time for wave energy to propagate across the entire fetch.

(b) If the period is $T = 10$ s then the number of waves per hour is

$$n = \frac{3600}{T} \quad = 360$$

(i) For a narrow-banded sea state a Rayleigh distribution is appropriate:

$$P(\text{height} > H) = \mathrm{e}^{-(H/H_{\mathrm{rms}})^2}$$

Hence,

$$P(\text{height} > 3.5) = \mathrm{e}^{-(3.5/2.5)^2} \quad = 0.1409$$

For 360 waves, the expected number exceeding this is

$$360 \times 0.1409 \quad = 50.72$$

Answer: 51 waves.

(ii)

$$P(\text{height} > 5) = \mathrm{e}^{-(5/2.5)^2} \quad = 0.01832$$

Corresponding to once in every

$$\frac{1}{0.01832} = 54.59 \text{ waves}$$

With a period of 10 s, this represents a time of

$$10 \times 54.59 = 545.9 \text{ s}$$

(Alternatively, this could be expressed as 6.59 times per hour.)

Answer: 546 s (about 9.1 min) or, equivalently, 6.59 times per hour.

(iii) In deep water the dispersion relation reduces to:

$$\omega^2 = gk$$

or

$$\left(\frac{2\pi}{T}\right)^2 = g\left(\frac{2\pi}{L}\right)$$

Hence,

$$L = \frac{gT^2}{2\pi}$$

With $T = 10$ s,

$$L = 156.1 \text{ m}$$

$$c = \frac{L}{T} \quad = 15.61 \text{ m s}^{-1}$$

and, in deep water

$$c_g = \frac{1}{2}c \quad = 7.805 \text{ m s}^{-1}$$

Answer: wavelength = 156.1 m; celerity = 15.6 m s^{-1}; group velocity = 7.81 m s^{-1}.

(c) Given

$$U = 20 \text{ m s}^{-1}$$
$$F = 10^5 \text{ m}$$
$$t = 6 \text{ hours} \quad = 21600 \text{ s}$$

Then

$$\hat{F} \equiv \frac{gF}{U^2} \quad = 2453$$

$$\hat{t}_{\min} \equiv \frac{gt_{\min}}{U} \quad = 68.8F^{2/3} \quad = 12513$$

But the non-dimensional time of the storm is

$$\hat{t} = \frac{gt}{U} \quad = 10590$$

This is less than $\hat{t}_{\min}$, hence the sea state is duration-limited.

Hence, we must calculate an effective fetch by working back from $\hat{t}$:

$$\hat{F}_{\text{eff}} = \left(\frac{\hat{t}}{68.8}\right)^{3/2} \quad = 1910$$

The non-dimensional wave height and peak period then follow from the JONSWAP formulae:

$$\hat{H}_s \equiv \frac{gH_s}{U^2} \quad = 0.0016\hat{F}_{\text{eff}}^{1/2} \quad = 0.06993$$

$$\hat{T}_p \equiv \frac{gT_p}{U} \quad = 0.2857\hat{F}_{\text{eff}}^{1/3} \quad = 3.545$$

whence, extracting the dimensional variables:

$$H_s = \frac{U^2}{g}\hat{H}_s \quad = 2.851\text{ m}$$

$$T_p = \frac{U}{g}\hat{T}_p \quad = 7.227\text{ s}$$

Answer: duration-limited; significant wave height = 2.85 m; peak period = 7.23 s.

Q22.
(a)
(i) Two of:

spilling breakers: steep waves and/or mild beach slopes;
plunging breakers: moderately steep waves and moderate beach slopes;
collapsing breakers: long waves and/or steep beach slopes.

(ii)

$$T = 7 \text{ s}$$

$$\omega = \frac{2\pi}{T} \quad = 0.8976 \text{ rad s}^{-1}$$

Shoaling from 100 m depth to 12 m depth. Need wave properties at these two depths.

The dispersion relation is

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

This may be iterated as either

$$kh = \frac{\omega^2 h/g}{\tanh kh} \qquad \text{or} \qquad kh = \frac{1}{2}(kh + \frac{\omega^2 h/g}{\tanh kh})$$

	$h = 12$ m	$h = 100$ m
$\frac{\omega^2 h}{g}$	0.9855	8.213
Iteration:	$kh = \frac{0.9855}{\tanh kh}$	$kh = \frac{8.213}{\tanh kh}$
kh	1.188	8.213
k	0.099 m^{-1}	0.08213 m^{-1}
$c = \frac{\omega}{k}$	9.067 m s^{-1}	10.93 m s^{-1}
$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]$	0.7227	0.5000
θ	0º (necessarily)	0º
H	?	1.8 m

From the shoaling equation:

$$(H^2 nc)_{12\text{ m}} = (H^2 nc)_{100\text{ m}}$$

Hence:

$$H_{12\text{ m}} = H_{100\text{ m}} \sqrt{\frac{(nc)_{100\text{ m}}}{(nc)_{12\text{ m}}}} \quad = 1.8 \times \sqrt{\frac{0.5 \times 10.93}{0.7227 \times 9.067}} \quad = 1.644\text{ m}$$

Answer: wave height = 1.64 m.

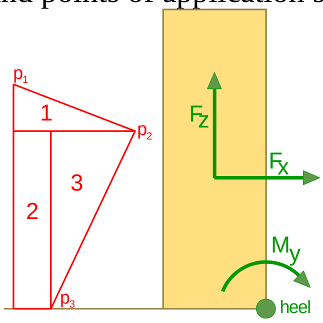
(iii)
The relevant pressures are:

$$p_1 = 0$$

$$p_2 = \rho g H \quad = 1025 \times 9.81 \times 1.644 \quad = 16530\text{ Pa}$$

$$p_3 = \frac{\rho g H}{\cosh kh} \quad = \frac{1025 \times 9.81 \times 1.644}{\cosh(1.188)} \quad = 9221\text{ Pa}$$

This yields the equivalent forces and points of application shown in the diagram.



Region	Force, F_x or F_z	Moment arm
1	$F_{x1} = \frac{1}{2} p_2 \times H \quad = 13590\text{ N/m}$	$z_1 = h + \frac{1}{3} H \quad = 12.55\text{ m}$
2	$F_{x2} = p_3 \times h \quad = 110700\text{ N/m}$	$z_2 = \frac{1}{2} h \quad = 6\text{ m}$
3	$F_{x3} = \frac{1}{2}(p_2 - p_3) \times h \quad = 43850\text{ N/m}$	$z_3 = \frac{2}{3} h \quad = 8\text{ m}$

The sum of all clockwise moments about the heel:

$$F_{x1} z_1 + F_{x2} z_2 + F_{x3} z_3 = 1186000\text{ N m/m}$$

Answer: overturning moment = 1186 kN m per metre of breakwater.

(b) New waves have revised properties:

$$T = 13\text{ s}$$

$$\omega = \frac{2\pi}{T} \quad = 0.4833 \text{ rad s}^{-1}$$

	$h = 12$ m	$h = 100$ m
$\frac{\omega^2 h}{g}$	0.2857	2.381
Iteration:	$kh = \frac{1}{2}\left(kh + \frac{0.2857}{\tanh kh}\right)$	$kh = \frac{2.381}{\tanh kh}$
kh	0.5613	2.419
k	0.04678 m^{-1}	0.02419 m^{-1}
$c = \frac{\omega}{k}$	10.33 m s^{-1}	19.98 m s^{-1}
$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]$	0.9086	0.5383
θ	?	30º
H	?	1.3 m

To estimate travel time of wave groups consider the group velocity.

The previous waves had group velocity

$$c_g = nc \quad = 5.465 \text{ m s}^{-1}$$

These longer-period waves have group velocity

$$c_g = nc \quad = 10.76 \text{ m s}^{-1}$$

The latter have a larger group velocity and hence have a shorter travel time.

(ii) Refraction:

$$(k \sin\theta)_{12\text{ m}} = (k \sin\theta)_{100\text{ m}}$$

$$0.04678 \sin\theta = 0.02419 \sin 30°$$

Hence,

$$\theta = 14.98°$$

From the shoaling equation:

$$(H^2 nc \cos\theta)_{12\text{ m}} = (H^2 nc \cos\theta)_{100\text{ m}}$$

Hence:

$$H_{12\text{m}} = H_{100\text{m}} \sqrt{\frac{(nc\cos\theta)_{100\text{ m}}}{(nc\cos\theta)_{12\text{ m}}}} = 1.3 \times \sqrt{\frac{0.5383 \times 19.98 \times \cos 30°}{0.9086 \times 10.33 \times \cos 14.98°}} = 1.318\text{ m}$$

The maximum freeboard for a combination of the two waves, allowing for reflection, is twice the sum of the amplitudes, i.e. the sum of the heights.

Hence, the freeboard must be

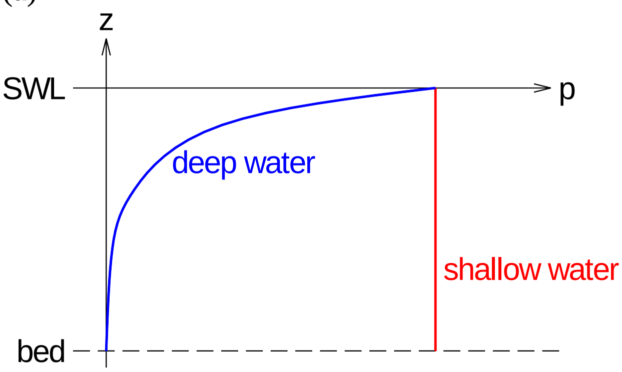
$$1.644 + 1.318 = 2.962\text{ m}$$

or height from the bed:

$$12 + 2.962 = 14.96\text{ m}$$

Answer: caisson height 14.96 m (from bed level), or 2.96 m (freeboard).

Q23.
(a)



(b) Given:

$$h = 8 \text{ m}$$
$$T = 5 \text{ s}$$

Then,

$$\omega = \frac{2\pi}{T} \quad = 1.257 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 1.289 = kh \tanh kh$$

Iterate as

$$kh = \frac{1.289}{\tanh kh}$$

to get

$$kh = 1.442$$

$$k = \frac{1.442}{h} \quad = 0.1803 \text{ m}^{-1}$$

$\pi/10 < kh < \pi$, so this is an intermediate-depth wave.

Answer: $kh = 1.44$; intermediate-depth wave.

(c)
(i) The velocity potential is

$$\phi = \frac{Ag}{\omega} \frac{\cosh k(h+z)}{\cosh kh} \sin(kx - \omega t)$$

Then

$$u \equiv \frac{\partial \phi}{\partial x} \quad = \frac{Agk}{\omega} \frac{\cosh k(h+z)}{\cosh kh} \cos{(kx - \omega t)}$$

At the SWL ($z = 0$) the amplitude of the horizontal velocity is

$$\frac{Agk}{\omega}$$

(ii) At the sensor height ($z = -6$ m) the amplitude of horizontal velocity is

$$\frac{Agk}{\omega} \frac{\cosh k(h+z)}{\cosh kh}$$

and hence, from the given data,

$$\frac{A \times 9.81 \times 0.1803}{1.257} \times \frac{\cosh[0.1803 \times 2]}{\cosh 1.442} \quad = 0.34$$

Hence,

$$A = 0.5062$$

$$H = 2A \quad = 1.012 \text{ m}$$

Answer: 1.01 m.

(iii) From

$$u = \frac{Agk}{\omega} \frac{\cosh k(h+z)}{\cosh kh} \cos{(kx - \omega t)}$$

we have

$$a_x = \frac{\partial u}{\partial t} + \text{non} - \text{linear terms} \quad = Agk \frac{\cosh k(h+z)}{\cosh kh} \sin{(kx - \omega t)}$$

The maximum horizontal acceleration at the bed ($z = -h$) is

$$\frac{Agk}{\cosh kh} = \frac{0.5062 \times 9.81 \times 0.1803}{\cosh 1.442} \quad = 0.4010 \text{ m s}^{-2}$$

Answer: 0.401 m s^{-2}.

Q24.

(a) “Significant wave height” is the average of the highest 1/3 of waves. For irregular waves,

$$H_s = 4\sqrt{\overline{\eta^2}}$$

whilst for a regular wave

$$\overline{\eta^2} = \frac{1}{2}A^2$$

Hence

$$H_s = 4\sqrt{\frac{A^2}{2}} \quad = 2\sqrt{2}A \quad = 2\sqrt{2} \times 0.8 \quad = 2.263 \text{ m}$$

Answer: significant wave height = 2.26 m.

(b) “Shoaling” is the change in wave height as a wave moves into shallower water. Linear theory can be applied provided the height-to-depth and height-to-wavelength ratios remain “small” and, in practice, up to the point of breaking.

(c) (i) No current.

Given:

$$h = 35 \text{ m}$$
$$A = 0.8 \text{ m} \quad (H = 1.6 \text{ m})$$
$$T = 9 \text{ s}$$

Then,

$$\omega = \frac{2\pi}{T} \quad = 0.6981 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 1.739 = kh \tanh kh$$

Iterate as

$$kh = \frac{1.739}{\tanh kh}$$

to get

$$kh = 1.831$$

$$k = \frac{1.831}{h} \quad = 0.05231\ \text{m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 120.1\ \text{m}$$

$$c = \frac{\omega}{k}\ \left(\text{or}\ \frac{L}{T}\right) \quad = 13.35\ \text{m s}^{-1}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.5941$$

$$P = \frac{1}{8}\rho g H^2 (nc) = \frac{1}{8} \times 1025 \times 9.81 \times 1.6^2 \times (0.5941 \times 13.35) \quad = 25520\ \text{W m}^{-1}$$

Answer: wavelength = 120 m;   power = 25.5 kW per metre of crest.

(ii) With current –0.5 m s^{-1}.

$$\omega_a = 0.6981\ \text{rad s}^{-1}$$

Dispersion relation:

$$(\omega_a - kU)^2 = \omega_r^2 \ = gk \tanh kh$$

Rearrange as

$$k = \frac{(\omega_a - kU)^2}{g \tanh kh}$$

or here:

$$k = \frac{(0.6981 + 0.5k)^2}{9.81 \tanh 35k}$$

to get

$$k = 0.05592\ \text{m}^{-1}$$

$$kh = 1.957$$

$$L = \frac{2\pi}{k} \quad = 112.4\ \text{m}$$

$$\omega_r = \omega_a - kU \quad = 0.6981 + 0.05592 \times 0.5 \quad = 0.7261\ \text{rad s}^{-1}$$

$$c_r = \frac{\omega_r}{k} \quad = 12.98\ \text{m s}^{-1}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.5782$$

$$P = \frac{1}{8}\rho g H^2 (nc_r) = \frac{1}{8} \times 1025 \times 9.81 \times 1.6^2 \times (0.5782 \times 12.98) = 24150\ \text{W m}^{-1}$$

(Since power comes from integrating the product of wave fluctuating properties pu to get rate of working, the group velocity here is that in a frame moving with the current; i.e. the 'r' suffix).

Answer: wavelength = 112 m;   power = 24.2 kW per metre of crest.

(d) For the heading change (refraction), find the new wavenumber and then apply Snell's Law. For the wavenumber at $h = 5$ m depth, ω as in part c(i), but

$$\frac{\omega^2 h}{g} = 0.2484 = kh \tanh kh$$

This is small, so iterate as

$$kh = \frac{1}{2}\left(kh + \frac{0.2484}{\tanh kh}\right)$$

to get

$$kh = 0.5200$$

$$k = \frac{0.5200}{h} \quad = 0.104 \text{ m}^{-1}$$

From Snell's Law:

$$k \sin\theta \ = k_1 \sin\theta_1$$

Hence:

$$\sin\theta = \frac{k_1}{k}\sin\theta_1 \quad = \frac{0.05231}{0.104}\sin 25° \quad = 0.2126$$

$$\theta = 12.27°$$

For the shoaling, the shoreward component of power is constant; i.e.

$$P\cos\theta = P_1 \cos\theta_1$$

Hence,

$$P = P_1 \frac{\cos\theta_1}{\cos\theta} \quad = 25520 \frac{\cos 25°}{\cos 12.27°} \quad = 23020 \text{ W m}^{-1}$$

Answer: heading = 12.3°;   power = 23.0 kW per metre of crest.

Q25.
(a) Given

$$h = 24 \text{ m}$$

$$T = 6 \text{ s}$$

Then,

$$\omega = \frac{2\pi}{T} \quad = 1.047 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 2.682 = kh \tanh kh$$

Iterate as

$$kh = \frac{2.682}{\tanh kh}$$

to get

$$kh = 2.706$$

$$k = \frac{2.706}{h} \quad = 0.1128 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 55.70 \text{ m}$$

$$c = \frac{\omega}{k} \left(\text{or } \frac{L}{T}\right) \quad = 9.282 \text{ m s}^{-1}$$

Answer: wavelength 55.7 m, phase speed 9.28 m s^{-1}.

(b)
(i) From the velocity potential

$$\phi = \frac{Ag}{\omega} \frac{\cosh k(h+z)}{\cosh kh} \sin(kx - \omega t)$$

then the dynamic pressure is

$$p = -\rho \frac{\partial \phi}{\partial t} \quad = \rho g A \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t)$$

(ii) At any particular depth the amplitude of the pressure variation is

$$\rho g A \frac{\cosh k(h+z)}{\cosh kh}$$

and hence, from the given conditions at $z = -22.5$ m,

$$1025 \times 9.81A \times \frac{\cosh(0.1128 \times 1.5)}{\cosh 2.706} = 1220$$

Hence,

$$A = 0.8993 \text{ m}$$

$$H = 2A \quad = 1.799 \text{ m}$$

Answer: 1.80 m.

(iii) From the velocity potential

$$\phi = \frac{Ag}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx - \omega t)$$

the horizontal velocity is

$$u = \frac{\partial \phi}{\partial x} \quad = \frac{Agk}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx - \omega t)$$

The maximum horizontal velocity occurs at the surface ($z = 0$), and is

$$u_{\text{max}} = \frac{Agk}{\omega} \quad = \frac{0.8993 \times 9.81 \times 0.1128}{1.047} \quad = 0.9505 \text{ m s}^{-1}$$

Answer: 0.950 m s^{-1}.

(c) *Kinematic boundary condition* – the boundary is a material surface (i.e. always composed of the same particles), or, equivalently, there is no flow through the boundary.

Dynamic boundary condition – stress (here, pressure) is continuous across the boundary.

(d) Determine the wave period that would result in the same absolute period if this were recorded in the presence of a uniform flow of 0.6 m s^{-1} in the wave direction.

With current 0.6 m s^{-1}.

$$\omega_a = 1.047 \text{ rad s}^{-1}$$

Dispersion relation:

$$(\omega_a - kU)^2 = \omega_r^2 \; = gk\tanh kh$$

Rearrange as

$$k = \frac{(\omega_a - kU)^2}{g\tanh kh}$$

or here:

$$k = \frac{(1.047 - 0.6k)^2}{9.81 \tanh 24k}$$

to get

$$k = 0.1008 \text{ m}^{-1}$$

$$\omega_r = \omega_a - kU \quad = 1.047 - 0.1008 \times 0.6 \quad = 0.9865 \text{ rad s}^{-1}$$

$$T_r = \frac{2\pi}{\omega_r} \quad = 6.369 \text{ s}$$

Answer: 6.37 s.

Q26.
(a) Given:

$$h = 1.2 \text{ m}$$
$$T = 1.5 \text{ s}$$

Then,

$$\omega = \frac{2\pi}{T} \qquad = 4.189 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 2.147 = kh \tanh kh$$

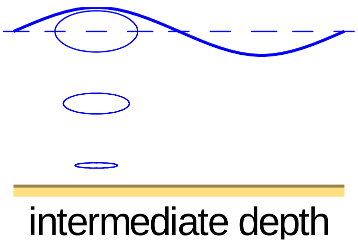
Iterate as

$$kh = \frac{2.147}{\tanh kh}$$

to get

$$kh = 2.200$$

These are intermediate-depth waves – particle trajectories sketched below.



(b) From

$$w = \frac{Agk}{\omega} \frac{\sinh k(h+z)}{\cosh kh} \sin(kx - \omega t)$$

we have

$$a_z = \frac{\partial w}{\partial t} + \text{non} - \text{linear terms} \qquad = -Agk \frac{\sinh k(h+z)}{\cosh kh} \cos(kx - \omega t)$$

At the free surface ($z = 0$) the amplitude of the vertical acceleration is

$$a_{z,\max} = Agk \tanh kh$$

Substituting the dispersion relation $\omega^2 = gk \tanh kh$,

$$a_{z,\max} = A\omega^2$$

Here,

$$k = \frac{2.200}{h} \quad = 1.833 \text{ m}^{-1}$$

Hence

$$A = \frac{a_{z,\max}}{\omega^2} \quad = \frac{0.88}{4.189^2} \quad = 0.05015 \text{ m}$$

$$H = 2A \quad = 0.1003$$

Also

$$L = \frac{2\pi}{k} \quad = 3.428 \text{ m}$$

$$c = \frac{\omega}{k} \quad = 2.285 \text{ m s}^{-1}$$

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right] \quad = 0.5540$$

$$P = \frac{1}{8}\rho g H^2 nc = \frac{1}{8} \times 1025 \times 9.81 \times 0.1003^2 \times 0.5540 \times 2.285 = 16.01 \text{ W m}^{-1}$$

Answer: wavelength = 3.43 m; wave height = 0.100 m; power = 16.0 W per metre crest.

(c)
With current –0.3 m s^{-1}.

$$\omega_a = 4.189 \text{ rad s}^{-1}$$

Dispersion relation:

$$(\omega_a - kU)^2 = \omega_r^2 \;= gk \tanh kh$$

Rearrange as

$$k = \frac{(\omega_a - kU)^2}{g \tanh kh}$$

or here:

$$k = \frac{(4.189 + 0.3k)^2}{9.81 \tanh(1.2k)}$$

to get

$$k = 2.499 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 2.514 \text{ m}$$

Answer: 2.51 m.

(d) Spilling breakers occur for steep waves and/or mildly-sloped beaches. Waves gradually dissipate energy as foam spills down the front faces.

Miche criterion in deep water ($\tanh kh \to 1$):

$$\frac{H_b}{L} = 0.14$$

where, in deep water, and with period $T = 1$ s:

$$L = \frac{gT^2}{2\pi} \quad = 1.561 \text{ m}$$

Hence,

$$H_b = 0.14 \times 1.561 \quad = 0.2185 \text{ m}$$

Answer: 0.219 m.

Q27.
(a) Period is unchanged:

$$T = 5.5 \text{ s}$$

Deep-water wavelength

$$L_0 = \frac{gT^2}{2\pi} \quad = 47.23 \text{ m}$$

Answer: period = 5.5 s; wavelength = 47.2 m.

(b)
Wave properties:

$$h = 4 \text{ m}$$

$$\omega = \frac{2\pi}{T} \quad = 1.142 \text{ rad s}^{-1}$$

The dispersion relation is

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 0.5318 = kh \tanh kh$$

Since the LHS is small this may be iterated as

$$kh = \frac{1}{2}(kh + \frac{0.5318}{\tanh kh})$$

to give

$$kh = 0.8005$$

Hydrodynamic pressure:

Crest: $p_1 = 0$

Still-water line: $p_2 = \rho g H \quad = 1025 \times 9.81 \times 0.75 \quad = 7541 \text{ Pa}$

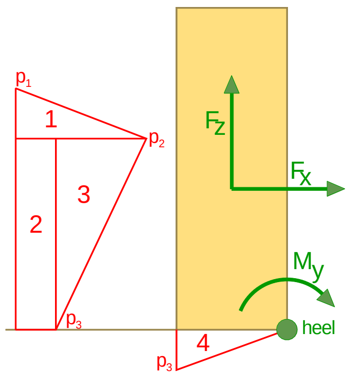
Bed: $P_3 = \dfrac{\rho g H}{\cosh kh} \quad = \dfrac{p_2}{\cosh 0.8005} \quad = 5637 \text{ Pa}$

(c) Decompose the pressure forces on the breakwater as shown. Relevant dimensions are:

$$h = 4 \text{ m}$$

$$H = 0.75 \text{ m}$$

$$b = 3 \text{ m}$$



Region	Force, F_x or F_z	Moment arm
1	$F_{x1} = \frac{1}{2}p_2 \times H \quad = 2828 \text{ N/m}$	$z_1 = h + \frac{1}{3}H \quad = 4.25 \text{ m}$
2	$F_{x2} = p_3 \times h \quad = 22548 \text{ N/m}$	$z_2 = \frac{1}{2}h \quad = 2 \text{ m}$
3	$F_{x3} = \frac{1}{2}(p_2 - p_3) \times h \quad = 3808 \text{ N/m}$	$z_3 = \frac{2}{3}h \quad = 2.667 \text{ m}$
4	$F_{z4} = \frac{1}{2}p_3 \times b \quad = 8456 \text{ N/m}$	$z_4 = \frac{2}{3}b \quad = 2 \text{ m}$

The sum of all clockwise moments about the heel:

$$F_{x1}z_1 + F_{x2}z_2 + F_{x3}z_3 + F_{z4}x_4 = 84180 \text{ N m/m}$$

Answer: 84.2 kN m per metre of breakwater.

Q28.
(a)

$$h = 23 \text{ m}$$

For deep water:

$$kh > \pi$$

$$\Rightarrow \qquad k > \frac{\pi}{h} \quad = 0.1366 \text{ m}^{-1}$$

$$\Rightarrow \qquad L < \frac{2\pi}{0.1366} \quad = 46.00 \text{ m}$$

Also, from the dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \omega > \sqrt{9.81 \times 0.1366 \times \tanh \pi} \quad = 1.155 \text{ rad s}^{-1}$$

$$\Rightarrow \qquad T < \frac{2\pi}{1.155} \quad = 5.440 \text{ s}$$

Answer: largest wavelength = 46 m; largest period = 5.44 s.

(b)
(i) Given

$$h = 23 \text{ m}$$
$$T = 9 \text{ s}$$

Then,

$$\omega = \frac{2\pi}{T} \quad = 0.6981 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 1.143 = kh \tanh kh$$

Iterate as

$$kh = \frac{1.143}{\tanh kh}$$

to get

$$kh = 1.319$$

$$k = \frac{1.319}{h} \quad = 0.05735 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 109.6 \text{ m}$$

Answer: wavenumber = 0.0573 m^{-1};   wavelength = 110 m.

(ii) From the velocity potential

$$\phi = \frac{Ag}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx - \omega t)$$

the horizontal velocity is

$$u = \frac{\partial \phi}{\partial x} \quad = \frac{Agk}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx - \omega t)$$

Hence, the amplitude of the horizontal particle velocity at height z is

$$u_{\max} = \frac{Agk}{\omega}\frac{\cosh k(h+z)}{\cosh kh}$$

From the given data, $u_{\max} = 0.31 \text{ m s}^{-1}$ when $z = -20$ m,

$$A = \frac{\omega u_{\max}}{gk}\frac{\cosh kh}{\cosh k(h+z)} = \frac{0.6981 \times 0.31}{9.81 \times 0.05735} \times \frac{\cosh 1.319}{\cosh[0.05735 \times 3]} = 0.7594 \text{ m}$$

$$H = 2A \quad = 1.519 \text{ m}$$

Answer: 1.52 m.

(iii) At $z = 0$,

$$u_{\max} = \frac{Agk}{\omega} \quad = \frac{0.7594 \times 9.81 \times 0.05735}{0.6981} \quad = 0.6120 \text{ m s}^{-1}$$

The wave speed is

$$c = \frac{\omega}{k} \quad = \frac{0.6981}{0.05735} \quad = 12.17 \text{ m s}^{-1}$$

Answer: particle velocity = 0.612 m s^{-1}, much less than the wave speed.

Q29.
Given

$$H_s = 1.5 \text{ m}$$

$$U = 14 \text{ m s}^{-1}$$

$$t = 6 \times 3600 \quad = 21600 \text{ s}$$

then

$$\hat{t} \equiv \frac{gt}{U} \quad = 15140$$

If duration-limited then this would imply an effective fetch related to time t by the non-dimensional relation

$$\hat{t} = 68.8\hat{F}_{\text{eff}}^{2/3}$$

or

$$\hat{F}_{\text{eff}} = \left(\frac{\hat{t}}{68.8}\right)^{3/2} \quad = 3264$$

and consequent wave height

$$\frac{gH_s}{U^2} \equiv \hat{H}_s \quad = 0.0016\hat{F}_{\text{eff}}^{1/2} \quad = 0.09141$$

$$H_s = 0.09141 \times \frac{U^2}{g} \quad = 1.826$$

But the actual wave height is less than this, so it is limited by fetch, not duration.

Q30.

(a) *Kinematic boundary condition* – the boundary is a material surface (i.e. always composed of the same particles), or, equivalently, there is no flow through the boundary.

Dynamic boundary condition – stress (here, pressure) is continuous across the boundary.

(b)

(i) Given:

$$h = 20 \text{ m}$$
$$T = 8 \text{ s}$$

Then,

$$\omega = \frac{2\pi}{T} \quad = 0.7854 \text{ rad s}^{-1}$$

Dispersion relation:

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

$$\Rightarrow \qquad 1.258 = kh \tanh kh$$

Iterate as

$$kh = \frac{1.258}{\tanh kh}$$

to get

$$kh = 1.416$$

$$k = \frac{1.416}{h} \quad = 0.07080 \text{ m}^{-1}$$

$$L = \frac{2\pi}{k} \quad = 88.75 \text{ m}$$

$$c = \frac{\omega}{k} \quad \left(\text{or } \frac{L}{T}\right) \quad = 11.09 \text{ m s}^{-1}$$

Answer: wavelength = 88.7 m; speed = 11.1 m s^{-1}.

(ii) From the velocity potential

$$\phi = \frac{Ag}{\omega} \frac{\cosh k(h+z)}{\cosh kh} \sin(kx - \omega t)$$

then the dynamic pressure is

$$p = -\rho \frac{\partial \phi}{\partial t} \quad = \rho g A \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t)$$

From the given magnitude of the pressure fluctuations at the bed ($z = -h$):

$$\frac{\rho g A}{\cosh kh} = 6470$$

Hence

$$A = \frac{6470 \times \cosh kh}{\rho g} \quad = \frac{6470 \times \cosh 1.416}{1025 \times 9.81} \quad = 1.404 \text{ m}$$

$$H = 2A \quad = 2.808 \text{ m}$$

Answer: wave height = 2.81 m.

(c)

(i) For $T < 5$ s,

$$\omega = \frac{2\pi}{T} \quad > 1.257 \text{ rad s}^{-1}$$

At the limiting value on the RHS,

$$\frac{\omega^2 h}{g} = 3.221$$

Solving the dispersion relationship in the form

$$\frac{\omega^2 h}{g} = kh \tanh kh$$

by iteration:

$$kh = \frac{3.221}{\tanh kh}$$

produces

$$kh = 3.220$$

This is a deep-water wave ($kh > \pi$), hence negligible wave dynamic pressure is felt at the bed.

(ii) From the velocity potential

$$\phi = \frac{Ag}{\omega} \frac{\cosh k(h+z)}{\cosh kh} \sin(kx - \omega t)$$

the horizontal velocity is

$$u = \frac{\partial \phi}{\partial x} \quad = \frac{Agk}{\omega} \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t)$$

and the particle acceleration is

$$a_x = \frac{\partial u}{\partial t} + \text{non} - \text{linear terms} \quad = Agk \frac{\cosh k(h+z)}{\cosh kh} \sin(kx - \omega t)$$

and the amplitude of horizontal acceleration at $z = 0$ is

$$Agk$$

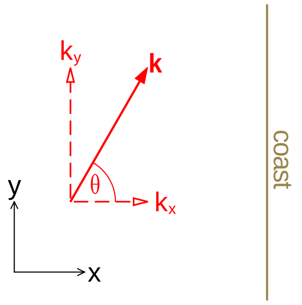
Q31.

(a) The irrotationality condition (given in that year's exam paper) is

$$\frac{\partial k_y}{\partial x} - \frac{\partial k_x}{\partial y} = 0$$

Wave behaviour is the same all the way along the coast; hence $\partial/\partial y = 0$ for all variables. This leaves

$$\frac{\partial k_y}{\partial x} = 0$$



But

$$k_y = k \sin\theta$$

(see diagram). Hence

$$k \sin\theta = \text{constant}$$

or

$$(k \sin\theta)_1 = (k \sin\theta)_2$$

for two locations on a wave ray.

(b)

$$T = 7 \text{ s}$$

Hence

$$\omega = \frac{2\pi}{T} \quad = 0.8976 \text{ rad s}^{-1}$$

The dispersion relation is

$$\omega^2 = gk \tanh kh$$

$$\Rightarrow \qquad \frac{\omega^2 h}{g} = kh \tanh kh$$

This may be iterated as either

$$kh = \frac{\omega^2 h/g}{\tanh kh} \qquad \text{or} \qquad kh = \frac{1}{2}(kh + \frac{\omega^2 h/g}{\tanh kh})$$

	$h = 5$ m	$h = 28$ m
$\frac{\omega^2 h}{g}$	0.4106	2.300
Iteration:	$kh = \frac{1}{2}\left(kh + \frac{0.4106}{\tanh kh}\right)$	$kh = \frac{2.300}{\tanh kh}$
kh	0.6881	2.343
k	0.1376 m^{-1}	0.08368 m^{-1}

$c = \dfrac{\omega}{k}$	6.523 m s^{-1}	10.73 m s^{-1}
$n = \dfrac{1}{2}\left[1 + \dfrac{2kh}{\sinh 2kh}\right]$	0.8712	0.5432
θ	?	35°
H	?	1.2 m

Refraction:

$$(k \sin\theta)_{5\text{ m}} = (k \sin\theta)_{28\text{ m}}$$

$$0.1376 \sin\theta = 0.08368 \sin 35°$$

Hence,

$$\theta = 20.41°$$

From the shoaling equation:

$$(H^2 nc \cos\theta)_{5\text{ m}} = (H^2 nc \cos\theta)_{28\text{ m}}$$

Hence:

$$H_{5\text{ m}} = H_{28\text{ m}} \sqrt{\frac{(nc\cos\theta)_{28\text{ m}}}{(nc\cos\theta)_{5\text{ m}}}} = 1.2 \times \sqrt{\frac{0.5432 \times 10.73 \times \cos 35°}{0.8712 \times 6.523 \times \cos 20.41°}} = 1.136\text{ m}$$

The power per metre of crest at depth 5 m is

$$P = \frac{1}{8}\rho g H^2 (nc) = \frac{1}{8} \times 1025 \times 9.81 \times 1.136^2 \times (0.8712 \times 6.523) = 9218\text{ W m}^{-1}$$

Answer: direction 20.4°; wave height 1.14 m; power = 9.22 kW m^{-1}.

(c) Given

$$T = 7\text{ s}$$

and at depth 5 m:

$$H = 2.8\text{ m}$$

$$\theta = 0°$$

Breaker Height

Breaker height index (on formula sheet):

$$H_b = 0.56 H_0 \left(\frac{H_0}{L_0}\right)^{-1/5}$$

First find deep-water conditions H_0 and L_0. By shoaling (with no refraction):

$$(H^2nc)_0 = (H^2nc)_{5\text{ m}}$$

In deep water,

$$n_0 = \frac{1}{2}$$

$$c_0 = \frac{gT}{2\pi} \quad = 10.93\text{ m s}^{-1}$$

whilst n and c at depth 5 m come from part (b). Hence,

$$H_0^2 \times 0.5 \times 10.93 = 2.8^2 \times 0.8712 \times 6.523$$

whence

$$H_0 = 2.855\text{ m}$$

Also in deep water,

$$L_0 = \frac{gT^2}{2\pi} \quad = 76.50\text{ m}$$

Then from the formula for breaker height:

$$H_b = 0.56H_0\left(\frac{H_0}{L_0}\right)^{-1/5} \quad = 3.086\text{ m}$$

<u>Breaking Depth</u>

From the formula sheet the breaker depth index is

$$\gamma_b \equiv \left(\frac{H}{h}\right)_b = b - a\frac{H_b}{gT^2}$$

where, with a beach slope $m = 1/40 = 0.025$:

$$a = 43.8(1 - \mathrm{e}^{-19m}) \quad = 43.8(1 - \mathrm{e}^{-19\times0.025}) \quad = 16.56$$

$$b = \frac{1.56}{1 + \mathrm{e}^{-19.5m}} \quad = \frac{1.56}{1 + \mathrm{e}^{-19.5\times0.025}} \quad = 0.9664$$

Hence, from the breaker depth index:

$$\frac{3.086}{h_b} = 0.9664 - 16.56 \times \frac{3.086}{9.81 \times 7^2}$$

giving depth of water at breaking:

$$h_b = 3.588\text{ m}$$

Answer: breaker height = 3.09 m; breaking depth = 3.59 m.