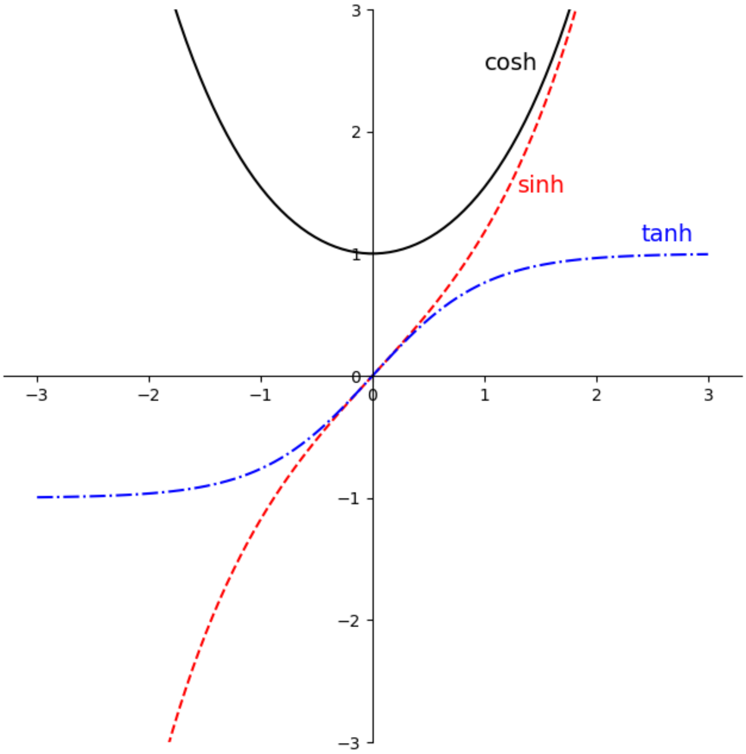
A1. Hyperbolic Functions

Definitions:

$$\sinh(x) \equiv \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) \equiv \frac{e^x + e^{-x}}{2}$$

$$\tanh x \equiv \frac{\sinh x}{\cosh x}$$



Basic Formulae

These bear more than a passing resemblance to those for trigonometric functions (with corresponding definitions for sech, cosech and coth), but beware of signs:

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x \quad = 2\cosh^2 x - 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

Derivatives

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

Asymptotic Behaviour

Small x: $\sinh x \sim \tanh x \sim x, \quad \cosh x \to 1$ as $x \to 0$

Large x: $\sinh x \sim \cosh x \sim \frac{1}{2}e^x, \quad \tanh x \to 1$ as $x \to \infty$

A2. Fluid-Flow Equations

To start, we need mathematical equations for three fluid-dynamical principles: continuity, irrotationality (or, equivalently, use of a velocity potential) and the time-dependent Bernoulli equation (mechanical energy). In two dimensions, with coordinates (x, z) and velocity (u, w), and assuming incompressible and inviscid flow, these are:

(i) Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

(ii) Irrotationality:

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \qquad \text{or} \qquad u = \frac{\partial \phi}{\partial x}\,, \quad w = \frac{\partial \phi}{\partial z}$$

where ϕ is a *velocity potential*.

(iii) Time-dependent Bernoulli equation:

$$\rho\frac{\partial \phi}{\partial t} + p + \frac{1}{2}\rho U^2 + \rho g z = C(t), \quad \text{along a streamline}$$

where U is the speed, or magnitude of velocity: $U^2 = u^2 + w^2$.

Heuristic justifications for these follow.

<u>Continuity</u>

Consider flow in the $x - z$ plane and flow through the sides of a small rectangular control volume, with sides Δx, Δz. In incompressible flow, the net volume flux out of this volume is zero.

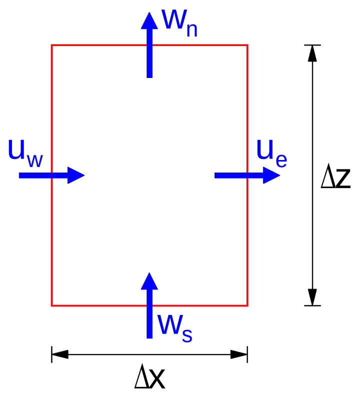
Since the volume flux through any surface is (velocity) × (area) then, per unit depth normal to this plane, and with the notation in the figure:

$$\text{net volume outflow} = u_e\Delta z - u_w\Delta z + w_n\Delta x - w_s\Delta x = 0$$

Dividing by area $\Delta x \Delta z$:

$$\frac{u_e - u_w}{\Delta x} + \frac{w_n - w_s}{\Delta z} = 0$$

$$\frac{\Delta u}{\Delta x} + \frac{\Delta w}{\Delta z} = 0$$



In the limit as $\Delta x, \Delta z \to 0$:

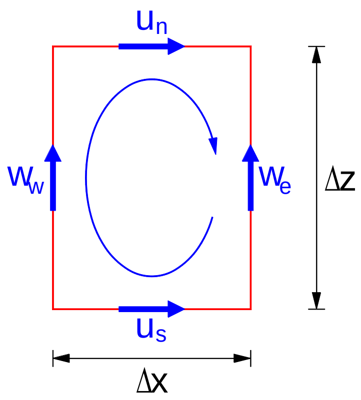
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Irrotationality

Irrotationality is a consequence of the inviscid approximation. In the absence of viscous forces (and density gradients) the only forces acting on a fluid element are pressure and a constant gravity force. The former acts perpendicular to the boundary of the element and the latter is constant, so neither can impart rotation (or "circulation"). Hence (see figure):

$$\text{circulation}(\equiv \text{line integral of velocity}) = 0$$

For our 2-d volume, and working clockwise from the top edge:



$$u_n\Delta x - w_e\Delta z - u_s\Delta x + w_w\Delta z = 0$$

Dividing by area $\Delta x\Delta z$:

$$\frac{u_n - u_s}{\Delta z} - \frac{w_e - w_w}{\Delta x} = 0$$

$$\frac{\Delta u}{\Delta z} - \frac{\Delta w}{\Delta x} = 0$$

In the limit as $\Delta x, \Delta z \to 0$:

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \qquad (1)$$

If we define the *velocity potential* ϕ as the line integral of velocity from an arbitrary reference point to (x,z), then ϕ is well-defined because the no-circulation condition makes this independent of route. In particular, considering the change from bottom-left to top-right of our representative control volume, by either route, and working directly in infinitesimals:

$$\mathrm{d}\phi = u\,\mathrm{d}x + w\,\mathrm{d}z$$

But, comparing the expansion for any 2-d function:

$$\mathrm{d}\phi = \frac{\partial\phi}{\partial x}\,\mathrm{d}x + \frac{\partial\phi}{\partial z}\,\mathrm{d}z$$

we have

$$u = \frac{\partial\phi}{\partial x}\,, \qquad w = \frac{\partial\phi}{\partial z}$$

As a check, substitute in the LHS of (1) to get

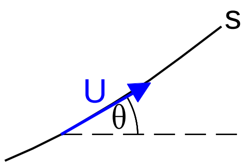
$$\frac{\partial^2\phi}{\partial z\,\partial x} - \frac{\partial^2\phi}{\partial x\,\partial z}$$

which vanishes by the symmetry of the second derivatives.

It is convenient to write all flow variables $(u,\ w,\ p)$ in terms of ϕ, because we then only have one equation to solve. However, the ability to do so relies on the assumption of irrotationality (here, due to the inviscid approximation) which does not hold in, for example, boundary layers.

Time-Dependent Bernoulli Equation

Writing "mass × acceleration = force", per unit volume, and in the direction of a streamline:



$$\rho\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial s}\right)=-\frac{\partial p}{\partial s}-\rho g\sin\theta$$

where U is the magnitude of velocity, s is the distance along a streamline and θ is the angle between streamline and horizontal.

From trigonometry, $\sin\theta=\Delta z/\Delta s$, or, along a streamline, $\sin\theta=\partial z/\partial s$, whilst, in terms of the velocity potential, $U=\partial\phi/\partial s$. Hence, after rearranging (and assuming ρ constant):

$$\frac{\partial}{\partial s}\left[\rho\frac{\partial\phi}{\partial t}+\tfrac{1}{2}\rho U^2+p+\rho gz\right]=0$$

Hence,

$$\rho\frac{\partial\phi}{\partial t}+p+\frac{1}{2}\rho U^2+\rho gz=C(t),\qquad \text{along any particular streamline}$$

We will use the time-dependent version here. However, note that, in the time-steady case, this reduces to the "normal" Bernoulli equation:

$$p+\rho gz+\frac{1}{2}\rho U^2=\text{constant, along a streamline}$$

3-d Generalisations (Not Examinable)

The above are very hand-waving arguments. Far better (but more mathematical) derivations can be found in any good fluid-mechanics textbook. There are also 3-d generalisations:

Continuity:

$$\nabla\bullet\mathbf{u}\equiv\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Irrotationality:

$$\nabla\wedge\mathbf{u}\equiv\begin{pmatrix}\dfrac{\partial w}{\partial y}-\dfrac{\partial v}{\partial z}\\[2ex] \dfrac{\partial u}{\partial z}-\dfrac{\partial w}{\partial x}\\[2ex] \dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\end{pmatrix}=\mathbf{0}$$

$\nabla\wedge\mathbf{u}$ is called *vorticity*. In this course we are only using the y component (i.e. the circulation in the $x-z$ plane).

In terms of the velocity potential:

$$\mathbf{u} = \nabla\phi = \begin{pmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \phi}{\partial z} \end{pmatrix}$$

The time-dependent Bernoulli equation is unchanged.

A3. Derivation of Wave Field and Dispersion Equation

A3.1 Laplace's Equation For the Velocity Potential

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

A3.2 Boundary Conditions

Boundary conditions are applied at the bed and the free surface. These are of two types:

- *kinematic* (no net flow through a boundary);
- *dynamic* (stress is continuous at a boundary).

<u>A3.2.1 Kinematic Boundary Conditions</u>

The condition that the curve

$$z = z_{\text{surf}}(x, t)$$

be a material surface (i.e. that the particles constituting it are always the same; or, there is no net flow through it) is that the *total* derivative following the flow (u, w) satisfies

$$\frac{\mathrm{D}}{\mathrm{D}t}(z - z_{\text{surf}}) = 0 \qquad \text{on} \qquad z = z_{\text{surf}}(x, t)$$

where, in 2 dimensions,

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}$$

Expanding and rearranging:

$$w = \frac{\partial z_{\text{surf}}}{\partial t} + u\frac{\partial z_{\text{surf}}}{\partial x} \qquad \text{on} \qquad z = z_{\text{surf}}(x, t)$$

This gives the following kinematic boundary conditions.

<u>Bed (KBBC – Kinematic Bed Boundary Condition)</u>

Assuming the bed is horizontal and rigid:

$$w = 0 \qquad \text{on} \qquad z = -h$$

<u>Free Surface (KFSBC – Kinematic Free-Surface Boundary Condition)</u>

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} \qquad \text{on} \qquad z = \eta(x, t)$$

A3.2.2 Dynamic Boundary Conditions

The stress on either side of a surface must be continuous (or the particles constituting the surface would have infinite acceleration).

Free Surface (DFSBC – Dynamic Free-Surface Boundary Condition)

Neglecting viscosity and surface tension, pressure must be atmospheric at the free surface. Hence, in terms of gauge pressure:

$$p = 0 \quad \text{on} \quad z = \eta(x, t)$$

Taken together with Bernoulli's equation, and noting that the free surface is a streamline,

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho U^2 + \rho g \eta = C(t) \quad \text{on} \quad z = \eta(x, t)$$

No dynamic boundary condition is applied at the bed, which can absorb any pressure distribution impressed on it.

A3.3 Linearised Equations

When an expression is expanded in terms of a small quantity ε it is common to drop quadratic and higher terms. i.e. if

$$y = a + b\varepsilon + c\varepsilon^2 + \cdots$$

then it is common to neglect terms in ε^2 and greater, so leaving the *linear* terms only:

$$y = a + b\varepsilon$$

Here, we assume that the wave amplitude A is small (compared with depth h and wavelength L) and hence neglect products and powers above the first of wave-related perturbations. Moreover, boundary conditions at $z = \eta(x, t)$ can effectively be applied at $z = 0$.

With this assumption, the governing equations are the following.

Laplace's equation for the velocity potential is already linear:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

The linearised boundary conditions, written in terms of the velocity potential ϕ are:
KBBC:

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{on} \qquad z = -h$$

KFSBC:

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{on} \qquad z = 0$$

DFSBC:

$$\frac{\partial \phi}{\partial t} + g\eta = C(t) \qquad \text{on} \qquad z = 0$$

A3.4 Solution for a Sinusoidal Wave

The linearised equations above are to be solved for a surface displacement that is a single harmonic wave component, noting that, under linearity assumptions, the general solution would be a sum over individual wave components:

$$\eta = A\cos(kx - \omega t)$$

Under appropriate boundary conditions, Laplace's equation has unique solutions. Look for *separable* solutions:

$$\phi = X(x, t)Z(z)$$

From the KFSBC:

$$X\left.\frac{\mathrm{d}Z}{\mathrm{d}z}\right|_{z=0} = A\omega\sin(kx - \omega t)$$

Hence, $X \propto \sin(kx - \omega t)$. WLOG, we can take the constant of proportionality as 1 and absorb the remainder of the multiplier into Z. Hence,

$$X = \sin(kx - \omega t)$$

and

$$\left.\frac{\mathrm{d}Z}{\mathrm{d}z}\right|_{z=0} = A\omega$$

From Laplace's equation:

$$-k^2XZ + X\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = 0$$

or

$$\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = k^2Z$$

This has general solution:

$$Z = \alpha\mathrm{e}^{kz} + \beta\mathrm{e}^{-kz}$$

From the KFSBC above and the KBBC:

$$\frac{\mathrm{d}Z}{\mathrm{d}z} = A\omega \quad \text{on} \quad z = 0$$

$$\frac{\mathrm{d}Z}{\mathrm{d}z} = 0 \quad \text{on} \quad z = -h$$

These fix α and β to give, after some messy algebra,

$$Z = \frac{A\omega}{k}\frac{\cosh k(h+z)}{\sinh kh}$$

The solution for the whole potential is then

$$\phi = \frac{A\omega}{k}\frac{\cosh k(h+z)}{\sinh kh}\sin(kx-\omega t)$$

Finally, we have to satisfy the DFSBC. This will gives a relationship between frequency ω and wavenumber k (and hence between period T and wavelength L):

$$-\frac{A\omega^2}{k}\frac{\cosh kh}{\sinh kh}\cos(kx-\omega t) + Ag\cos(kx-\omega t) = C(t)$$

The LHS has zero spatial mean; hence the RHS must be identically zero. Hence, after dividing out $A\cos(kx-\omega t)$ and rearranging, we arrive at the very famous

Dispersion Relationship:

$$\omega^2 = gk\tanh kh$$

From this last equation,

$$\frac{\omega}{k} = \left(\frac{g}{\omega}\right)\frac{\sinh kh}{\cosh kh}$$

and so the velocity potential can also be written

Velocity Potential:

$$\phi = \frac{Ag}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx-\omega t)$$

Once the velocity potential and dispersion relation are known, the velocity, pressure and other quantities follow immediately.

Velocity

$$u \equiv \frac{\partial\phi}{\partial x} \quad = \frac{Agk}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx-\omega t)$$

$$w \equiv \frac{\partial\phi}{\partial z} \quad = \frac{Agk}{\omega}\frac{\sinh k(h+z)}{\cosh kh}\sin(kx-\omega t)$$

Pressure

From the linearised form of Bernoulli's equation,

$$\begin{aligned} p &= -\rho g z - \rho \frac{\partial \phi}{\partial t} \\ &= -\rho g z + \rho g A \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t) \end{aligned}$$

This can also be written

$$p = -\rho g z + \rho g \eta \frac{\cosh k(h+z)}{\cosh kh}$$

where we have used the specified surface displacement, η.

Note:

(i) It is common to decompose the pressure field into *hydrostatic* and *hydrodynamic* components:

$$p = \underbrace{-\rho g z}_{\text{hydrostatic}} + \underbrace{\rho g A \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t)}_{\text{hydrodynamic (i.e. wave)}}$$

(ii) Both u and p demonstrate the characteristic depth dependence

$$\frac{\cosh k(h+z)}{\cosh kh}$$

which varies from 1 at the free surface ($z = 0$) to

$$\frac{1}{\cosh kh}$$

at the bed ($z = -h$). If kh is large (say, $kh > \pi$) then wave disturbances do not reach the bed.

A4. Wave Kinetic and Potential Energy

Kinetic Energy

Wave kinetic energy (per unit horizontal area) is, integrating over the water column:

$$\text{KE} = \frac{1}{2}\rho\int_{z=-h}^{\eta}(u^2 + w^2)\,\mathrm{d}z$$

The upper limit may be taken as 0 rather than η because of the linear approximation.

From the solution for the velocity components

$$u^2 + w^2 = \left(\frac{Agk}{\omega\cosh kh}\right)^2\{\cosh^2 k(h+z)\cos^2(kx-\omega t) + \sinh^2 k(h+z)\sin^2(kx-\omega t)\}$$

The average value of both $\cos^2(kx-\omega t)$ and $\sin^2(kx-\omega t)$ over a wave cycle is ½. Hence, the *average* wave kinetic energy (per unit horizontal area) is

$$\begin{aligned}
\overline{\text{KE}} &= \frac{1}{2}\rho\left(\frac{Agk}{\omega\cosh kh}\right)^2 \times \frac{1}{2}\int_{-h}^{0}\{\cosh^2 k(h+z) + \sinh^2 k(h+z)\}\mathrm{d}z \\
&= \frac{1}{2}\rho\left(\frac{Agk}{\omega\cosh kh}\right)^2 \times \frac{1}{2}\int_{-h}^{0}\cosh 2k(h+z)\,\mathrm{d}z \\
&= \frac{1}{2}\rho\left(\frac{Agk}{\omega\cosh kh}\right)^2 \times \frac{1}{2}\left[\frac{\sinh 2k(h+z)}{2k}\right]_{-h}^{0} \\
&= \frac{1}{2}\rho\left(\frac{Agk}{\omega\cosh kh}\right)^2 \times \frac{1}{2}\left[\frac{\sinh 2kh}{2k}\right]_{-h}^{0} \\
&= \frac{1}{2}\rho\left(\frac{Agk}{\omega\cosh kh}\right)^2 \times \frac{1}{2} \times \frac{2\sinh kh\cosh kh}{2k} \\
&= \frac{1}{4}\rho\frac{A^2g^2k}{\omega^2}\tanh kh
\end{aligned}$$

But, from the dispersion relation, $\omega^2 = gk\tanh kh$. Hence,

$$\overline{\text{KE}} = \frac{1}{4}\rho g A^2$$

Potential Energy

The potential energy (per unit horizontal area) is

$$\begin{aligned}
\int_{z=-h}^{\eta}\rho g z\,\mathrm{d}z &= \frac{1}{2}\rho g[z^2]_{-h}^{\eta} \\
&= \frac{1}{2}\rho g(\eta^2 - h^2) \\
&= \frac{1}{2}\rho g\eta^2 + \text{constant}
\end{aligned}$$

We are only interested in the wave-varying part of this, and hence the wave potential energy per unit area is

$$\frac{1}{2}\rho g A^2 \cos^2(kx - \omega t)$$

Again, the average value of $\cos^2(kx - \omega t)$ is ½. Hence, the average wave potential energy is

$$\overline{\text{PE}} = \frac{1}{4}\rho g A^2$$

A5. Group Velocity

The group velocity (for *any* wave type) is defined by

$$c_g \equiv \frac{\mathrm{d}\omega}{\mathrm{d}k}$$

The dispersion relationship for gravity waves on still water is

$$\omega^2 = gk \tanh kh$$

Differentiating (implicitly) with respect to k:

$$\begin{aligned} 2\omega \frac{\mathrm{d}\omega}{\mathrm{d}k} &= g \tanh kh + gkh \,\mathrm{sech}^2\, kh \\ &= \frac{\omega^2}{k} + \frac{\omega^2}{\tanh kh} \times \frac{h}{\cosh^2 kh} \\ &= \frac{\omega^2}{k}\left[1 + \frac{kh}{\sinh kh \cosh kh}\right] \end{aligned}$$

Hence, using $\sinh 2x = 2 \sinh x \cosh x$ (note the factor of 2):

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]\frac{\omega}{k}$$

or

$$c_g = nc$$

where

$$c \equiv \frac{\omega}{k}$$

is the phase velocity and

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]$$

is the ratio of group to phase velocities.

A6. Wave Power

The wave power is the (time-averaged) rate of working of pressure forces over a vertical surface. Integrating over the water column, per unit length of wave crest,

$$\text{power} = \int_{z=-h}^{\eta} pu \, \mathrm{d}z$$

We need only consider the hydrodynamic part of p, since the time-steady hydrostatic part will, in conjunction with u, give a component integrating to 0 over a cycle. Using the linear wave theory expressions for p (hydrodynamic) and u,

$$\begin{aligned} pu &= \rho g A \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t) \times \frac{Agk}{\omega} \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \omega t) \\ &= \frac{\rho g^2 A^2 k}{\omega} \frac{\cosh^2 k(h+z)}{\cosh^2 kh} \cos^2(kx - \omega t) \end{aligned}$$

Under the linear approximation, the upper limit of integration may be replaced by $z = 0$. The average value of $\cos^2(kx - \omega t)$ is ½, whilst integrating the z-dependent part over the water column gives

$$\begin{aligned} \int_{-h}^{0} \cosh^2 k(h+z) \, \mathrm{d}z &= \frac{1}{2} \int_{-h}^{0} (\cosh 2k(h+z) + 1) \, \mathrm{d}z \\ &= \frac{1}{2} \left[\frac{\sinh 2k(h+z)}{2k} + z \right]_{-h}^{0} \\ &= \frac{1}{2} \left(\frac{\sinh 2kh}{2k} + h \right) \end{aligned}$$

Hence, the wave power is

$$\begin{aligned} P &= \frac{\rho g^2 k A^2}{\omega \cosh^2 kh} \times \frac{1}{2} \left(\frac{\sinh 2kh}{2k} + h \right) \times \frac{1}{2} \\ &= \frac{1}{2} \rho g A^2 \times \frac{gk}{\omega \cosh^2 kh} \times \frac{\sinh 2kh}{2k} \left(1 + \frac{2kh}{\sinh 2kh} \right) \times \frac{1}{2} \\ &= \frac{1}{2} \rho g A^2 \times \frac{gk}{\omega \cosh^2 kh} \times \frac{2 \sinh kh \cosh kh}{2k} \times \left(1 + \frac{2kh}{\sinh 2kh} \right) \times \frac{1}{2} \\ &= \frac{1}{2} \rho g A^2 \times \left(\frac{gk \tanh kh}{\omega^2} \right) \times \frac{1}{2} \left(1 + \frac{2kh}{\sinh 2kh} \right) \frac{\omega}{k} \end{aligned}$$

But the wave energy density is

$$E = \frac{1}{2} \rho g A^2$$

the dispersion relationship is

$$\omega^2 = gk \tanh kh$$

and the group velocity

$$c_g = nc$$

where

$$n = \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right) , \qquad c = \frac{\omega}{k}$$

Hence, the wave power per unit length of crest is

$$P = Ec_g$$