Q1.
Cheng's formula:

$$\frac{w_s d}{\nu} = \left[(25 + 1.2d^{*2})^{1/2} - 5\right]^{3/2}, \qquad \text{where} \qquad d^* = d\left[\frac{(s-1)g}{\nu^2}\right]^{1/3}$$

Densities:

$\rho_{\text{sand}} = 2650$ kg m^{-3}
$\rho_{\text{air}} = 1.2$ kg m^{-3}
$\rho_{\text{water}} = 1000$ kg m^{-3}

Kinematic viscosities:

$\nu_{\text{air}} = 1.5 \times 10^{-5}$ m^2 s^{-1}
$\nu_{\text{water}} = 1.0 \times 10^{-6}$ m^2 s^{-1}

(a) For sand particles in air:

$$s = \frac{\rho_s}{\rho} \quad = \frac{2650}{1.2} \quad = 2208$$

$$d^* = d\left[\frac{(s-1)g}{\nu^2}\right]^{1/3} \quad = 0.001\left[\frac{(2208-1)\times 9.81}{(1.5\times 10^{-5})^2}\right]^{1/3} \quad = 45.82$$

$$\frac{w_s d}{\nu} = \left[(25 + 1.2d^{*2})^{1/2} - 5\right]^{3/2} \quad = \left[(25 + 1.2 \times 45.82^2)^{1/2} - 5\right]^{3/2} \quad = 306.3$$

$$w_s = 306.3 \times \frac{\nu}{d} \quad = 306.3 \times \frac{1.5 \times 10^{-5}}{10^{-3}} \quad = 4.595 \text{ m s}^{-1}$$

Answer: settling velocity in air = 4.60 m s^{-1}.

(b) For sand particles in water:

$$s = \frac{\rho_s}{\rho} \quad = \frac{2650}{1000} \quad = 2.65$$

$$d^* = d\left[\frac{(s-1)g}{\nu^2}\right]^{1/3} \quad = 0.001\left[\frac{(2.65-1)\times 9.81}{(1.0\times 10^{-6})^2}\right]^{1/3} \quad = 25.30$$

$$\frac{w_s d}{\nu} = \left[(25 + 1.2d^{*2})^{1/2} - 5\right]^{3/2} \quad = \left[(25 + 1.2 \times 25.30^2)^{1/2} - 5\right]^{3/2} \quad = 111.5$$

$$w_s = 111.5 \times \frac{\nu}{d} \quad = 111.5 \times \frac{1.0 \times 10^{-6}}{10^{-3}} \quad = 0.1115 \text{ m s}^{-1}$$

Answer: settling velocity in water = 0.112 m s^{-1}.

Q2.
As in Q1(b) above,

$$d^* = 25.30$$

Soulsby's formula,

$$\tau^*_{\text{crit}} = \frac{0.30}{1 + 1.2d^*} + 0.055\ [1 - \exp(-0.020d^*)] \qquad \text{where} \quad \tau^* = \frac{\tau_b}{(\rho_s - \rho)gd}$$

Hence,

$$\tau^*_{\text{crit}} = \frac{0.30}{1 + 1.2 \times 25.30} + 0.055\ [1 - \exp(-0.020 \times 25.30)] \quad = 0.03141$$

$$\tau_{\text{crit}} = \tau^*_{\text{crit}} \times (\rho_s - \rho)gd \ = 0.03141 \times (2650 - 1000) \times 9.81 \times 0.001 \ = 0.5084 \text{ N m}^{-2}$$

Answer: critical Shields parameter = 0.0314; critical shear stress = 0.508 N m^{-2}.

Q3.
From Q1 and Q2 above, the settling velocity and critical shear stress for incipient motion are given, respectively, by:

$$w_s = 0.1115 \text{ m s}^{-1}$$

$$\tau_{\text{crit}} = 0.5084 \text{ N m}^{-2}$$

(a) The bed shear stress is given, in general, by

$$\tau_b = c_f \left(\frac{1}{2}\rho V^2\right)$$

Rearranging for *V*:

$$V = \sqrt{\frac{2}{c_f}\frac{\tau_b}{\rho}}$$

At incipient motion, $\tau_b = \tau_{\text{crit}}$. Hence,

$$V = \sqrt{\frac{2}{0.005} \times \frac{0.5084}{1000}} \quad = 0.4510 \text{ m s}^{-1}$$

Answer: for incipient motion, velocity = 0.451 m s^{-1}.

(b) For incipient suspended load,

$$u_\tau = w_s$$

where u_τ is the friction velocity and w_s is the settling velocity.

By definition,

$$u_\tau = \sqrt{\frac{\tau_b}{\rho}} \quad = \sqrt{c_f \left(\frac{1}{2}V^2\right)} \quad = V\sqrt{\frac{c_f}{2}}$$

Hence,

$$V\sqrt{\frac{c_f}{2}} = w_s$$

$$\Rightarrow \qquad V = w_s\sqrt{\frac{2}{c_f}} \quad = 0.1115\sqrt{\frac{2}{0.005}} \quad = 2.23 \text{ m s}^{-1}$$

Answer: for incipient suspended load, velocity = 2.23 m s^{-1}.

Q4.
$q = 0.9 \text{ m}^2 \text{ s}^{-1}$
$d = 0.06 \text{ m}$
$\rho_s = 2650 \text{ kg m}^{-3}$
$\tau^*_{\text{crit}} = 0.056$
$c_f = 0.01$
$w_s = 0.8 \text{ m } s^{-1}$

From the critical Shields stress we can determine the critical bed stress for incipient motion:

$$\tau_{\text{crit}} = \tau^*(\rho_s - \rho)gd \quad = 0.056 \times (2650 - 1000) \times 9.81 \times 0.06$$
$$= 54.39 \text{ N m}^{-2}$$

(a) Upstream of the gate:

$h = 2.5$ m

$$V = \frac{q}{h} \quad = \frac{0.9}{2.5} \quad = 0.36 \text{ m s}^{-1}$$

$$\tau_b = c_f \times \frac{1}{2}\rho V^2 \quad = 0.01 \times \frac{1}{2} \times 1000 \times 0.36^2 \quad = 0.648 \text{ N m}^{-2}$$

The bed stress is (considerably) less than the critical value; the bed is stationary.

(b) Sluice gate assumption: total head the same on both sides of the gate.

$$z_{s1} + \frac{V_1^2}{2g} = z_{s2} + \frac{V_2^2}{2g}$$

For a flat bed:

$$h_1 + \frac{q^2}{2gh_1^2} = h_2 + \frac{q^2}{2gh_2^2}$$

Substituting values:

$$2.507 = h_2 + \frac{0.04128}{h_2^2}$$

Rearranging for the supercritical solution:

$$h_2 = \sqrt{\frac{0.04128}{2.507 - h_2}}$$

Iteration (from, e.g., $h_2 = 0$) gives

$$h_2 = 0.1318 \text{ m}$$

Then:

$$V = \frac{q}{h_2} \quad = \frac{0.9}{0.1318} \quad = 6.829 \text{ m s}^{-1}$$

$$\tau_b = c_f \times \frac{1}{2}\rho V^2 \quad = 0.01 \times \frac{1}{2} \times 1000 \times 6.829^2 \quad = 233.2 \text{ N m}^{-2}$$

τ_b exceeds τ_{crit}; the bed is mobile.

Answer: downstream depth = 0.132 m; bed stress exceeds the critical value.

(c) Scour will continue with the depth increasing and the velocity and stress decreasing, until the stress no longer exceeds the critical value. At this point:

$$\tau_b = 54.39 \text{ N m}^{-2}$$

$$c_f \times \frac{1}{2}\rho V^2 = 54.39$$

$$V = \sqrt{\frac{2}{c_f} \times \frac{54.39}{\rho}} = \sqrt{\frac{2}{0.01} \times \frac{54.39}{1000}} = 3.298 \text{ m s}^{-1}$$

The overall depth downstream is then

$$h = \frac{q}{V} = \frac{0.9}{3.298} = 0.2729 \text{ m}$$

Since the water level is fixed by the gate it is the same as before. The depth of scour is then

$$0.2729 - 0.1318 = 0.1411 \text{ m}$$

Sluice gate assumption: total head the same on both sides of the gate.

$$z_{s1} + \frac{V_1^2}{2g} = z_{s2} + \frac{V_2^2}{2g}$$

Upstream, $z_{s1} = h_1$, but downstream there is a distinction between water level $z_{s2} = 0.1318$ m and depth $h_2 = 0.2729$ m.

$$h_1 + \frac{q^2}{2gh_1^2} = z_{s2} + \frac{q^2}{2gh_2^2}$$

Substituting values:

$$h_1 + \frac{0.04128}{h_1^2} = 0.6861$$

Rearranging for the subcritical solution:

$$h_1 = 0.6861 - \frac{0.04128}{h_1^2}$$

Iteration (from, e.g., $h_1 = 0.6861$) gives

$$h_1 = 0.5493 \text{ m}$$

Answer: depth of scour = 0.141 m; depth upstream = 0.549 m; depth downstream = 0.273 m.

(d) To determine the significance of suspended load compare the settling velocity with the maximum friction velocity (which occurs in the initial downstream state).

Settling velocity:

$$w_s = 1.1 \text{ m s}^{-1} \text{ (given)}$$

Friction velocity:

$$u_\tau = \sqrt{\frac{\tau}{\rho}} = \sqrt{\frac{233.2}{1000}} = 0.4829 \text{ m s}^{-1}$$

Here, the settling velocity greatly exceeds the friction velocity, so suspended load is negligible.

Q5.
$b = 5$ m   (width of main channel)
$b_{\min} = 3$ m   (width of restricted section)
$d = 0.01$ m
$\rho_s = 2650$ kg m^{-3} ($s = 2.65$)
$\tau^*_{\text{crit}} = 0.05$
$c_f = 0.01$
$Q = 5$ m^3 s^{-1}

The bed is mobile if and only if the bed shear stress exceeds the critical stress for incipient motion. Alternatively, one may compare the velocity with a critical velocity found using the friction coefficient – this is more convenient here.

The critical shear stress for incipient motion is

$$\tau_{\text{crit}} = \tau^*_{\text{crit}}(\rho_s - \rho)gd \quad = 0.05 \times (2650 - 1000) \times 9.81 \times 0.01 \quad = 8.093 \text{ N m}^{-2}$$

and the corresponding velocity for incipient motion is given by

$$\tau_{\text{crit}} = c_f \left(\frac{1}{2}\rho V_{\text{crit}}^2\right)$$

whence

$$V_{\text{crit}} = \sqrt{\frac{2}{c_f}\frac{\tau_{\text{crit}}}{\rho}} \quad = \sqrt{\frac{2}{0.01} \times \frac{8.093}{1000}} \quad = 1.272 \text{ m s}^{-1}$$

This is a venturi so we need to determine velocities at various locations.

Upstream, $z_s = h = 1$ m and the velocity and total head are, respectively,

$$V = \frac{Q}{bh} \quad = \frac{5}{5 \times 1} \quad = 1.0 \text{ m s}^{-1}$$

$$H_a = z_s + \frac{V^2}{2g} \quad = 1 + \frac{1^2}{2 \times 9.81} \quad = 1.051 \text{ m}$$

If critical-flow conditions occur at the throat then the total head there would be

$$H_c = \frac{3}{2}h_c \quad = \frac{3}{2}\left(\frac{q_m^2}{g}\right)^{1/3} \quad = \frac{3}{2}\left(\frac{Q^2}{b_{\min}^2 g}\right)^{1/3} \quad = \frac{3}{2}\left(\frac{5^2}{3^2 \times 9.81}\right)^{1/3} \quad = 0.9850 \text{ m}$$

Since the upstream head exceeds that required for critical-flow conditions the flow remains subcritical throughout, with total head in the restricted section, $H = 1.051$ m. The depth h is given by:

$$H = h + \frac{V^2}{2g} \quad = h + \frac{Q^2}{2g b_{\min}^2 h^2}$$

Rearranging as an iterative formula for the subcritical value of h:

$$h = H - \frac{Q^2}{2gb_{\text{min}}^2 h^2}$$

Substituting numerical values:

$$h = 1.051 - \frac{0.1416}{h^2}$$

Iteration (from, e.g., $h = 1.051$) gives

$$h = 0.8592 \text{ m}$$

and corresponding velocity

$$V = \frac{Q}{b_{\text{min}} h} \quad = \frac{5}{3 \times 0.8592} \quad = 1.940 \text{ m s}^{-1}$$

Hence we have:

- in the 5 m width, $V = 1.0 \text{ m s}^{-1} \quad < V_{\text{crit}}$ and the bed is stationary;
- in the 3 m width, $V = 1.94 \text{ m s}^{-1} \quad > V_{\text{crit}}$ and the bed is mobile.

(b) The bed will erode in the restricted section until $V = V_{\text{crit}}$. Then the flow depth is given by

$$Q = V_{\text{crit}} \times hb_{\text{min}}$$

$$\Rightarrow \qquad h = \frac{Q}{b_{\text{min}} \times V_{\text{crit}}} \quad = \frac{5}{3 \times 1.272} \quad = 1.310 \text{ m}$$

If Δz_b is the change in height of the bed, then the total head is given by

$$H = \Delta z_b + E \quad = \Delta z_b + h + \frac{V^2}{2g}$$

Hence,

$$\Delta z_b = H - h - \frac{V^2}{2g} \quad = 1.051 - 1.310 - \frac{1.272^2}{2 \times 9.81} \quad = -0.3415 \text{ m}$$

Answer: depth of scour hole = 0.342 m.

Q6.
$b = 12$ m
$S = 0.003$
$Q = 200$ m^3 s^{-1}
$\tau^*_{\text{crit}} = 0.056$

Let the minimum stone diameter (corresponding to incipient motion) be d. Then the critical shear stress for incipient motion is given by

$$\tau_{\text{crit}} = \tau^*_{\text{crit}}(\rho_s - \rho)gd \quad = 0.056 \times (2650 - 1000) \times 9.81 \times d \quad = 906.4d$$

For normal flow:

$$\tau_b = \rho g R_h S$$

For incipient motion:

$$R_h = \frac{\tau_{\text{crit}}}{\rho g S} \quad = \frac{906.4d}{1000 \times 9.81 \times 0.003} \quad = 30.80d \qquad (*)$$

The corresponding depth of flow h is given by the expression for R_h in a rectangular channel,

$$30.80d = \frac{h}{1 + \frac{2h}{b}}$$

With $b = 12$ m this gives (after some rearrangement) an expression for h in terms of d:

$$h = \frac{30.8d}{1 - 5.133d} \qquad (**)$$

From Manning's equation:

$$Q = VA \quad = \frac{1}{n} R_h^{2/3} S^{1/2} \times bh, \qquad \text{where} \qquad n = \frac{d^{1/6}}{21.1} \quad \text{(Strickler's equation)}$$

Hence,

$$Q = \frac{21.1}{d^{1/6}} \times (30.80d)^{2/3} S^{1/2} \times b \times \frac{30.8d}{1 - 5.133d}$$

Subtituting numerical values:

$$200 = 4198 \frac{d^{3/2}}{1 - 5.133d}$$

Rearranging as an iterative formula for d:

$$d = \left[\frac{200(1 - 5.133d)}{4198}\right]^{2/3}$$

Iteration (from, e.g., $d = 0$) gives

$$d = 0.08802 \text{ m}$$

Substituting in (**) gives

$$h = \frac{30.8d}{1 - 5.133d} \quad = \frac{30.8 \times 0.08802}{1 - 5.133 \times 0.08802} \quad = 4.945 \text{ m}$$

Answer: minimum diameter of stone = 88 mm;   river depth = 4.95 m.

Q7.
(a) Assume rapidly-varied flow with critical conditions over the crest. Neglect upstream dynamic head.

Measuring head relative to the top of the weir, noting that the upstream head is just the freeboard and the head over the weir is 3/2 times critical depth:

$$h_0 = \frac{3}{2}(\frac{q^2}{g})^{1/3}$$

Hence, the flow rate (per metre width of embankment) is

$$q = (2/3)^{3/2} g^{1/2} h_0^{3/2} \quad = (2/3)^{3/2} \times 9.81^{1/2} \times 0.15^{3/2} \quad = 0.09905 \text{ m}^2 \text{ s}^{-1}$$

Critical depth:

$$h_c = \frac{2}{3} h_0 \quad = \frac{2}{3} \times 0.15 \quad = 0.1 \text{ m}$$

Answer: flow rate (per metre width) = 0.0990 $\text{m}^2 \text{ s}^{-1}$; depth over embankment = 0.1 m.

(b)
$S = 0.125$
$n = 0.013 \text{ m}^{-1/3} \text{ s}$

Normal Depth

$$q = Vh, \qquad \text{where} \qquad V = \frac{1}{n} R_h^{2/3} S^{1/2}, \qquad R_h = h \qquad \text{(wide channel)}$$

$$\Rightarrow \qquad q = \frac{1}{n} h^{5/3} S^{1/2}$$

$$\Rightarrow \qquad h = \left(\frac{nq}{\sqrt{S}}\right)^{3/5} \quad = \left(\frac{0.013 \times 0.09905}{\sqrt{0.125}}\right)^{3/5} \quad = 0.03442 \text{ m}$$

Velocity

$$V = \frac{q}{h} \quad = \frac{0.09905}{0.03442} \quad = 2.878 \text{ m s}^{-1}$$

Bed shear stress
For normal flow,

$$\tau = \rho g R_h S, \qquad \text{where} \qquad R_h = h \qquad \text{(wide channel)}$$

$$\Rightarrow \qquad \tau = 1000 \times 9.81 \times 0.03442 \times 0.125 \quad = 42.21 \text{ N m}^{-2}$$

Answer: depth = 0.034 m; velocity = 2.88 m s^{-1}; bed stress = 42.2 N m^{-2}.

(c) For the given bed material,

$$d^* = d\left[\frac{(s-1)g}{\nu^2}\right]^{1/3} \quad = 0.002 \times \left[\frac{(2.65-1)\times 9.81}{(1.0\times 10^{-6})^2}\right]^{1/3} \quad = 50.59$$

$$\tau^*_{\text{crit}} = \frac{0.30}{1+1.2d^*} + 0.055\,[1-\exp(-0.020d^*)] \quad = 0.03987$$

For the actual flow down the slope,

$$\tau^* = \frac{\tau}{(\rho_s - \rho)gd} \quad = \frac{42.21}{(2650-1000)\times 9.81 \times 0.002} \quad = 1.304$$

τ^* is far in excess of τ^*_{crit}; hence the surface will erode.

(Alternatively, one could compute the *absolute* critical stress τ_{crit} to compare with τ).

Answer: slope erodes.

(d) Cost-effective examples include:

- increasing the size of bed material; e.g. rock armour;
- semi-immobilising by vegetating the slope or using geotextiles.

Q8.

(a)

$$n = \frac{d^{1/6}}{21.1} \quad = 0.01800\ \ \mathrm{m}^{-1/3}\ \mathrm{s}$$

Answer: Manning's $n =$ 0.0180 m$^{-1/3}$ s.

(b)

$$q = Vh, \qquad \text{where} \qquad V = \frac{1}{n} R_h^{2/3} S^{1/2}, \qquad R_h = h$$

Hence,

$$q = \frac{1}{n} h^{5/3} S^{1/2}$$

or

$$h = \left(\frac{nq}{\sqrt{S}}\right)^{3/5} \quad = 1.532\ \mathrm{m}$$

Answer: depth of flow = 1.53 m.

(c)

$$\tau_b = \rho g R_h S \quad = 18.79\ \mathrm{N\ m}^{-2}$$

Answer: bed shear stress = 18.8 N m^{-2}.

(d)

$$d^* = d\left[\frac{(s-1)g}{\nu^2}\right]^{1/3} \quad = 75.89$$

By Soulsby,

$$\tau^*_{\mathrm{crit}} = \frac{0.30}{1 + 1.2d^*} + 0.055\,[1 - \exp(-0.020 d^*)] \quad = 0.04620$$

Here,

$$\tau^* = \frac{\tau_b}{(\rho_s - \rho) g d} \quad = 0.3869$$

As $\tau^* > \tau^*_{\mathrm{crit}}$ the bed is mobile.

Meyer-Peter and Müller

$$q^* = 8(\tau^* - \tau^*_{\text{crit}})^{3/2} \qquad = 1.591$$

Hence,

$$q_b = q^* \sqrt{(s-1)gd^3} \qquad = 1.052 \times 10^{-3} \text{ m}^2 \text{ s}^{-1}$$

Van Rijn

$$q^* = \frac{0.053}{(d^{*0.3})\left(\frac{\tau^*}{\tau^*_{\text{crit}}} - 1\right)^{2.1}} \qquad = 0.9604$$

Hence,

$$q_b = q^* \sqrt{(s-1)gd^3} \qquad = 6.349 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$$

Answer: bed load = 1.05×10^{-3} m^2 s^{-1} (Meyer-Peter and Müller); 6.35×10^{-4} m^2 s^{-1} (Van Rijn)

(e) By Cheng's formula,

$$w_s = \frac{\nu}{d\left[\left(25 + 1.2d^{*2}\right)^{1/2} - 5\right]^{3/2}} \qquad = 0.2309 \text{ m s}^{-1}$$

By definition,

$$u_\tau = \sqrt{\frac{\tau_b}{\rho}} \qquad = 0.1371 \text{ m s}^{-1}$$

$u_\tau < w_s$, so no significant suspended load occurs.

Q9.
$S = 1/800 = 0.00125$
$d = 0.0005$ m
$q = 5$ m^2 s^{-1}
Assume water has density $\rho = 1000$ kg m^{-3} and kinematic viscosity $\nu = 1.0 \times 10^{-6}$ m^2 s^{-1}.

(a)

$$n = \frac{d^{1/6}}{21.1} \quad = \frac{(5 \times 10^{-4})^{1/6}}{21.1} \quad = 0.01335 \text{ m}^{-1/3} \text{ s}$$

Answer: Manning's $n = 0.0134$ m$^{-1/3}$ s.

(b)

$$q = Vh, \qquad \text{where} \qquad V = \frac{1}{n} R_h^{2/3} S^{1/2}, \qquad R_h = h$$

Hence,

$$q = \frac{1}{n} h^{5/3} S^{1/2}$$

$$\Rightarrow \qquad h = \left(\frac{nq}{\sqrt{S}}\right)^{3/5} \quad = \left(\frac{0.01335 \times 5}{\sqrt{0.00125}}\right)^{3/5} \quad = 1.464 \text{ m}$$

Answer: depth of flow = 1.46 m.

(c)

$$\tau_b = \rho g R_h S \quad = 1000 \times 9.81 \times 1.464 \times 0.00125 \quad = 17.95 \text{ N m}^{-2}$$

Answer: bed shear stress = 18.0 N m^{-2}.

(d)

$$d^* = d \left[\frac{(s-1)g}{\nu^2}\right]^{1/3} \quad = 0.0005 \times \left[\frac{(2.65-1) \times 9.81}{(1.0 \times 10^{-6})^2}\right]^{1/3} \quad = 12.65$$

By Soulsby, the critical Shields stress is

$$\tau^*_{\text{crit}} = \frac{0.30}{1 + 1.2d^*} + 0.055 \;\; [1 - \exp(-0.020 d^*)]$$
$$= \frac{0.30}{1 + 1.2 \times 12.65} + 0.055 \;\; [1 - \exp(-0.020 \times 12.65)] \quad = 0.03084$$

For this flow the actual Shields stress is

$$\tau^* = \frac{\tau_b}{(\rho_s - \rho)gd} \quad = \frac{17.95}{(2650 - 1000) \times 9.81 \times 0.0005} \quad = 2.218$$

As $\tau^* > \tau^*_{\text{crit}}$ the bed is mobile.

Meyer-Peter and Müller

$$q^* = 8(\tau^* - \tau^*_{\text{crit}})^{3/2} \quad = 8 \times (2.218 - 0.03084)^{3/2} \quad = 25.88$$

Hence,

$$q_b = q^* \sqrt{(s-1)gd^3} \; = 25.88 \times \sqrt{1.65 \times 9.81 \times 0.0005^3} \; = 1.164 \times 10^{-3} \text{ m}^2 \text{ s}^{-1}$$

Nielsen

$$q^* = 12(\tau^* - \tau^*_{\text{crit}})\sqrt{\tau^*} \quad = 12 \times (2.218 - 0.03084)\sqrt{2.218} \quad = 39.09$$

Hence,

$$q_b = q^* \sqrt{(s-1)gd^3} \; = 39.09 \times \sqrt{1.65 \times 9.81 \times 0.0005^3} = 1.758 \times 10^{-3} \text{ m}^2 \text{ s}^{-1}$$

Answer: bed load = 1.16×10^{-3} m^2 s^{-1} (Meyer-Peter and Müller); 1.76×10^{-3} m^2 s^{-1} (Nielsen).

(e) By Cheng's formula, the settling velocity is given by

$$w_s = \frac{\nu}{d\left[\left(25 + 1.2d^{*2}\right)^{1/2} - 5\right]^{3/2}} \quad = \frac{1.0 \times 10^{-6}}{0.0005[(25 + 1.2 \times 12.65^2)^{1/2} - 5]^{3/2}}$$
$$= 0.06072 \text{ m s}^{-1}$$

By definition, the friction velocity is

$$u_\tau = \sqrt{\frac{\tau_b}{\rho}} \quad = \sqrt{\frac{17.95}{1000}} \quad = 0.1340 \text{ m s}^{-1}$$

$u_\tau > w_s$, so suspended load occurs.

Answer: settling velocity = 0.0607 m s^{-1}.

(f)

$$C_{\text{ref}} = \min\left\{\frac{0.117}{d^*}\left(\frac{\tau^*}{\tau^*_{\text{crit}}} - 1\right), \quad 0.65\right\} \quad = \min\left\{\frac{0.117}{12.65}\left(\frac{2.218}{0.03084} - 1\right), \quad 0.65\right\}$$
$$= \min\{0.6559, \quad 0.65\} \quad = 0.65$$

$$z_{\text{ref}} = d \times 0.3 d^{*0.7}\left(\frac{\tau^*}{\tau^*_{\text{crit}}} - 1\right)^{1/2}$$

$$= 0.0005 \times 0.3 \times 12.65^{0.7}\left(\frac{2.218}{0.03084} - 1\right)^{1/2} \quad = 7.463 \times 10^{-3}\text{ m}$$

The concentration profile is then

$$\frac{C}{C_{\text{ref}}} = \left(\frac{\frac{h}{z} - 1}{\frac{h}{z_{\text{ref}}} - 1}\right)^{\frac{w_s}{\kappa u_\tau}}$$

or, with $\kappa = 0.41$,

$$C = 0.65\left(\frac{\frac{1.464}{z} - 1}{195.2}\right)^{1.105} \qquad (*)$$

The velocity profile is

$$u(z) = \frac{u_\tau}{\kappa}\ln\left(33\frac{z}{k_s}\right)$$

Taking the roughness length k_s equal to a diameter, i.e. $k_s = 0.0005$ m, gives

$$u(z) = 0.3268\ln(66000z) \qquad (**)$$

The total suspended load (per unit width) is:

$$q_s = \int_{z_{\text{ref}}}^{h} CU\,\mathrm{d}z$$

With equations (*) and (**) this becomes

$$q_s = \int_{0.007463}^{1.464}(6.255 \times 10^{-4})\left(\frac{1.464}{z} - 1\right)^{1.105}\ln(66000z)\,\mathrm{d}z$$

and this can be evaluated using numerical integration (e.g. trapezium rule) to give

$$q_s = 0.044\text{ m}^2\text{ s}^{-1}$$

Answer: suspended-load flux = 0.044 $\text{m}^3\ \text{s}^{-1}$ per metre width.

A simple Fortran code for doing the numerical integration is given on the following page. The code can be modified for any alternative values of k_s or velocity and concentration profiles.

A significant number of trapezia is necessary (probably 200+), although this could be reduced by using Simpson's rule instead. For any numerical method *you should always perform the calculations with more, shorter, subintervals to confirm that numerical accuracy is sufficient*. The need to be able to vary the number of intervals is one reason why a spreadsheet alone is not particularly good at this, as it would require intervention to change the number of trapezia.

```
PROGRAM SUSPENDED_LOAD
   IMPLICIT NONE
   DOUBLE PRECISION, PARAMETER :: KAPPA = 0.41
   DOUBLE PRECISION CREF, ZREF
   DOUBLE PRECISION H, KS, WS, UTAU
   DOUBLE PRECISION ROUSE
   DOUBLE PRECISION DZ, Z, INTEGRAL
   INTEGER N, J

   ! Particulate and flow data
   DATA  CREF,     ZREF,     H,     KS,      WS,   UTAU &
       / 0.65, 0.007463, 1.464, 0.0005, 0.06072, 0.1340 /

   ROUSE = WS / (KAPPA * UTAU)

   ! Number of intervals and size
   PRINT *, 'Input N';   READ *, N
   DZ = (H - ZREF) / N

   ! Integral by trapezium rule
   INTEGRAL = C(ZREF) *  U(ZREF) + C(H) * U(H)
   DO J = 1, N - 1
      Z = ZREF + J * DZ
      INTEGRAL = INTEGRAL + 2.0 * C(Z) * U(Z)
   END DO
   INTEGRAL = INTEGRAL * DZ / 2.0

   WRITE( *, "('Suspended load: ', ES10.3, ' m3/s per metre')" ) INTEGRAL


   ! Internal functions giving profiles of concentration and velocity
   CONTAINS

      !--------------------------------------

      DOUBLE PRECISION FUNCTION C( Z )
         IMPLICIT NONE
         DOUBLE PRECISION Z

         C = CREF * ( (H / Z - 1.0) / (H / ZREF - 1.0) ) ** ROUSE

      END FUNCTION C

      !--------------------------------------

      DOUBLE PRECISION FUNCTION U( Z )
         IMPLICIT NONE
         DOUBLE PRECISION Z

         U = (UTAU / KAPPA ) * LOG( 33.0 * Z / KS )

      END FUNCTION U


END PROGRAM SUSPENDED_LOAD
```

Q10.
(a)

$$n = \frac{d^{1/6}}{21.1} \quad = \frac{(0.0025)^{1/6}}{21.1} \quad = 0.01746 \ \ \mathrm{m}^{-1/3}\ \mathrm{s}$$

Answer: Manning's $n = 0.0175$ m$^{-1/3}$ s.

(b)
For normal flow:

$$q = Vh, \qquad \text{where} \qquad V = \frac{1}{n} R_h^{2/3} S^{1/2}, \qquad R_h = h$$

$$\Rightarrow \qquad q = \frac{1}{n} h^{5/3} S^{1/2}$$

$$\Rightarrow \qquad h = \left(\frac{nq}{\sqrt{S}}\right)^{3/5} \quad = \left(\frac{0.01746 \times 3.5}{\sqrt{\frac{1}{1500}}}\right)^{3/5} \quad = 1.677\ \mathrm{m}$$

Average velocity:

$$V = \frac{q}{h} \quad = \frac{3.5}{1.677} \quad = 2.087\ \mathrm{m\ s}^{-1}$$

Answer: depth of flow = 1.68 m;   average velocity = 2.09 m s^{-1}.

(c) Using the normal-flow relation:

$$\tau_b = \rho g R_h S \quad = 1000 \times 9.81 \times 1.677 \times \left(\frac{1}{1500}\right) \quad = 10.97\ \mathrm{N\ m}^{-2}$$

The Shields stress is

$$\tau^* = \frac{\tau_b}{(\rho_s - \rho) g d} \quad = \frac{10.97}{(2650 - 1000) \times 9.81 \times 0.0025} \quad = 0.2711$$

To find the critical Shields stress:

$$d^* = d \ \left[\frac{(s-1)g}{\nu^2}\right]^{1/3} \quad = 0.0025 \times \left[\frac{(2.65-1) \times 9.81}{(1.0 \times 10^{-6})^2}\right]^{1/3} \quad = 63.24$$

$$\tau^*_{\mathrm{crit}} = \frac{0.30}{1 + 1.2 d^*} + 0.055\ \ [1 - \exp(-0.020 d^*)]$$

$$= \frac{0.30}{1 + 1.2 \times 63.24} + 0.055\ \ [1 - \exp(-0.020 \times 63.24)] \quad = 0.04338$$

As $\tau^* > \tau^*_{\text{crit}}$ the bed is mobile.

Answer: bed shear stress = 11.0 N m^{-2} and exceeds the critical shear stress.

(d) For the bed-load flux:

$$q^* = 8(\tau^* - \tau^*_{\text{crit}})^{3/2} \quad = 8 \times (0.2711 - 0.04338)^{3/2} \quad = 0.8693$$

whence

$$q_b = q^* \sqrt{(s-1)gd^3} \quad = 0.8693 \times \sqrt{1.65 \times 9.81 \times 0.0025^3}$$
$$= 4.372 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$$

Answer: bed load = 4.37×10^{-4} m^3 s^{-1} per metre width.

(e)
By Cheng's formula:

$$w_s^* = \left[\left(25 + 1.2d^{*2}\right)^{1/2} - 5\right]^{3/2} = \left[(25 + 1.2 \times 63.24^2)^{\frac{1}{2}} - 5\right]^{3/2} \quad = 517.5$$

$$\Rightarrow \qquad w_s = \frac{\nu}{d} \times w_s^* \quad = \frac{1.0 \times 10^{-6}}{0.0025} \times 517.5 \quad = 0.2070 \text{ m s}^{-1}$$

By definition, the friction velocity is

$$u_\tau = \sqrt{\frac{\tau_b}{\rho}} \quad = \sqrt{\frac{10.97}{1000}} \quad = 0.1047 \text{ m s}^{-1}$$

$u_\tau < w_s$ so significant suspended load would not be expected to occur.

Answer: settling velocity = 0.207 m s^{-1}.

(f) If suspended load does occur then it may be computed by summing

concentration (C) × volumetric flow rate (U dz per unit width)

over the water column. i.e.

$$q_s = \int_0^h CU \, \mathrm{d}z$$

Q11.

$$c_D = \frac{24}{\text{Re}}$$

Expanding:

$$\frac{\text{drag}}{\frac{1}{2}\rho w_s^2 \times (\text{projected area})} = 24 \times \frac{\nu}{w_s d}$$

But, at terminal velocity,

$$\text{drag} = \text{reduced weight} \quad = (m - m_w)g \quad = (\rho_s - \rho) \times \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 \times g$$
$$= \frac{1}{6}\pi(\rho_s - \rho)gd^3$$

whilst

$$\text{projected area} = \frac{\pi d^2}{4}$$

Hence,

$$\frac{\frac{1}{6}\pi(\rho_s - \rho)gd^3}{\frac{1}{2}\rho w_s^2 \times \frac{1}{4}\pi d^2} = 24\frac{\nu}{w_s d}$$

$$\Rightarrow \qquad \frac{4}{3}\frac{\left(\frac{\rho_s}{\rho} - 1\right)gd}{w_s^2} = 24\frac{\nu}{w_s d}$$

$$\Rightarrow \qquad \frac{1}{18}\frac{(s-1)gd^2}{\nu} = w_s$$

Q12.
Consider the velocity profile

$$U(z) = \frac{u_\tau}{\kappa} \ln\left(33 \frac{z}{k_s}\right)$$

The total flow per unit span is given by

$$q = \int_0^h U(z)\, \mathrm{d}z$$

Since $q = Vh$ (by definition of average velocity V):

$$Vh = \int_0^h \frac{u_\tau}{\kappa} \ln\left(33 \frac{z}{k_s}\right)\, \mathrm{d}z$$

Integrating by parts:

$$Vh = \frac{u_\tau}{\kappa}\left\{\left[z \ln\left(33 \frac{z}{k_s}\right)\right]_0^h - \int_0^h \mathrm{d}z\right\}$$

$$\Rightarrow \qquad Vh = \frac{u_\tau}{\kappa}\left\{h \ln\left(33 \frac{h}{k_s}\right) - h\right\}$$

$$\Rightarrow \qquad V = \frac{u_\tau}{\kappa}\left\{\ln\left(33 \frac{h}{k_s}\right) - 1\right\}$$

or, since $1 = \ln \mathrm{e}$,

$$V = \frac{u_\tau}{\kappa} \ln\left(\frac{33}{e} \frac{h}{k_s}\right) \quad = \frac{u_\tau}{\kappa} \ln\left(12 \frac{h}{k_s}\right)$$

(with 2 sig. fig. accuracy for the constant)

Q13.
(a)

$$u_\tau = \sqrt{\frac{\tau_b}{\rho}}$$

where τ_b is the bed shear stress and ρ is the fluid density.

(b) Linear shear stress profile with $\tau = \tau_b \equiv \rho u_\tau^2$ at $z = 0$ and $\tau = 0$ at $z = h$:

$$\tau = \rho u_\tau^2 \left(1 - \frac{z}{h}\right)$$

From the given velocity profile:

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \frac{u_\tau}{\kappa z}$$

By the definition of eddy viscosity ($\mu_t = \rho \nu_t$):

$$\tau = \rho \nu_t \frac{\mathrm{d}U}{\mathrm{d}z}$$

Rearranging for ν_t:

$$\nu_t = \frac{\tau/\rho}{\frac{\mathrm{d}U}{\mathrm{d}z}} \quad = \frac{u_\tau^2 \left(1 - \frac{z}{h}\right)}{\frac{u_\tau}{\kappa z}} \quad = \kappa u_\tau z \left(1 - \frac{z}{h}\right)$$

Thus, the kinematic eddy viscosity ν_t is a quadratic function of z.

(c) According to Fick's gradient-diffusion law the net *upward* flux of sediment volume across a horizontal area A is

$$-K \frac{\mathrm{d}C}{\mathrm{d}z} A$$

whilst the net *downward* flux due to settling is volume flux × concentration, or

$$w_s A C$$

At equilibrium these are equal and hence

$$-K \frac{\mathrm{d}C}{\mathrm{d}z} A = w_s A C$$

Dividing by A and rearranging:

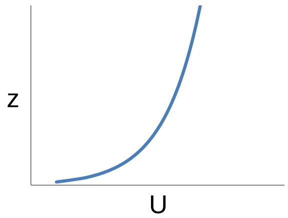
$$K \frac{\mathrm{d}C}{\mathrm{d}z} + w_s C = 0$$

It is assumed that, as the same turbulent eddies are responsible for the transport of both momentum and particulate material the kinematic eddy viscosity (ν_t) and eddy diffusivity (K) are equal. Hence, $K = \kappa u_\tau z\left(1-\frac{z}{h}\right)$ and so

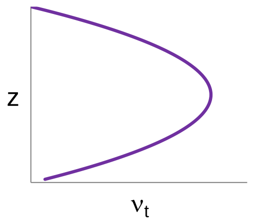
$$\frac{\kappa u_\tau z\left(1-\frac{z}{h}\right)\mathrm{d}C}{\mathrm{d}z} + w_s C = 0$$

(d)
(Note that all sketches below have the independent variable – here the vertical coordinate – on the vertical axis.)

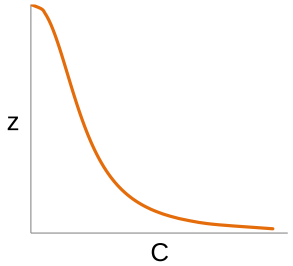
Velocity:



Eddy viscosity:



Concentration (Rouse number = 0.5 here)



(e)
Bed-load transport:
- set in motion by the fluid stress;
- main mechanisms: sliding, rolling, saltating (small jumps).

Suspended-load transport:
- occurs for sufficiently vigorous turbulent fluid motion;
- balance between net upward transport by turbulent eddies and downward settling;
- usually quantified by solving a diffusion equation (see the following question), then integrating the resultant flux density (CU) over the flow cross-section.

Q14.

$$-K\frac{\mathrm{d}C}{\mathrm{dz}} = w_s C$$

Substituting the eddy diffusivity $K = \kappa u_\tau z\left(1-\frac{z}{h}\right)$:

$$-\frac{\kappa u_\tau z\left(1-\frac{z}{h}\right)\mathrm{d}C}{\mathrm{d}z} = w_s C$$

Separating variables:

$$\frac{\mathrm{d}C}{C} = -\frac{w_s}{\kappa u_\tau}\frac{\mathrm{d}z}{z\left(1-\frac{z}{h}\right)}$$

Using partial fractions:

$$\frac{\mathrm{d}C}{C} = -\frac{w_s}{\kappa u_\tau}\left(\frac{1}{z}+\frac{1}{h-z}\right)\mathrm{d}z$$

Integrating between z_{ref}, C_{ref} and a general z, C pair:

$$\int_{C_{\mathrm{ref}}}^{\mathrm{C}}\frac{\mathrm{d}C}{C} = -\frac{w_s}{\kappa u_\tau}\int_{z_{\mathrm{ref}}}^{z}\left(\frac{1}{z}+\frac{1}{h-z}\right)\mathrm{d}z$$

$$\Rightarrow \qquad \ln\frac{C}{C_{\mathrm{ref}}} = -\frac{w_s}{\kappa u_\tau}\left(\ln\frac{z}{z_{\mathrm{ref}}} - \ln\frac{h-z}{h-z_{\mathrm{ref}}}\right)$$

$$\Rightarrow \qquad \ln\frac{C}{C_{\mathrm{ref}}} = \frac{w_s}{\kappa u_\tau}\ln\left(\frac{z_{\mathrm{ref}}}{z}\frac{h-z}{h-z_{\mathrm{ref}}}\right)$$

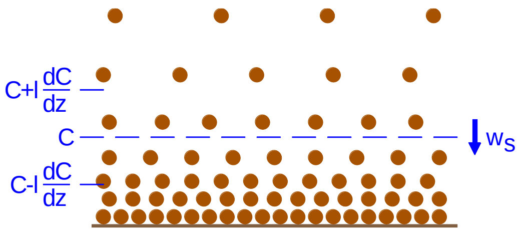
On the RHS, inside the logarithm divide both numerator and denominator by $z \times z_{\mathrm{ref}}$:

$$\ln\frac{C}{C_{\mathrm{ref}}} = \ln\left(\frac{\frac{h}{z}-1}{\frac{h}{z_{\mathrm{ref}}}-1}\right)^{\frac{w_s}{\kappa u_\tau}}$$

$$\Rightarrow \qquad \frac{C}{C_{\mathrm{ref}}} = \left(\frac{\frac{h}{z}-1}{\frac{h}{z_{\mathrm{ref}}}-1}\right)^{\frac{w_s}{\kappa u_\tau}}$$

Q15.

(a) Due to settling, sediment concentration is greater near the bed. As a result, upward turbulent velocity fluctuations tend to carry larger amounts of sediment than downward fluctuations, leading to a net upward diffusive flux. An equilibrium concentration distribution is attained when this balances the downward settling flux.



Consider a horizontal surface, where the concentration is C. An upward turbulent velocity u' for half the time carries material of concentration ($C - l\,\mathrm{d}C/\mathrm{d}z$), where l is a typical size of turbulent eddy. The corresponding downward velocity for the other half of the time carries material at concentration ($C + l\,\mathrm{d}C/\mathrm{d}z$). The average upward flux of sediment (volume flux × concentration) through horizontal area A is

$$\frac{1}{2}u'A\left(C - l\frac{\mathrm{d}C}{\mathrm{d}z}\right) - \frac{1}{2}u'A\left(C + l\frac{\mathrm{d}C}{\mathrm{d}z}\right) = -u'l\frac{\mathrm{d}C}{\mathrm{d}z}A$$

Writing $u'l$ as K, the net *upward* flux is

$$-K\frac{\mathrm{d}C}{\mathrm{d}z}A$$

At equilibrium this is balanced by a net *downward* flux of material w_sAC due to settling:

$$-K\frac{\mathrm{d}C}{\mathrm{d}z}A = w_sAC$$

Dividing by A:

$$-K\frac{\mathrm{d}C}{\mathrm{d}z} = w_sC$$

(b)

$$-\frac{\kappa u_\tau z\left(1-\frac{z}{h}\right)\mathrm{d}C}{\mathrm{d}z} = w_sC$$

Separating variables:

$$\frac{\mathrm{d}C}{C} = -\frac{w_s}{\kappa u_\tau}\frac{\mathrm{d}z}{z\left(1-\frac{z}{h}\right)}$$

$$\Rightarrow \qquad \frac{\mathrm{d}C}{C} = -\frac{w_s}{\kappa u_\tau}\left(\frac{1}{z}+\frac{1}{h-z}\right)\mathrm{d}z$$

Integrating between a reference height z_{ref} and z:

$$\int_{C_{\mathrm{ref}}}^{C}\frac{\mathrm{d}C}{C} = -\frac{w_s}{\kappa u_\tau}\int_{z_{\mathrm{ref}}}^{z}\left(\frac{1}{z}+\frac{1}{h-z}\right)\mathrm{d}z$$

$$\Rightarrow \qquad \ln\frac{C}{C_{\text{ref}}} = -\frac{w_s}{\kappa u_\tau}\left(\ln\frac{z}{z_{\text{ref}}} - \ln\frac{h-z}{h-z_{\text{ref}}}\right)$$

$$\Rightarrow \qquad \ln\frac{C}{C_{\text{ref}}} = \frac{w_s}{\kappa u_\tau}\ln\left(\frac{z_{\text{ref}}}{z}\frac{h-z}{h-z_{\text{ref}}}\right)$$

Dividing top and bottom of the last fraction by z and z_{ref}:

$$\ln\frac{C}{C_{\text{ref}}} = \ln\left(\frac{\frac{h}{z}-1}{\frac{h}{z_{\text{ref}}}-1}\right)^{\frac{w_s}{\kappa u_\tau}}$$

$$\Rightarrow \qquad \frac{C}{C_{\text{ref}}} = \left(\frac{\frac{h}{z}-1}{\frac{h}{z_{\text{ref}}}-1}\right)^{\frac{w_s}{\kappa u_\tau}}$$

(c) The sediment flux can be determined by summing contributions $C(U\,\mathrm{d}A)$ over a cross-section, where $\mathrm{d}A = b\,\mathrm{d}z$ and b is the width of the channel:

$$Q_s = b\int_{z_{\text{ref}}}^{h} CU\,\mathrm{d}z$$

Using the trapezium rule (no need to learn this formula – just sum trapezoidal areas if you prefer) on N intervals (here, $N = 3$) this is approximated by

$$Q_s = b\frac{\Delta z}{2(f_0 + 2\sum_{i=1}^{N-1} f_i + f_N)} \qquad (*)$$

where f is the integrand.

In this case,

$$b = 5\text{ m}$$

$$\Delta z = \frac{h - z_{\text{ref}}}{N} \quad = \frac{1.5 - 0.001}{3} \quad = 0.4997\text{ m}$$

$$C = C_{\text{ref}}\left(\frac{\frac{h}{z}-1}{\frac{h}{z_{\text{ref}}}-1}\right)^{\frac{w_s}{\kappa u_\tau}} \quad = 0.65\left(\frac{\frac{1.5}{z}-1}{1499}\right)^{0.3659}$$

$$U = \frac{u_\tau}{\kappa}\ln\left(33\frac{z}{k_s}\right) \quad = 0.4878\ln(33000z)$$

$$f = CU$$

z_i	C_i	U_i	f_i
0.0010	0.65000	1.706	1.1089
0.5007	0.05764	4.738	0.2731
1.0004	0.03472	5.075	0.1762
1.5000	0.00000	5.273	0.0000

Using (*),

$$Q_s = 5 \times \frac{0.4997}{2} \times (1.1089 + 2 \times (0.2731 + 0.1762) + 0) \quad = 2.508 \text{ m}^3 \text{ s}^{-1}$$

Answer: $Q_s = 2.5 \text{ m}^3 \text{ s}^{-1}$.