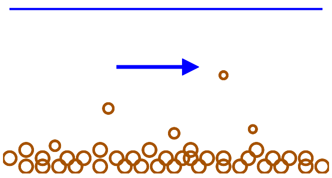
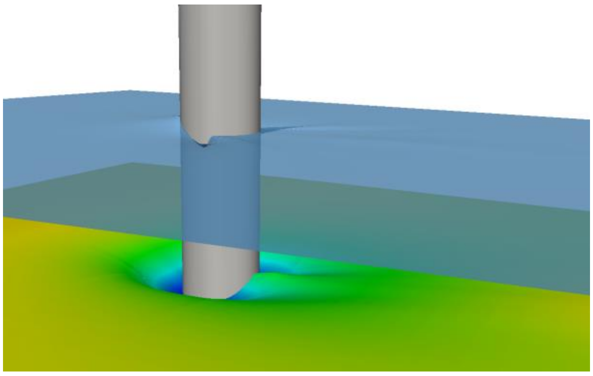
1.1 Introduction



In natural channels and bodies of water the bed is not fixed but is composed of mobile particles; e.g. gravel, sand or silt. These may be dislodged and moved by the flow – the process of *sediment transport*.

Many large rivers are famed for their sediment-carrying capacity: the Yellow River in China is coloured by its sediment content, and for centuries Egyptian farmers have relied on the rich deposits carried by the Nile. In many cases erosion occurs but is not noticed because there is a *dynamic equilibrium* established whereby, on average, as much sediment is supplied as is removed from an area. However, short-term events (such as severe storms) and man-made structures (such as dams) can severely disrupt this equilibrium. Chanson's book contains salutary details of reservoirs rendered useless by siltation and bridge failures because of scour around their foundations. The Nile delta is eroding because the sediment supply from upstream is being held up by the Aswan Dam. Bridge piers are highly susceptible to short-term or long-term scour around their foundations (see the CFD simulation right), whilst road and rail transport are often disrupted by surface damage in flash floods.



Models exist to address three basic questions.
- Does sediment transport occur? ("*Threshold of motion*").
- If it does, then at what rate? ("*Sediment load*").
- What effect does an imbalance have on bed morphology? ("*Scour vs accretion*").

In general, two modes of transport are recognised:
- *bed load*: particles sliding, rolling or saltating (making short jumps), but remaining essentially in contact with the bed;
- *suspended load*: finer particles carried along in suspension by the turbulent fluid flow.

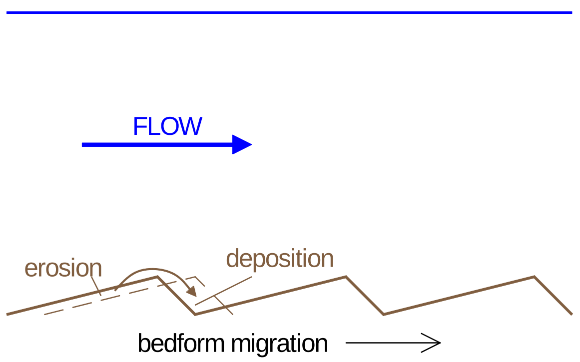
The combination of these two is called the *total load*.

To predict sediment transport we need to consider:
- *particle* properties: diameter, specific gravity, settling velocity, porosity;
- *flow* properties: bed stress, velocity and turbulence profiles.

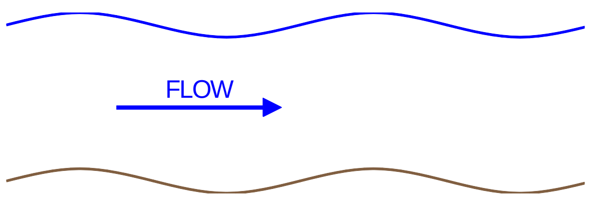
Finally, we note that, although our primary interest here is in sediment transport by water, similar processes can be observed in air ("*aeolian*" or *wind-borne* transport). Examples are the raising of dust clouds and sand-dune movement across deserts.

Besides overall transport of material, sediment transport gives rise to some classic *bedforms*:

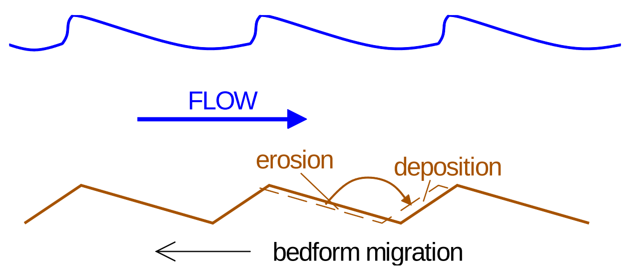
- *ripples* (fine particles; $\text{Fr} \ll 1$; wavelength depends on particle size, not flow depth);
- *dunes* ($\text{Fr} < 1$; migrate in the direction of flow);
- *standing waves* ($\text{Fr} = 1$; bed undulations in phase with free-surface standing waves);
- *antidunes* ($\text{Fr} > 1$; migrate in the opposite direction to flow).



Dunes



Standing waves



Antidunes

1.2 Particle Properties

1.2.1 Diameter, d

Since natural particles have very irregular shapes the concept of *diameter* is somewhat imprecise. Common definitions include:

- *sieve diameter* – the finest mesh that a particle can pass through;
- *sedimentation diameter* – diameter of a sphere with the same settling velocity;
- *nominal diameter* – diameter of a sphere with the same volume.

A typical size classification is given below.

Type	Diameter
Boulders	> 256 mm
Cobbles	64 mm – 256 mm
Gravel	2 mm – 64 mm
Sand	0.06 mm – 2 mm
Silt	0.002 mm – 0.06 mm
Clay	< 0.002 mm (cohesive)

Where there is a range of particle sizes the cumulative percentage is attached to the diameter; e.g. the *median* diameter d_{50} is that sieve size which passes 50% (by weight) of particulate, whilst a measure of spread is the *geometric standard deviation* $\sigma_g = (d_{84.1}/d_{15.9})^{1/2}$. (The percentiles assume a lognormal size distribution.)

This introductory course (and most models) will simply refer to a diameter d.

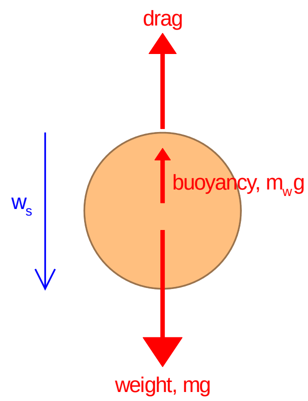
1.2.2 Specific Gravity, s

The *specific gravity* (or *relative density*) s is the ratio of particle density (ρ_s) to that of the fluid (ρ):

$$s = \frac{\rho_s}{\rho} \qquad (1)$$

Quartz-like minerals have density ≈ 2650 kg m^{-3}, so the relative density (in water) is 2.65.

1.2.3 Settling Velocity, w_s



The settling velocity is the terminal velocity in still fluid and may be found by balancing drag against submerged weight; i.e.

$$\text{drag} = \text{weight} - \text{buoyancy}$$

$$c_D(\frac{1}{2}\rho w_s^2)(\frac{\pi d^2}{4}) = (\rho_s - \rho)g\frac{\pi d^3}{6}$$

where c_D is the drag coefficient. This rearranges to give

$$w_s = \left[\frac{4}{3}\frac{(s-1)gd}{c_D}\right]^{1/2} \qquad (2)$$

Stokes' law for the force on spherical particles at small Reynolds numbers gives

$$c_D = \frac{24}{\text{Re}} \qquad\qquad \text{Re} \equiv \frac{w_s d}{\nu} < 1$$

whence (after some algebra):

$$w_s = \frac{1}{18}\frac{(s-1)gd^2}{\nu} \tag{3}$$

or, in non-dimensional form:

$$\frac{w_s d}{\nu} = \frac{1}{18}\frac{(s-1)gd^3}{\nu^2} = \frac{1}{18}d^{*3} \qquad \text{where} \qquad d^* = d\left[\frac{(s-1)g}{\nu^2}\right]^{1/3}$$

However, this is valid only for very small, spherical particles (diameter < 0.1 mm in water). For larger grains (and natural shapes) a useful empirical formula is that of Cheng (1997):

$$\frac{w_s d}{\nu} = \left[\,(25 + 1.2d^{*2})^{1/2} - 5\right]^{3/2} \tag{4}$$

One of the major uses of the settling velocity is in determining whether *suspended load* occurs, and the concentration profile in the water column that results (see Section 4).

1.2.4 Porosity, *P*

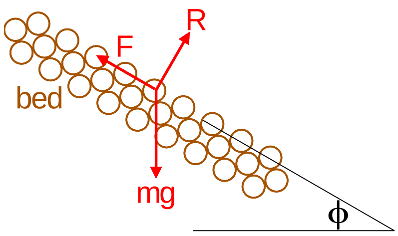
The *porosity* P is the ratio of voids to total volume of material; i.e. in a volume V of space there will actually be a volume $(1-P)V$ of sediment and volume PV of fluid.

Porosity is important in modelling changes to bed morphology and the leaching of pollutants through the bed. For natural uncompacted sediment P is typically about 0.4.

1.2.5 Angle of Repose, ϕ

The angle of repose is the maximum angle (to the horizontal) which a pile of sediment may adopt before it begins to avalanche. It is easily measured in the laboratory.

The resistance to incipient motion may be quantified by an effective *coefficient of friction* μ_f. Consider the basic mechanics problem of a particle on a slope. Incipient motion occurs when the downslope component of weight equals the maximum friction force ($\mu_f \times$ normal reaction). Then



$$mg\sin\phi = \mu_f(mg\cos\phi)$$

or

$$\mu_f = \tan\phi$$

Although the mechanism for causing motion is not the same, μ_f can then be used to estimate the effect of gravitational assistance on sloping beds, where both fluid drag and downslope component of weight act on the particles of the bed (see Section 2).

1.3 Flow Properties.

1.3.1 Friction Velocity, u_τ

The *bed shear stress* τ_b is the drag (per unit area) of the flow on the granular bed. It is responsible for setting the sediment in motion.

As stress has dimensions of [density] × [velocity]^2 it is possible to define from τ_b an important stress-related velocity scale called the *friction velocity* (or *shear velocity*) u_τ such that:

$$\tau_b = \rho u_\tau^2 \qquad \text{or} \qquad u_\tau = \sqrt{\tau_b/\rho} \tag{5}$$

1.3.2 Mean-Velocity Profile

It may be shown (elsewhere!) that a fully-developed turbulent boundary layer adopts a logarithmic mean-velocity profile. For a *rough* boundary this is of the form

$$U(z) = \frac{u_\tau}{\kappa}\ln(33\frac{z}{k_s}) \tag{6}$$

where u_τ is the friction velocity, κ is von Kármán's constant (a famous number with a value of about 0.41) and k_s is the roughness height (typically 1 – 2.5 times particle diameter). z is the distance from the bed.

1.3.3 Eddy-Viscosity Profile

A classical model for the effective shear stress τ in a turbulent flow is to assume, by analogy with laminar flow, that it is proportional to the mean-velocity gradient:

$$\tau = \mu_t \frac{\mathrm{d}U}{\mathrm{d}z} \qquad \text{or} \qquad \tau = \rho \nu_t \frac{\mathrm{d}U}{\mathrm{d}z} \tag{7}$$

μ_t is the *eddy viscosity*. ($\nu_t = \mu_t/\rho$ is the corresponding *kinematic* eddy viscosity). μ_t is not a true viscosity, but a means of modelling the effect of turbulent motion on momentum transport. Such models are called *eddy-viscosity models* and they are widely used in fluid mechanics. In a fully-turbulent flow μ_t is many times larger than the molecular viscosity μ.

At the bed, the shear stress is $\tau = \tau_b \equiv \rho u_\tau^2$. At the free surface ($z = h$), in the absence of significant wind stress, $\tau = 0$. In fully-developed flow it may be shown that the shear stress varies linearly across the channel; hence, to meet these boundary conditions it is given by

$$\tau = \rho u_\tau^2 (1 - \frac{z}{h}) \tag{8}$$

Differentiating the mean-velocity profile (6) we find:

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \frac{u_\tau}{\kappa z} \qquad (9)$$

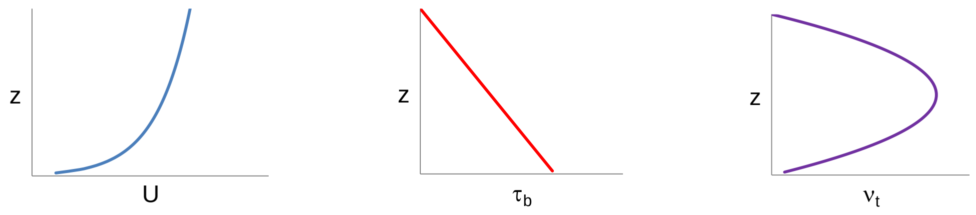
Substituting these expressions for stress and mean-velocity gradient into (7) gives

$$\rho u_\tau^2 (1 - z/h) = \rho \nu_t \frac{u_\tau}{\kappa z}$$

whence:

$$\nu_t = \kappa u_\tau z (1 - z/h) \qquad (10)$$

The kinematic eddy viscosity ν_t therefore has a parabolic profile (in channel flow).



1.3.4 Formulae For Bed Shear Stress

Many sediment-transport formulae rely on knowledge of the bed shear stress τ_b, which is what sets the particles in motion and determines the bed load.

Recall that in normal flow there is a balance between the downslope component of gravity and bed friction, leading to:

$$\tau_b = \rho g R_h S \qquad (R_h \text{ is the hydraulic radius; } S \text{ is the slope}) \qquad (11)$$

Alternatively, by definition of the (skin-)friction coefficient c_f:

$$\tau_b = c_f (\tfrac{1}{2} \rho V^2) \qquad (V \text{ is the channel-average velocity}) \qquad (12)$$

If you are lucky, c_f may be given; (probably the only possibility in air). Otherwise one could adopt one of the following approaches. Both assume fully-developed flow (although this assumption is regularly stretched.)

In normal flow, if the discharge is known then Manning's equation:

$$V = \frac{1}{n} R_h^{2/3} S^{1/2} \qquad (13)$$

may be used to determine the hydraulic radius and thence, from (11), the bed shear stress. A useful correlation when the bed consists of granular material is (the dimensionally-inconsistent) Strickler's equation:

$$n = \frac{d^{1/6}}{21.1} \qquad (14)$$

where d is the particle diameter in m. This gives $n = 0.015$ m$^{-1/3}$ s for a grain size of 1 mm.

Alternatively, one can find an analytical expression for depth-averaged velocity V by integrating the velocity profile (6) to get (*exercise*: do it!):

$$V = \frac{1}{h}\int_0^h U(z)\,\mathrm{d}z \quad = \frac{u_\tau}{\kappa}\ln(\frac{12h}{k_s}) \tag{15}$$

The skin friction coefficient c_f is then, from its definition:

$$c_f \equiv \frac{\tau_b}{\frac{1}{2}\rho V^2} \quad = \frac{\rho u_\tau^2}{\frac{1}{2}\rho V^2} \quad = 2\left(\frac{u_\tau}{V}\right)^2$$

whence

$$c_f = \frac{0.34}{[\ln(12h/k_s)]^2} \tag{16}$$

Typical values of c_f are in the range 0.003 – 0.01.

2. THRESHOLD OF MOTION

2.1 Shields Parameter

In general, a granular bed will remain still until the flow is sufficient to move it. This point is called the *threshold of motion* and the bed stress that initiates it the *critical stress*, $\tau_{\rm crit}$.

According to a simple frictional model, on a flat bed:

critical stress × representative area = friction coefficient × normal reaction

$$\tau_{\rm crit} \times c\frac{\pi d^2}{4} = \mu_{\rm frict} \times (\rho_s - \rho) g \frac{\pi d^3}{6}$$

where c is a constant of order unity. Hence,

$$\frac{\tau_{\rm crit}}{(\rho_s - \rho)gd} = \frac{2\mu_{\rm frict}}{3c}$$

i.e.

$$\frac{\tau_{\rm crit}}{(\rho_s - \rho)gd} = \text{dimensionless function of particle size and shape} \qquad (17)$$

In practice, the dependence on shape is not very significant.

The non-dimensional stress

$$\tau^* = \frac{\tau_b}{(\rho_s - \rho)gd} \qquad (18)$$

is called the *Shields parameter* (in the sediment-transport literature often denoted θ) after American engineer A.F. Shields, who, in 1936, plotted his results on the initiation of sediment motion in the form of a graph of $\tau^*_{\rm crit}$ against the *particle Reynolds number* Re_p:

$$\tau^*_{\rm crit} = f(\mathrm{Re}_p) \qquad (19)$$

in what became known as a *Shields diagram*. Here,

$$\mathrm{Re}_p = \frac{u_\tau d}{\nu} \qquad (20)$$

In practice, the choice of dimensionless groups in the original Shields diagram is not convenient for predicting the threshold of motion because the bed stress τ_b effectively appears on both sides of the equation. (Remember that the friction velocity in Re_p is given by $u_\tau = \sqrt{\tau_b/\rho}$.) You will recall from dimensional analysis in Hydraulics 2 that one is at liberty to replace either of the dimensionless Π groups by another independent combination. In this case we can eliminate the dependence on bed stress (or u_τ) by forming

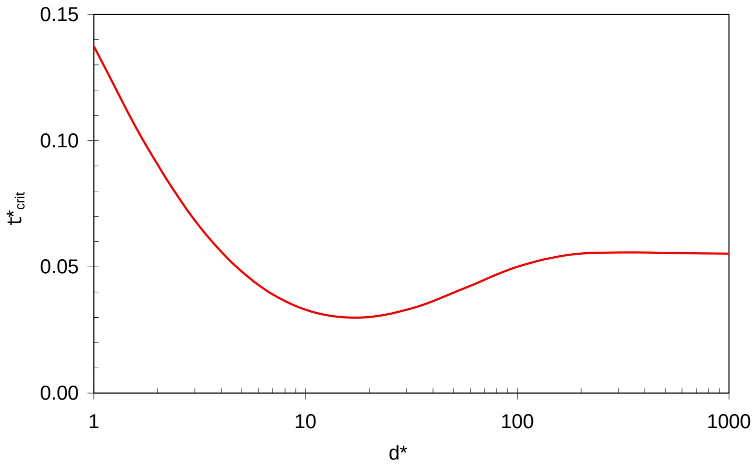
$$\frac{\mathrm{Re}_p^2}{\tau^*} = \frac{(s-1)gd^3}{\nu^2}$$

where $s = \rho_s/\rho$ is the relative density. Taking the cube root in order to give a dimensionless parameter proportional to diameter:

$$d^* = d\left[\frac{(s-1)g}{\nu^2}\right]^{1/3} \qquad (21)$$

Shields' threshold line may then be replotted as a function of $\tau^*_{\rm crit}$ against d^*:

$$\tau^*_{\rm crit} = f(d^*) \qquad (22)$$



A convenient curve fit to experimental data is provided by Soulsby (1997)[1]:

$$\tau^*_{\rm crit} = \frac{0.30}{1 + 1.2d^*} + 0.055\,[1 - \exp(-0.020d^*)] \qquad (23)$$

For large diameters the critical Shields parameter tends to a constant value; this is 0.055 for Soulsby's formula, although 0.056 is actually a more popular figure in the literature.

Summary of Calculation Formulae For Threshold of Motion

$$\tau^*_{\rm crit} = f(d^*)$$

e.g. graphically, or

$$\tau^*_{\rm crit} = \frac{0.30}{1 + 1.2d^*} + 0.055\;[1 - \exp(-0.020d^*)]$$

where

$$\tau^* = \frac{\tau_b}{(\rho_s - \rho)gd} \qquad \text{(Shields parameter)}$$

$$d^* = d\left[\frac{(s-1)g}{\nu^2}\right]^{1/3} \qquad (s = \rho_s/\rho)$$

[1] Soulsby, R., 1997, "Dynamics of Marine Sands", Thomas Telford.

Example. (Exam 2007 – part)

An undershot sluice is placed in a channel with a horizontal bed covered by gravel with a median diameter of 5 cm and density 2650 kg m^{-3}. The flow rate is 4 m^3 s^{-1} per metre width and initially the depth below the sluice is 0.5 m. Assuming a critical Shields parameter $\tau^*_{\rm crit}$ of 0.06 and friction coefficient c_f of 0.01:

(a) find the depth just upstream of the sluice and show that the bed there is stationary;

(b) show that the bed below the sluice will erode and determine the depth of scour.

2.2 Inception of Motion in Normal Flow

In the large-diameter limit, particles will move if

$$\frac{\tau_b}{(\rho_s-\rho)gd} > 0.056$$

But for normal flow, with hydraulic radius R_h and slope S:

$$\tau_b = \rho g R_h S$$

Putting these together, and noting that $\rho_s/\rho = 2.65$ for sand in water shows that, *for large particles in normal flow*, the bed will be mobile if

$$d < 10.8 R_h S \qquad (24)$$

For example, a uniform flow of depth 1 m on a slope of 10^{-4} will move sediment of diameter about 1 mm or less.

Note that this is an estimate, applying for large particles of a particular density in water and assuming normal flow. If possible it is better to compare τ with $\tau_{\rm crit}$ (or τ^* with $\tau^*_{\rm crit}$) to establish whether the bed is mobile.

2.3 Effect of Slopes

The main effect of slopes is to add (vectorially) a downslope component of gravity to the fluid stress acting on the particles. For a downward slope the gravitational component will assist in the initiation of motion, whereas for an adverse slope it will oppose it.

Let $\tau_{\rm crit}$ be the critical stress on a slope and $\tau_{\rm crit,0}$ be the critical stress on a flat bed. Let the slope angle be β and the angle of repose ϕ.

For slopes aligned with the flow:

$$\tau_{\text{crit}} = \frac{\sin(\phi + \beta)}{\sin \phi} \tau_{\text{crit},0} \qquad \text{(where } \beta \text{ is positive for upslope flow)}$$

For slopes at right angles to the flow:

$$\tau_{\text{crit}} = \cos \beta \sqrt{1 - \frac{\tan^2 \beta}{\tan^2 \phi}} \; \tau_{\text{crit},0}$$

For arbitary alignment see, e.g., Soulsby (1997)[1] or Apsley and Stansby (2008)[2].

[2] Apsley, D.D. and Stansby, P.K., 2008, Bed-load sediment transport with large slopes, model formulation and implementation within a RANS flow solver, Journal of Hydraulic Engineering, 134, 1440-1451.

3. BED LOAD

3.1 Dimensionless Groups

Bed-load transport of sediment is quantified by the bed-load flux q_b, the volume of non-suspended sediment crossing unit width of bed per unit time. The tendency of particles to move is determined by the drag imposed by the fluid flow and countered by the submerged particle weight, so that q_b may be taken as a function of bed shear stress τ_b, reduced gravity $(s-1)g$ (where $s = \rho_s/\rho$), particle diameter d, and the fluid density ρ and kinematic viscosity ν. A formal dimensional analysis (6 variables, 3 independent dimensions) dictates that there is a relationship between three non-dimensional groups, conveniently taken as

$$q^* = \frac{q_b}{\sqrt{(s-1)gd^3}} \qquad \text{(dimensionless bed-load flux)} \qquad (25)$$

$$\tau^* = \frac{\tau_b}{\rho(s-1)gd} \qquad \text{(dimensionless bed stress, or \textit{Shields parameter})} \qquad (26)$$

$$d^* = d\left[\frac{(s-1)g}{\nu^2}\right]^{1/3} \qquad \text{(dimensionless particle diameter)} \qquad (27)$$

3.2 Bed-Load Transport Models

Some of the more common bed-load transport formulae are listed in the table below. (For those containing τ^*_{crit}, bed-load sediment transport is zero until τ^* exceeds this.)

Reference	Formula	Comments
Meyer-Peter and Müller (1948)	$q^* = 8(\tau^* - \tau^*_{\text{crit}})^{3/2}$	
Nielsen (1992)	$q^* = 12(\tau^* - \tau^*_{\text{crit}})\sqrt{\tau^*}$	
Van Rijn (1984)	$q^* = \frac{0.053}{d^{*0.3}}(\frac{\tau^*}{\tau^*_{\text{crit}}} - 1)^{2.1}$	
Einstein[3]-Brown (Brown, 1950)	$q^* = \begin{cases} \frac{K\exp(-0.391/\tau^*)}{0.465} & \tau^* < 0.182 \\ 40K\tau^{*3} & \tau^* \geq 0.182 \end{cases}$	$K = \sqrt{\frac{2}{3} + \frac{36}{d^{*3}}} - \sqrt{\frac{36}{d^{*3}}}$
Yalin (1963)	$q^* = 0.635r\sqrt{\tau^*}[1 - \frac{1}{\sigma r}\ln(1 + \sigma r)]$	$r = \frac{\tau^*}{\tau^*_{\text{crit}}} - 1, \quad \sigma = 2.45\frac{\sqrt{\tau^*_{\text{crit}}}}{s^{0.4}}$

[3] Prof. H.A. Einstein was a noted hydraulic engineer – he was the son of Albert Einstein!

References

Brown, C.B. (1950). “Sediment Transport”, in *Engineering Hydraulics*, Ch. 12, Rouse, H. (ed.), Wiley.

Meyer-Peter, E. and Müller, R. (1948). “Formulas for bed-load transport.” *Rept 2nd Meeting Int. Assoc. Hydraul. Struct. Res.*, Stockholm, 39-64.

Nielsen, P., (1992). *Coastal Bottom Boundary Layers and Sediment Transport*, World Scientific.

Van Rijn, L.C., (1984). “Sediment transport. Part I: bed load transport.” *ASCE J. Hydraulic Eng.*, 110, 1431-1456.

Yalin, M.S., (1963). “An expression for bed-load transportation”, *Proc. ASCE,* 89, 221-250.

Note.

(i) Models have been written here in a common notation, which may differ a lot from that in the original papers.

(ii) The actual derivations of the original models vary considerably. Some are mechanical models of particle movement; others are probabilistic.

(iii) Several models contain a factor $(\tau^* - \tau^*_{\text{crit}})$ indicating, as expected, that sediment does not move until a certain bed stress has been exceed. Formulae for the threshold of motion:

$$\tau^*_{\text{crit}} = f(d^*)$$

have already been covered in Section 2.

4. SUSPENDED LOAD

4.1 Inception of Suspended Load

Particles cannot be swept up and maintained in the flow until the typical updraft of turbulent motions exceeds their natural settling velocity w_s. A typical turbulent velocity fluctuation is of the order of the friction velocity u_τ. Thus, suspended load will occur if

$$\frac{u_\tau}{w_s} > 1 \qquad (28)$$

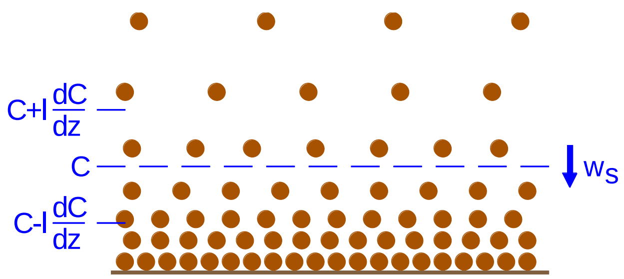
In practice, what is deemed to be suspended load rather than bed load is not precise, and different authors give slightly different numbers on the RHS.

For much gravelly sediment, suspended load does not occur and bed load is the only mode of sediment transport.

4.2 Turbulent Diffusion

The *sediment concentration* C is the volume of sediment per total volume of material (fluid + sediment).

Once sediment has been swept up by the flow it is distributed throughout the fluid depth. However, since it is always tending to settle out there is a greater concentration near the bed. Because the concentration is greater near the bed, any upward turbulent velocity fluctuation will tend to be carrying a larger amount of sediment than the corresponding downward fluctuation. Thus, where a concentration gradient exists turbulent diffusion will tend to lead to a net upward flux of material, whereas settling leads to a net downward flux. An equilibrium distribution is attained when these opposing effects are equal.



Suppose that the average concentration at some level z is C. In a simple model an upward turbulent velocity u' for half the time carries material of concentration $(C - l\,\mathrm{d}C/\mathrm{d}z)$, where l is a *mixing length* – a typical size of turbulent eddy. The corresponding downward velocity for the other half of the time carries material at concentration $(C + l\,\mathrm{d}C/\mathrm{d}z)$. The *average* upward flux of sediment (volume flux × concentration) through a horizontal area A is

$$\frac{1}{2}u'A(C - l\frac{\mathrm{d}C}{\mathrm{d}z}) - \frac{1}{2}u'A(C + l\frac{\mathrm{d}C}{\mathrm{d}z}) = -u'l\frac{\mathrm{d}C}{\mathrm{d}z}A$$

The quantity $u'l$ is written as a *diffusivity* K, so that the net upward flux is

$$-K\frac{\mathrm{d}C}{\mathrm{d}z}A \qquad (29)$$

This is referred to a *gradient diffusion* (because it is proportional to a gradient!) or *Fick's law of diffusion*. The minus sign indicates, as expected, that there is a net flux from high concentration to low.

At the same time there is a net *downward* flux of material $w_s AC$ due to settling. When the concentration profile has reached equilibrium the upward diffusive flux and downward settling flux are equal in magnitude; i.e.

$$-K\frac{\mathrm{d}C}{\mathrm{d}z}A = w_s AC$$

or, dividing by area:

$$-K\frac{\mathrm{d}C}{\mathrm{d}z} = w_s C \qquad (30)$$

4.3 Concentration Profile

Because it is the same turbulent eddies that are responsible for transporting both mean momentum and suspended particulate it is commonly assumed that the diffusivity K is the same as the eddy viscosity ν_t; i.e. for a wide channel and fully-developed flow:

$$K = \nu_t \quad = \kappa u_\tau z(1 - z/h)$$

where κ is von Kármán's constant (0.41), u_τ is the friction velocity ($= \sqrt{\tau_b/\rho}$), z is the vertical height above the bed and h is the channel depth. Hence, substituting in (30):

$$-\kappa u_\tau z(1 - z/h)\frac{\mathrm{d}C}{\mathrm{d}z} = w_s C$$

Separating variables and using partial fractions,

$$\frac{\mathrm{d}C}{C} = -\frac{w_s}{\kappa u_\tau}\left(\frac{1}{z} + \frac{1}{h - z}\right)\mathrm{d}z$$

Integrating between a reference height z_{ref} and general z gives, after some algebra:

$$\ln\frac{C}{C_{\mathrm{ref}}} = \frac{w_s}{\kappa u_\tau}\ln\frac{z_{\mathrm{ref}}(h - z)}{z(h - z_{\mathrm{ref}})}$$

whence

$$\frac{C}{C_{\mathrm{ref}}} = \left(\frac{h/z - 1}{h/z_{\mathrm{ref}} - 1}\right)^{\frac{w_s}{\kappa u_\tau}} \qquad (31)$$

This is called the *Rouse profile*, and the exponent

$$\frac{w_s}{\kappa u_\tau} \qquad (32)$$

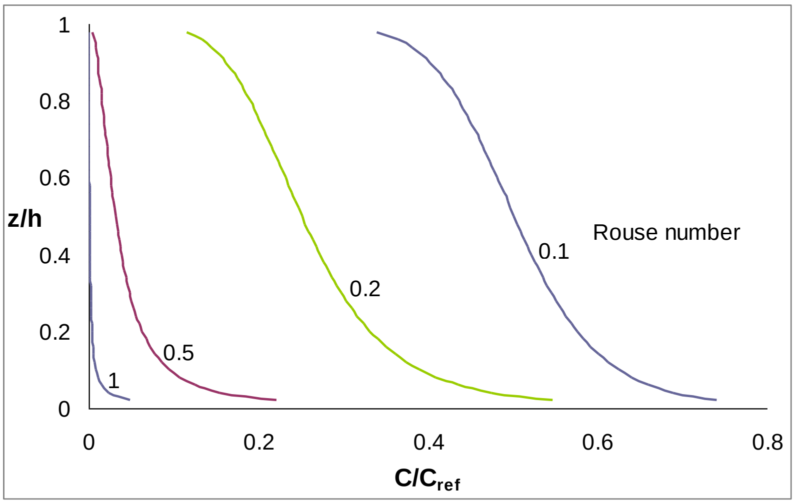
is called the *Rouse number* after H. Rouse (1937). Typical concentration profiles are shown below.

To be of much predictive use it is necessary to specify C_{ref} at some depth z_{ref}, typically at a height representative of the bed load. There are many such formulae but one of the simplest is that of Van Rijn (see, e.g., Chanson's book):

$$C_{\text{ref}} = \min\left\{\frac{0.117}{d^*}\left(\frac{\tau^*}{\tau^*_{\text{crit}}} - 1\right),\quad 0.65\right\}$$

$$\frac{z_{\text{ref}}}{d} = 0.3 d^{*0.7}\left(\frac{\tau^*}{\tau^*_{\text{crit}}} - 1\right)^{1/2} \qquad (33)$$

where d^* and τ^* are the dimensionless diameter and stress defined in Section 2.



4.4 Calculation of Suspended Load

Equation (31) describes the concentration profile, but usually the most important quantity is the total sediment flux.

The volume flux (per unit width) of fluid through a depth $\mathrm{d}z$ in a wide channel is

$$u\ \mathrm{d}z$$

Since concentration C is the volume of sediment per volume of fluid, the volume flux of sediment through the same depth is

$$Cu\ \mathrm{d}z$$

Hence, the total suspended flux through the entire depth of the channel is (per unit width):

$$q_s = \int_{z_{\text{ref}}}^{h} Cu\,\mathrm{d}z \qquad (34)$$

This must be obtained from C and u profiles by numerical integration – see the Examples.

Summary of Suspended-Load Formulae

Incipient suspended load:

$$\frac{u_\tau}{w_s} > 1$$

Concentration profile:

$$\frac{C}{C_{\text{ref}}} = \left(\frac{h/z - 1}{h/z_{\text{ref}} - 1}\right)^{\frac{w_s}{\kappa u_\tau}}$$

where

$$C_{\text{ref}} = \min\left\{\frac{0.117}{d^*}\left(\frac{\tau^*}{\tau^*_{\text{crit}}} - 1\right), \quad 0.65\right\}$$

$$\frac{z_{\text{ref}}}{d} = 0.3 d^{*0.7}\left(\frac{\tau^*}{\tau^*_{\text{crit}}} - 1\right)^{1/2}$$

Velocity profile:

$$U(z) = \frac{u_\tau}{\kappa} \ln(33 \frac{z}{k_s})$$

Sediment flux (per unit width):

$$q_s = \int_{z_{\text{ref}}}^{h} CU \, \mathrm{d}z$$