Section 1.2

Example.
The discharge in a channel with bottom width 3 m is 12 m$^3$ s$^{-1}$. If Manning’s $n$ is 0.013 m$^{-1/3}$ s$^{-1}$ and the streamwise slope is 1 in 200, find the normal depth if:
(a) the channel has vertical sides (i.e. rectangular channel);
(b) the channel is trapezoidal with side slopes 2H:1V.

\[ b = 3 \text{ m (base width)} \]
\[ Q = 12 \text{ m}^3 \text{ s}^{-1} \]
\[ n = 0.013 \text{ m}^{-1/3} \text{ s}^{-1} \]
\[ S = 0.005 \]

(a) Discharge:
\[ Q = VA \]
where, in normal flow,
\[ V = \frac{1}{n} R_h^{2/3} S^{1/2}, \quad A = bh, \quad R_h = \frac{bh}{b+2h} = \frac{h}{1+2h/b} \]
Hence,
\[ Q = \frac{1}{n} \frac{bh^{5/3}}{(1+2h/b)^{2/3}} S^{1/2} \]
Re-arranging as an iterative formula for $h$:
\[ h = \left( \frac{nQ}{bS} \right)^{3/5} \left(1 + 2h/b \right)^{2/5} \]
Here, with lengths in metres,
\[ h = 0.8316 \left(1 + 2h/3 \right)^{2/5} \]
Iteration (from, e.g., $h = 0.8316$) gives
\[ h_n = 1.024 \text{ m} \]

**Answer:** normal depth = 1.02 m.

(b) Geometry: trapezoidal cross-section with base width $b$, surface width $b + 2 \times (2h)$ and two sloping side lengths $\sqrt{h^2 + (2h)^2} = h\sqrt{5}$.

Area and wetted perimeter:
\[ A = \frac{1}{2} (b + b + 4h)h = h(b + 2h) = hb(1 + 2h/b) \]
\[ P = b + 2h\sqrt{5} \]
Hydraulic radius:

\[ R_h \equiv \frac{A}{P} = \frac{h(b + 2h)}{b + 2h\sqrt{5}} = h \left( \frac{1 + 2h/b}{1 + 2\sqrt{5}h/b} \right) \]

Discharge:

\[ Q = V\bar{A} = \frac{1}{n} R_h^{2/3} S^{1/2} A \]

Hence,

\[ Q = \frac{1}{n} h^{2/3} \left( \frac{1 + 2h/b}{1 + 2\sqrt{5}h/b} \right)^{2/3} S^{1/2} h b (1 + 2h/b) \]

\[ \Rightarrow \frac{nQ}{b\sqrt{5}} = h^{5/3} \frac{(1 + 2h/b)^{5/3}}{(1 + 2\sqrt{5}h/b)^{2/3}} \]

\[ \Rightarrow h = \left( \frac{nQ}{b\sqrt{5}} \right)^{3/5} \frac{(1 + 2\sqrt{5}h/b)^{2/5}}{1 + 2h/b} \]

Here, with lengths in metres,

\[ h = 0.8316 \frac{(1 + 1.491h)^{2/5}}{1 + 2h/3} \]

Iteration (from, e.g., \( h = 0.8316 \)) gives

\[ h_n = 0.7487 \text{ m} \]

**Answer:** normal depth = 0.749 m.
Section 1.4

Example.
The discharge in a rectangular channel of width 6 m with Manning’s $n = 0.012 \text{ m}^{-1/3} \text{ s}$ is $24 \text{ m}^3 \text{ s}^{-1}$. If the streamwise slope is 1 in 200 find:
(a) the normal depth;
(b) the Froude number at the normal depth;
(c) the critical depth.
State whether the normal flow is subcritical or supercritical.

$b = 6 \text{ m}$
$n = 0.012 \text{ m}^{-1/3} \text{ s}$
$Q = 24 \text{ m}^3 \text{ s}^{-1}$
$S = 0.005$

(a) Discharge:

$$Q = VA$$

where, in normal flow,

$$V = \frac{1}{n} R_h^{2/3} S^{1/2}, \quad A = bh, \quad R_h = \frac{bh}{b + 2h} = \frac{h}{1 + 2h/b}$$

Hence,

$$Q = \frac{1}{n} \frac{bh^{5/3}}{(1 + 2h/b)^{2/3}} S^{1/2}$$

or, rearranging as an iterative formula for $h$:

$$h = \left( \frac{nQ}{b \sqrt{S}} \right)^{3/5} (1 + 2h/b)^{2/5}$$

Here, with lengths in metres,

$$h = 0.7926 (1 + h/3)^{2/5}$$

Iteration (from, e.g., $h = 0.7926$) gives

$$h_n = 0.8783 \text{ m}$$

**Answer:** normal depth $= 0.878 \text{ m}$.

(b) At the normal depth, $h = 0.8783 \text{ m}$:

$$V = \frac{Q}{A} = \frac{24}{6 \times 0.8783} = 4.554 \text{ m s}^{-1}$$

$$Fr = \frac{V}{\sqrt{gh}} = \frac{4.554}{\sqrt{9.81 \times 0.8783}} = 1.551$$

**Answer:** Froude number $= 1.55$. 

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(c) The critical depth is that depth (at the given flow rate) for which \( \text{Fr} = 1 \). It is not normal flow, and does not depend on the slope \( S \) or the roughness \( n \).

\[
\text{Fr} = \frac{V}{\sqrt{gh}}
\]

where

\[
V = \frac{Q}{A} \quad \text{(in general)}
\]

or \( V = \frac{Q}{bh} = \frac{q}{h} \) (for a rectangular channel; \( q \) is the flow per unit width)

Hence, for a rectangular channel,

\[
\text{Fr}^2 = \frac{(q/h)^2}{gh} = \frac{q^2}{gh^3}
\]

For critical flow, \( \text{Fr} = 1 \) and so

\[
h_c = \left( \frac{q^2}{g} \right)^{1/3}
\]

Here, the flow per unit width is \( q = 24/6 = 4 \) m\(^2\) s\(^{-1}\), so that

\[
h_c = \left( \frac{4^2}{9.81} \right)^{1/3} = 1.177 \text{ m}
\]

**Answer:** critical depth = 1.18 m.

The normal depth is supercritical because, when \( h = h_n \), then \( \text{Fr} > 1 \) (part (b)).

Alternatively (and often more conveniently), the normal depth here is supercritical because \( h_n < h_c \); so speed \( V \) is larger, and depth \( h \) is smaller in normal flow than critical flow, so that \( \text{Fr} \equiv V/\sqrt{gh} \) must be greater than 1.
Section 2.2

Example.
A 3-m wide channel carries a total discharge of 12 m³ s⁻¹. Calculate:
(a) the critical depth;
(b) the minimum specific energy;
(c) the alternate depths when $E = 4$ m.

\[ b = 3 \text{ m} \]
\[ Q = 12 \text{ m}^3 \text{ s}^{-1} \]

(a)
Discharge per unit width:
\[ q = \frac{Q}{b} = \frac{12}{3} = 4 \text{ m}^2 \text{ s}^{-1} \]
Then, for a rectangular channel:
\[ h_c = \left( \frac{q^2}{g} \right)^{1/3} = \left( \frac{4^2}{9.81} \right)^{1/3} = 1.177 \text{ m} \]

Answer: critical depth = 1.18 m.

(b) For a rectangular channel,
\[ E_c = \frac{3}{2}h_c = \frac{3}{2} \times 1.177 = 1.766 \text{ m} \]

Answer: minimum specific energy = 1.77 m.

(c) As $E > E_c$, there are two possible depths for a given specific energy.
\[ E \equiv h + \frac{V^2}{2g} \quad \text{where} \quad V = \frac{Q}{A} = \frac{q}{h} \quad \text{(for a rectangular channel)} \]
\[ \Rightarrow E \equiv h + \frac{q^2}{2gh^2} \]
Substituting values in metre-second units:
\[ 4 \equiv h + \frac{0.8155}{h^2} \]
For the subcritical (slow, deep) solution, the first term, associated with potential energy, dominates, so rearrange as:
\[ h = 4 - \frac{0.8155}{h^2} \]
Iteration (from, e.g., $h = 4$) gives $h = 3.948 \text{ m}$.

For the supercritical (fast, shallow) solution, the second term, associated with kinetic energy, dominates, so rearrange as:

$$h = \sqrt{\frac{0.8155}{4 - h}}$$

Iteration (from, e.g., $h = 0$) gives $h = 0.4814 \text{ m}$.

**Answer:** alternate depths are 3.95 m and 0.48 m.
Section 2.3.1

Example. (Exam 2020)
(a) Define:
(i) specific energy
(ii) Froude number
for open-channel flow. What is special about these quantities in critical conditions?

A long, wide channel has a slope of 1:1000, a Manning’s n of 0.015 m−1/3 s and a discharge of 3 m³ s⁻¹ per metre width.
(b) Calculate the normal and critical depths.
(c) In a region of the channel the bed is raised by a height of 0.8 m over a length sufficient for the flow to be parallel to the bed over this length. Determine the depths upstream, downstream and over the raised bed, ignoring frictional losses. Sketch the key features of the flow, indicating all hydraulic transitions caused by the bed rise.
(d) In the same channel, the bed is lowered by 0.8 m from its original level. Determine the depths upstream, downstream and over the lowered bed, ignoring frictional losses. Sketch the flow.

(a)
(i) Specific energy is the head relative to the local bed of the channel:

\[ E = h + \frac{V^2}{2g} \]

(ii) The Froude number is

\[ Fr = \frac{V}{\sqrt{gh}} \]

In critical conditions Fr = 1 and the specific energy is the minimum for that discharge.

(b)
\[ S = 0.001 \]
\[ n = 0.015 \text{ m}^{-1/3} \text{ s} \]
\[ q = 3 \text{ m}^3 \text{ s}^{-1} \]

Normal depth:
Discharge per unit width:
\[ q = Vh, \quad \text{where} \quad V = \frac{1}{n} R_n^{2/3} S^{1/2} \quad \text{(Manning),} \quad R_n = h \quad \text{ (“wide” channel)} \]

\[ \Rightarrow \quad q = \frac{1}{n} h^{2/3} S^{1/2} \]

\[ \Rightarrow \quad q = \frac{h^{5/3} \sqrt{S}}{n} \]
\[ h_n = \left( \frac{nq}{\sqrt{S}} \right)^{3/5} = \left( \frac{0.015 \times 3}{\sqrt{0.001}} \right)^{3/5} = 1.236 \text{ m} \]

Critical depth:

\[ h_c = \left( \frac{q^2}{g} \right)^{1/3} = \left( \frac{3^2}{9.81} \right)^{1/3} = 0.9717 \text{ m} \]

**Answer:** normal depth = 1.24 m; critical depth = 0.972 m.

(c) To determine the type of behaviour over the raised bed, compare the total head under critical conditions (the minimum energy necessary to get over the weir at this flow rate) with that available in the approach flow.

**Critical**

\[ h_c = 0.9717 \text{ m} \]

\[ E_c = \frac{3}{2} h_c = 1.458 \text{ m} \]

\[ z_b = 0.8 \text{ m} \]

\[ H_c = z_b + E_c = 2.258 \text{ m} \]

**Approach Flow**

Because the channel is described as “long” it will have sufficient fetch to develop normal flow; hence the approach-flow head is that for the normal depth \((h = 1.236 \text{ m})\):

\[ H_a = E_a = h_n + \frac{V_n^2}{2g} \]

\[ = h_n + \frac{q^2}{2gh_n^2} \]

\[ = 1.236 + \frac{3^2}{2 \times 9.81 \times 1.236^2} \]

\[ = 1.536 \text{ m} \]

At the normal depth the available head \((H_a)\) is less than the minimum required to get over the bed rise \((H_c)\). Hence the water depth must increase upstream (“back up”), to raise the head immediately upstream. Thus:

- critical conditions *do* occur;
- the total head in the vicinity is the critical head \((H = H_c = 2.258 \text{ m})\).

Over the raised bed there is a hydraulic transition, so the depth over this is critical: \(h = h_c = 0.9717 \text{ m}\).

Just up- or downstream,
\[ H = E = h + \frac{V^2}{2g} \quad \text{where} \quad V = \frac{q}{h} \]

\[ \Rightarrow \quad H = h + \frac{q^2}{2gh^2} \]

\[ \Rightarrow \quad 2.258 = h + \frac{0.4587}{h^2} \]

**Upstream**, rearrange for the deep, subcritical, solution:

\[ h = 2.258 - \frac{0.4587}{h^2} \]

Iteration (from, e.g., \( h = 2.258 \)) gives \( h = 2.160 \) m.

**Downstream**, rearrange for the shallow, supercritical solution:

\[ h = \sqrt{\frac{0.4587}{2.258 - h}} \]

Iteration (from, e.g., \( h = 0 \)) gives \( h = 0.5127 \) m.

Since the preferred (i.e. normal) depth is subcritical, there must be a downstream hydraulic jump. (A quick calculation shows that the upstream depth for this jump is greater than \( h_2 \), so there is indeed a length of GVF between the area of bed rise and the jump.)

**Answer:** depths upstream, over, downstream of the raised bed: 2.16 m, 0.972 m, 0.513 m.

(d) The flow does not require additional energy to pass a depressed section; hence, the total head throughout is that supplied by the approach flow \( (H = H_a = 1.536 \text{ m}) \) and the flow remains subcritical. The depths just upstream and downstream of the lowered section are those in the approach flow; i.e. normal depth.

As bed height \( z_b \) decreases, specific energy \( E \) must increase to maintain the same total head. In the lowered section:

\[ H = z_b + E \]

\[ \Rightarrow \quad 1.536 = -0.8 + E \]

\[ \Rightarrow \quad E = 2.336 \text{ m} \]
Then

\[ E = h + \frac{V^2}{2g} \]

where \( V = \frac{q}{h} \)

\( \Rightarrow \)

\[ E = h + \frac{q^2}{2gh^2} \]

\( \Rightarrow \)

\[ 2.336 = h + \frac{0.4587}{h^2} \]

As we require the subcritical solution, rearrange as

\[ h = 2.336 - \frac{0.4587}{h^2} \]

Iteration (from, e.g., \( h = 2.336 \)) gives \( h = 2.245 \) m.

(Note that this is the depth of the water column. The actual surface level here is

\[ z_s = -0.8 + h = 1.445 \] m

so the overall water level also rises in this section.)

**Answer:** depths upstream, within, downstream of the lowered section: 1.24 m, 2.24 m, 1.24 m.
Section 2.3.1

Example.
A long channel of rectangular cross-section with width 3.5 m and streamwise slope 1 in 800 carries a discharge of 15 m$^3$ s$^{-1}$. Manning’s $n$ may be taken as 0.016 m$^{-1/3}$ s. A broad-crested weir of height 0.7 m is constructed at the centre of the channel. Determine:
(a) the depth far upstream of the weir;
(b) the depth just upstream of the weir;
(c) whether or not a region of supercritical gradually-varied flow exists downstream of the weir.

\[ b = 3.5 \text{ m} \]
\[ S = 0.00125 \]
\[ Q = 15 \text{ m}^3 \text{ s}^{-1} \]
\[ n = 0.016 \text{ m}^{-1/3} \text{ s} \]
\[ z_{\text{weir}} = 0.7 \text{ m} \]

(a) The depth far upstream is normal since the channel is described as “long”. For normal flow in a rectangular channel:
\[ Q = VA \]
where:
\[ V = \frac{1}{n} R_h^{2/3} S^{1/2} \quad A = bh \]
\[ R_h = \frac{bh}{b + 2h} = \frac{h}{1 + 2h/b} \]
Hence,
\[ Q = \frac{1}{n} \frac{bh^{5/3}}{(1 + 2h/b)^{2/3} S^{1/2}} \]
or, rearranging as an iterative formula for $h$:
\[ h = \left( \frac{nQ}{b \sqrt{S}} \right)^{3/5} \left( 1 + 2h/b \right)^{2/5} \]

Here, with lengths in metres,
\[ h = 1.488 \left( 1 + 0.5714h \right)^{2/5} \]
Iteration (from, e.g., $h = 1.488$) gives
\[ h = 2.023 \text{ m} \]

**Answer:** depth far upstream = 2.02 m.

(b) To establish depths near the weir we need to know the flow behaviour at the weir. Compare the energy in the approach flow with that under critical conditions.

Approach flow
\( h = 2.023 \text{ m} \) (from part (a))

\[
V = \frac{Q}{A} = \frac{Q}{bh} = \frac{15}{3.5 \times 2.023} = 2.118 \text{ m s}^{-1}
\]

Specific energy in the approach flow:

\[
E_a = h + \frac{V^2}{2g} = 2.023 + \frac{2.118^2}{2 \times 9.81} = 2.252 \text{ m}
\]

Referring heads to the undisturbed bed near the weir:

\[
H_a = E_a = 2.252 \text{ m}
\]

Critical conditions

\[
h_c = \left(\frac{q^2}{g}\right)^{1/3}, \quad q = \frac{Q}{b} = \frac{15}{3.5} = 4.286 \text{ m}^2 \text{ s}^{-1}
\]

\[
\Rightarrow \quad h_c = \left(\frac{4.286^2}{9.81}\right)^{1/3} = 1.233 \text{ m}
\]

\[
E_c = \frac{3}{2}h_c = \frac{3}{2} \times 1.233 = 1.850 \text{ m}
\]

\[
H_c = z_{\text{weir}} + E_c = 0.7 + 1.850 = 2.550 \text{ m}
\]

Since the head required to flow over the weir \( (H_c = 2.550 \text{ m}) \) exceeds that in the approach flow \( (H_a = 2.252 \text{ m}) \), the depth just upstream of the weir must increase and the flow back up. The total head at any position in the vicinity of the weir is \( H = H_c = 2.550 \text{ m} \).

Just upstream and downstream of the weir (i.e. at undisturbed bed level):

\[
H = E = h + \frac{V^2}{2g} \quad \text{\&} \quad V = \frac{Q}{bh}
\]

\[
\Rightarrow \quad H = h + \frac{Q^2}{2gb^2h^2}
\]

\[
\Rightarrow \quad 2.550 = h + \frac{0.9362}{h^2} \quad (*)
\]

The depth just upstream is the deep, subcritical solution. Hence, rearrange as

\[
h = 2.550 - \frac{0.9362}{h^2}
\]

Iteration (from, e.g., \( h = 2.550 \)) gives

\[
h = 2.385 \text{ m}
\]

**Answer:** depth just upstream of the weir = 2.39 m.
(c) Since the normal flow is subcritical, the flow must return to it via a hydraulic jump on the downstream side of the weir.

If the flow in the vicinity of the weir is unaffected by the hydraulic jump the flow goes smoothly supercritical on the downstream side, with total head $H = 2.550$ m (equation (*)). Rearranging to get an iterative formula for the supercritical solution:

$$ h = \sqrt{\frac{0.9362}{2.550 - h}} $$

Iteration (from, e.g., $h = 0$) gives

$$ h = 0.7141 \text{ m} $$

Denote by subscripts A and B respectively the conditions upstream and downstream of the hydraulic jump. On the downstream side conditions may be assumed normal, since the channel is “long” and hence there is sufficient fetch to develop the preferred depth:

$$ h_B = 2.023 \text{ m} $$
$$ V_B = 2.118 \text{ m s}^{-1} \quad \text{(from part (b))} $$
$$ \text{Fr}_B = \frac{V_B}{\sqrt{gh_B}} = \frac{2.118}{\sqrt{9.81 \times 2.023}} = 0.4754 $$

Hence, from the hydraulic-jump relation for the sequent depths:

$$ h_A = \frac{h_B}{2} \left( -1 + \sqrt{1 + 8\text{Fr}^2_B} \right) = \frac{2.023}{2} \left( -1 + \sqrt{1 + 8 \times 0.4754^2} \right) = 0.6835 \text{ m} $$

Any gradually-varied supercritical flow downstream of the weir would increase in depth until a hydraulic jump occurred (see the lectures on GVF). Since the depth downstream of the weir is already greater than any sequent depth upstream of the hydraulic jump, no such increasing-depth GVF is possible and the hydraulic jump must actually occur at (or just before) the downstream end of the weir.
Example.
A reservoir has a plan area of 50 000 m². The outflow passes over a broad-crested weir of width 8 m and discharge coefficient 0.9. Calculate:
(a) the discharge when the level in the reservoir is 0.6 m above the top of the weir;
(b) the time taken for the level of water in the reservoir to fall by 0.3 m.

(a) Total head:

\[ H = z_s + \frac{v^2}{2g} = z_b + E \]

where levels \( z \) can be measured relative to any convenient datum. Relative to the top of the weir, assuming constant head, still water in the reservoir and critical conditions over the weir:

head upstream = head over weir

\[ h_0 = \frac{3}{2} \left( \frac{q^2}{g} \right)^{1/3} \]

Hence, in ideal flow,

\[ q = \sqrt{(2/3)^3 gh_0^3} \]

\[ Q_{ideal} = qb = b \sqrt{(2/3)^3 gh_0^3} = 1.705bh_0^{3/2} \quad \text{(in metre-second units)} \]

Representing non-ideal behaviour via a discharge coefficient \( c_d \), and taking \( b = 8 \) m,

\[ Q = c_dQ_{ideal} = 0.9 \times 1.705 \times 8h_0^{3/2} \]

\[ Q = 12.28h_0^{3/2} \]

When \( h_0 = 0.6 \) m,

\[ Q = 12.28 \times 0.6^{3/2} = 5.707 \text{ m}^3 \text{ s}^{-1} \]

**Answer:** initial discharge = 5.71 m³ s⁻¹.

(b) Drop the subscript 0 and write the freeboard as \( h \).

Consider the change in volume of a tank, water surface area \( A_{ws} \). When the water level changes by \( dh \) the change in volume is

\[ \text{d(volume)} = A_{ws} dh \]

Hence, by continuity,
\[
\frac{d}{dt} \text{(volume)} = Q_{\text{in}} - Q_{\text{out}}
\]

\[\Rightarrow A_{ws} \frac{dh}{dt} = 0 - 12.28h^{3/2}\]

\[\Rightarrow 50000 \frac{dh}{dt} = -12.28h^{3/2}\]

Separating variables,

\[50000 \frac{dh}{h^{3/2}} = -12.28 \, dt\]

Apply boundary conditions \(h = 0.6\) when \(t = 0\) and \(h = 0.3\) when \(t = T\), and integrate:

\[50000 \int_{0.6}^{0.3} h^{-3/2} \, dh = -12.28 \int_0^T dt\]

\[\Rightarrow 50000 \left[ \frac{h^{-1/2}}{(-1/2)} \right]_{0.6}^{0.3} = -12.28T\]

\[\Rightarrow \frac{50000 \times 2}{12.28} \left( \frac{1}{\sqrt{0.3}} - \frac{1}{\sqrt{0.6}} \right) = T\]

\[\Rightarrow T = 4355 \, \text{s}\]

**Answer:** time = 4360 s (about 73 min).
Section 2.3.2

Example (Exam 2008 – modified, including a change in the value of \( n \))

A venturi flume is placed near the middle of a long rectangular channel with Manning’s \( n = 0.012 \text{ m}^{-1/3} \text{ s} \). The channel has a width of 5 m, a discharge of 12.5 m\(^3\) s\(^{-1}\) and a slope of 1:2500.

(a) Determine the critical depth and the normal depth in the main channel.
(b) Determine the venturi flume width which will just make the flow critical at the contraction.
(c) If the contraction width is 2 m find the depths just upstream, downstream and at the throat of the venturi flume (neglecting friction in this short section).
(d) Sketch the surface profile.

\( n = 0.012 \text{ m}^{-1/3} \text{ s} \)
\( b = 5 \text{ m} \) (main channel)
\( Q = 12.5 \text{ m}^3 \text{ s}^{-1} \)
\( S = 4 \times 10^{-4} \)

(a)

In the main channel,
\[ q = \frac{Q}{b} = \frac{12.5}{5} = 2.5 \text{ m}^2 \text{ s}^{-1} \]

Critical Depth
\[ h_c = \left( \frac{q^2}{g} \right)^{1/3} = \left( \frac{2.5^2}{9.81} \right)^{1/3} = 0.8605 \text{ m} \]

Normal Depth
\[ Q = VA \]
where:
\[ V = \frac{1}{n}R_h^{2/3}S^{1/2}, \quad A = bh, \quad R_h = \frac{bh}{b + 2h} = \frac{h}{1 + 2h/b} \]

Hence,
\[ Q = \frac{1}{n} \frac{bh^{5/3}}{(1 + 2h/b)^{2/3}S^{1/2}} \]
or, rearranging as an iterative formula for \( h \):
\[ h = \left( \frac{nQ}{b\sqrt{S}} \right)^{3/5} (1 + 2h/b)^{2/5} \]

Here, with lengths in metres,
\[ h = 1.275 \left(1 + 0.4h\right)^{2/5} \]

Iteration (from, e.g., \( h = 1.275 \)) gives

\[ h_n = 1.546 \text{ m} \]

**Answer:** critical depth = 0.860 m; normal depth = 1.55 m.

(b) The flow will just go critical if the head in the throat \( (H_c) \) is exactly equal to that in the approach flow \( (H_a) \). Measure heads relative to the bed of the channel in the vicinity of the venturi.

**Critical Head**

\[
H_c = E_c = \frac{3}{2} h_c = \frac{3}{2} \left(\frac{q_m^2}{g}\right)^{1/3}, \quad \text{where} \quad q_m = \frac{Q}{b_m} 
\]

(Note that the critical depth is different at the throat to that in the main channel, due to the narrower width.)

\[
\Rightarrow \quad H_c = \frac{3}{2} \left(\frac{Q^2}{g b_m^2}\right)^{1/3}
\]

**Approach Flow**

The approach flow is normal, since the channel is “long”. Hence,

\[
h_a = 1.546 \quad \text{m}
\]

\[
V_a = \frac{Q}{b h_a} = \frac{12.5}{5 \times 1.546} = 1.617 \quad \text{m s}^{-1}
\]

\[
H_a = E_a = h_a + \frac{V_a^2}{2g} = 1.546 + \frac{1.617^2}{2 \times 9.81} = 1.679 \quad \text{m}
\]

For the flow just to go critical at the throat,

\[
H_c = H_a
\]

\[
\Rightarrow \quad \frac{3}{2} \left(\frac{Q^2}{g b_m^2}\right)^{1/3} = 1.679
\]

\[
\Rightarrow \quad b_m = \frac{12.5}{\left(\frac{2}{3} \times 1.679\right)^{3/2} \sqrt{9.81}} = 3.370 \quad \text{m}
\]

**Answer:** throat width = 3.37 m.
(c) If the throat width is reduced further, then the flow will back up and undergo a critical transition at the throat.

At the throat,
\[ q_m = \frac{Q}{b_m} = \frac{12.5}{2} = 6.25 \text{ m}^2 \text{ s}^{-1} \]

\[ h_c = \frac{3}{2} \left( \frac{q_m^2}{g} \right)^{1/3} = \left( \frac{6.25^2}{9.81} \right)^{1/3} = 1.585 \text{ m} \]

\[ E_c = \frac{3}{2} h_c = \frac{3}{2} \times 1.585 = 2.378 \text{ m} \]

\[ H_c = z_b + E_c = 0 + 2.378 = 2.378 \text{ m} \]

The head throughout the venturi will be the critical head (\( H = H_c = 2.378 \text{ m} \)).

Anywhere in the flume,
\[ H = z_s + \frac{V^2}{2g} \quad \text{where} \quad z_s = h \quad V = \frac{Q}{bh} \]

\[ \Rightarrow \quad H = h + \frac{Q^2}{2gb^2h^2} \]

At the throat the depth will be the critical depth there; i.e. \( h = h_c = 1.585 \text{ m} \).

Just upstream and downstream, \( b = 5 \text{ m} \); hence,
\[ 2.378 = h + \frac{0.3186}{h^2} \]

**Upstream**
Rearrange for the deep, subcritical solution:
\[ h = 2.378 - \frac{0.3186}{h^2} \]

Iteration (from, e.g., \( h = 2.378 \)) gives \( h = 2.319 \text{ m} \).

**Downstream**
Rearrange for the shallow, supercritical solution:
\[ h = \sqrt{\frac{0.3186}{2.378 - h}} \]

Iteration (from, e.g., \( h = 0 \)) gives \( h = 0.4015 \text{ m} \).

**Answer:** depths upstream, in the throat, downstream = 2.32 m, 1.59 m, 0.401 m.
(d) PLAN VIEW

\[ b_{\text{min}} \]

WATER PROFILE

\[ \text{critical} \]
**Example.**
The water depth upstream of a sluice gate is 0.8 m and the depth just downstream (at the vena contracta) is 0.2 m. Calculate:
(a) the discharge per unit width;
(b) the Froude numbers upstream and downstream.

\[ h_1 = 0.8 \text{ m}\]
\[ h_2 = 0.2 \text{ m}\]

(a) Assuming total head the same on either side of the gate:
\[ z_{s1} + \frac{V_1^2}{2g} = z_{s2} + \frac{V_2^2}{2g} \]
Substituting \( z_s = h \) and \( V = q/h \):
\[ h_1 + \frac{q^2}{2gh_1^2} = h_2 + \frac{q^2}{2gh_2^2} \]
From the given data, in metre-second units:
\[ 0.8 + 0.0796q^2 = 0.2 + 1.2742q^2 \]
\[ \Rightarrow 0.6 = 1.1946q^2 \]
\[ \Rightarrow q = 0.7087 \text{ m}^2 \text{ s}^{-1} \]
**Answer:** discharge per unit width = 0.709 m$^2$s$^{-1}$.

(b) Use, on each side of the gate,
\[ V = \frac{q}{h} \]
\[ Fr = \frac{V}{\sqrt{gh}} \]
to get
\[ V_1 = 0.8859 \text{ m s}^{-1} \]
\[ V_2 = 3.544 \text{ m s}^{-1} \]
and then
\[ Fr_1 = 0.3162 \]
\[ Fr_2 = 2.530 \]
**Answer:** Froude numbers upstream, downstream = 0.316, 2.53.
Section 2.3.3

Example.
A sluice gate controls the flow in a channel of width 2 m. If the discharge is 0.5 m$^3$ s$^{-1}$ and the upstream water depth is 1.5 m, calculate the downstream depth and velocity.

$b = 2$ m  
$Q = 0.5$ m$^3$ s$^{-1}$  
$h_1 = 1.5$ m

Use upstream conditions to get total head. Then, assuming no losses, find the supercritical flow with the same head.

Total head (either side):

$$H = z_s + \frac{V^2}{2g}$$

where $z_s = h$ and $V = \frac{Q}{bh}$

$$\Rightarrow \quad H = h + \frac{Q^2}{2gb^2h^2} = h + \frac{3.186 \times 10^{-3}}{h^2}$$

The upstream depth $h_1 = 1.5$ m gives

$$H = 1.5 + \frac{3.186 \times 10^{-3}}{1.5^2} = 1.501 \text{ m} \quad \text{(dominated by } h_1\text{)}$$

Hence,

$$1.501 = h + \frac{3.186 \times 10^{-3}}{h^2}$$

Rearrange for the shallow, supercritical solution:

$$h = \sqrt{\frac{3.186 \times 10^{-3}}{1.501 - h}}$$

Iteration (from, e.g., $h = 0$) gives

$$h_2 = 0.04681 \text{ m}$$

$$V_2 = \frac{Q}{bh_2} = \frac{0.5}{2 \times 0.04681} = 5.341 \text{ m s}^{-1}$$

Answer: downstream depth = 0.0468 m; velocity = 5.34 m s$^{-1}$. 

---

Hydraulics 3  
Dr David Apsley
Section 2.4

Example. (Exam 2018)
Water flows at 0.8 m³ s⁻¹ per metre width down a long, wide spillway of slope 1 in 30 onto a wide apron of slope 1 in 1000. Manning’s roughness coefficient \( n = 0.014 \text{ m}^{-1/3} \text{ s} \) on both slopes.

(a) Find the normal depths in both sections and show that normal flow is supercritical on the spillway and subcritical on the apron.

(b) Baffle blocks are placed a short distance downstream of the slope transition to provoke a hydraulic jump. Assuming that flow is normal on both the spillway and downstream of the hydraulic jump, calculate the force per metre width of channel that the blocks must impart.

(c) Find the head loss across the blocks.

\[ S_1 = 1/30; \quad S_2 = 1/1000 \]
\[ q = 0.8 \text{ m}^2 \text{ s}^{-1} \]
\[ n = 0.014 \text{ m}^{-1/3} \text{ s} \]

(a) Normal flow:

\[ q = Vh, \quad \text{where} \quad V = \frac{1}{n} R_h^{2/3} S^{1/2} \quad \text{(Manning)}, \quad R_h = h \quad \text{("wide" channel)} \]

\[ \Rightarrow q = \frac{1}{n} h^{2/3} S^{1/2} \]

\[ \Rightarrow q = \frac{h^{5/3}}{n} \sqrt{S} \]

\[ \Rightarrow h = \left(\frac{nq}{\sqrt{S}}\right)^{3/5} \]

For the two slopes this gives

\[ h_1 = 0.1874 \text{ m} \]
\[ h_2 = 0.5365 \text{ m} \]

**Answer:** depth on spillway = 0.187 m; depth on apron = 0.536 m.

For subcritical/supercritical:

**Method 1** (use critical depth)
The critical depth is

\[ h_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{0.8^2}{9.81}\right)^{1/3} = 0.4026 \text{ m} \]
On the spillway, $h_1 < h_c$: this is shallower than critical flow (where $Fr = 1$) and hence faster; both ensure $Fr_1 > 1$, so supercritical.

On the apron, $h_2 > h_c$: this is deeper than critical flow and hence slower; both ensure $Fr_2 < 1$, so subcritical.

Method 2 (find Froude numbers)

$$Fr = \frac{V}{\sqrt{gh}} = \frac{q}{\sqrt{gh^3}}$$

Applying this for both depths we find $Fr_1 = 3.148$ (supercritical) and $Fr_2 = 0.650$ (subcritical).

(b)
The corresponding velocities are deduced from $V = q/h$, whence:

$$V_1 = 4.269 \text{ m s}^{-1}$$
$$V_2 = 1.491 \text{ m s}^{-1}$$

Let $f$ be the magnitude of the force per unit width exerted by the fluid on the blocks and, by reaction, the blocks on the fluid, which is clearly acts in the upstream direction.

On each side of the blocks the hydrostatic pressure force is given by

average pressure $\times$ area

or $\frac{1}{2} \rho gh \times h$ (per unit width)

Hence, from the steady-state momentum principle:

force $= \text{rate of change of momentum (mass flux} \times \text{change in velocity)}$

$$-f + \frac{1}{2} \rho gh_1^2 - \frac{1}{2} \rho gh_2^2 = \rho q(V_2 - V_1)$$

Hence,

$$f = \frac{1}{2} \rho g(h_1^2 - h_2^2) + \rho q(V_1 - V_2)$$

$$= -1240 + 2222$$
$$= 982 \text{ N}$$

Answer: force (per metre width) $= 982 \text{ N}$.

(c) Where hydrostatic, head in open-channel flow is given by

$$H = z_s + \frac{V^2}{2g}$$

Here, relative to the bed of the apron, $z_s = h$. Hence,
head loss = \( h_1 - h_2 + \frac{V_1^2 - V_2^2}{2g} \)
= \(-0.3491 + 0.8156\)
= 0.4665 m

**Answer:** head loss = 0.467 m.
Section 2.4

Example.
A downward step of height 0.5 m causes a hydraulic jump in a wide channel when the depth and velocity of the flow upstream are 0.5 m and 10 m s$^{-1}$, respectively.

(a) Find the downstream depth.
(b) Find the head lost in the jump.

(a) The downstream depth can be deduced from the momentum principle if the reaction force from the step is known. The approximation is that this is the same as would occur if it were in equilibrium with a hydrostatic pressure distribution here.

Flow rate per unit width:

\[ q = V_1 h_1 = 10 \times 0.5 = 5 \text{ m}^2 \text{s}^{-1} \]

Steady-state momentum principle

force = rate of change of momentum (mass flux \times change in velocity)

Per unit width:

\[ \frac{1}{2} \rho g (h_1 + \Delta)^2 - \frac{1}{2} \rho g h_2^2 = \rho q (V_2 - V_1) \]

Since

\[ V_2 = \frac{q}{h_2} = \frac{5}{h_2} \]

this gives, in metre-second units,

\[ 4905(1 - h_2^2) = 5000\left(\frac{5}{h_2} - 10\right) \]

\[ \Rightarrow 1 - h_2^2 = 1.019\left(\frac{5}{h_2} - 10\right) \]

\[ \Rightarrow 11.19 = h_2^2 + \frac{5.095}{h_2} \]

Rearrange for the deep, subcritical solution:

\[ h_2 = \sqrt{11.19 - \frac{5.095}{h_2}} \]

Iterating (from, e.g., \( h_2 = \sqrt{11.19} \)) gives \( h_2 = 3.089 \) m.

**Answer:** downstream depth = 3.09 m.
(b) Head either side is given by:

\[ H = z_s + \frac{V^2}{2g} \]

The datum is not important as it is only the difference in head that is required. For convenience, measure \( z \) relative to the bed of the expanded part. Then,

\[ z_{s1} = 1 \text{ m (note: water surface level, not depth), } V_1 = 10 \text{ m s}^{-1} \implies H_1 = 6.097 \text{ m} \]

\[ z_{s2} = 3.089 \text{ m, } V_2 = \frac{q}{h_2} = 1.619 \text{ m s}^{-1} \implies H_2 = 3.223 \text{ m} \]

Hence,

\[ \text{head lost} = 6.097 - 3.223 = 2.874 \text{ m} \]

**Answer:** head lost = 2.87 m.
Section 3.6.2

Example. (Exam 2008 – modified)

A long, wide channel has a slope of 1:2747 with a Manning’s $n$ of 0.015 m$^{1/3}$ s. It carries a discharge of 2.5 m$^3$ s$^{-1}$ per metre width, and there is a free overfall at the downstream end. An undershot sluice is placed a certain distance upstream of the free overfall which determines the nature of the flow between sluice and overfall. The depth just downstream of the sluice is 0.5 m.

(a) Determine the critical depth and normal depth.

(b) Sketch, with explanation, the two possible gradually-varied flows between sluice and overfall.

(c) Calculate the particular distance between sluice and overfall which determines the boundary between these two flows. Use one step in the gradually-varied-flow equation.

\[
S_0 = 1/2747 = 3.640 \times 10^{-4}
\]
\[
n = 0.015 \text{ m}^{-1/3} \text{ s}
\]
\[
q = 2.5 \text{ m}^2 \text{ s}^{-1}
\]

(a) Critical Depth

\[
h_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{2.5^2}{9.81}\right)^{1/3} = 0.8605 \text{ m}
\]

Normal Depth

\[
q = Vh, \quad \text{where} \quad V = \frac{1}{n}R_h^{2/3}S_0^{1/2}, \quad \text{(Manning)}, \quad R_h = h \quad (\text{"wide" channel})
\]

\[
\Rightarrow \quad q = \frac{1}{n} h^{2/3} S_0^{1/2} h
\]

\[
\Rightarrow \quad q = \frac{h^{5/3}}{n} S_0^{1/2}
\]

\[
\Rightarrow \quad h = \left(\frac{nq}{S_0^{1/2}}\right)^{3/5} = \left(\frac{0.015 \times 2.5}{\sqrt{1/2747}}\right)^{3/5} = 1.500 \text{ m}
\]

Answer: critical depth = 0.860 m; normal depth = 1.50 m.

(b) The depth just downstream of the sluice is supercritical (0.5 m < $h_c$). However, the preferred depth is subcritical ($h_n > h_c$). Hence, if the channel is long enough then there will be a downstream hydraulic jump, with the flow depth then decreasing to pass through critical again near the overfall.
If the channel is too short, however, the region of supercritical flow from the sluice will extend to the overfall.

(c) As the channel shortens, the depth change across the hydraulic jump diminishes. The boundary between the two possible flow behaviours occurs when the supercritical GVF just reaches critical depth at the overfall (i.e. the limiting depth change across the hydraulic jump is zero).

As the flow is supercritical, integrate the GVF equation forward from the downstream side of the sluice gate (where \( h = 0.5 \) m) to the overfall (where \( h = h_c = 0.8605 \) m). Use 1 step.

GVF equation:

\[
\frac{dh}{dx} = \frac{S_0 - S_f}{1 - Fr^2}
\]

For the direct-step method invert the GVF equation:

\[
\frac{dx}{dh} = \frac{1 - Fr^2}{S_0 - S_f} \quad \text{and} \quad \Delta x \approx \left(\frac{dx}{dh}\right) \Delta h
\]

For the working, write the derivative as a function of \( h \); (all lengths in metres).

\[
Fr = \frac{V}{\sqrt{gh}} = \frac{q}{\sqrt{gh^3}} \quad \Rightarrow \quad Fr^2 = \frac{q^2}{gh^3} = \frac{0.6371}{h^3}
\]

\[
S_f = \left(\frac{nh}{h^{5/3}}\right)^2 = \frac{1.406 \times 10^{-3}}{h^{10/3}}
\]

\[
\Delta h = 0.8605 - 0.5 = 0.3605
\]
Working formulae:

\[
\Delta x = \left( \frac{dx}{dh} \right)_{\text{mid}} \Delta h
\]

where

\[
\frac{dx}{dh} = \frac{1 - \frac{0.6371}{h^3}}{3.640 \times 10^{-4} - \frac{1.406 \times 10^{-3}}{h^{10/3}}}
\]

\[\Delta h = 0.3605\]

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**Answer:** length to overfall = 78 m.
Example. (Exam 2018)
An undershot sluice controls the flow in a long rectangular channel of width 2.5 m, Manning’s roughness coefficient $n = 0.012 \text{ m}^{-1/3} \text{ s}$ and streamwise slope 0.002. The depths of parallel flow upstream and downstream of the gate are 1.8 m and 0.3 m, respectively.

(a) Assuming no losses at the sluice, find the volume flow rate, $Q$.

(b) Find the normal and critical depths in the channel.

(c) Compute the distance from the sluice gate to the hydraulic jump, assuming normal depth downstream of the jump. Use two steps in the gradually-varied-flow equation.

$b = 2.5 \text{ m} \\
h_1 = 1.8 \text{ m} \\
h_2 = 0.3 \text{ m} \\
n = 0.012 \text{ m}^{-1/3} \text{ s} \\
S_0 = 0.002$

(a) Assuming the same total head on either side of the gate:

$$z_{s1} + rac{V_1^2}{2g} = z_{s2} + rac{V_2^2}{2g}$$

$$\Rightarrow h_1 + rac{q^2}{2gh_1^2} = h_2 + rac{q^2}{2gh_2^2}$$

$$\Rightarrow h_1 - h_2 = rac{q^2}{2g} \left( \frac{1}{h_2^2} - \frac{1}{h_1^2} \right)$$

Substituting values:

$$1.5 = 0.5506q^2$$

Hence, the flow per unit width is

$$q = \sqrt{\frac{1.5}{0.5506}} = 1.651 \text{ m}^2 \text{ s}^{-1}$$

and the total discharge is

$$Q = qb = 1.651 \times 2.5 = 4.128 \text{ m}^3 \text{ s}^{-1}$$

Answer: discharge = 4.13 m$^3$ s$^{-1}$.

(b) Normal depth

$$Q = VA$$
where, in normal flow:

\[ V = \frac{1}{n}R_h^{2/3}S^{1/2} \quad A = bh \]

\[ R_h = \frac{bh}{b + 2h} = \frac{h}{1 + 2h/b} \]

Hence,

\[ Q = \frac{1}{n} \frac{bh^{5/3}}{(1 + 2h/b)^{2/3}S_0^{1/2}} \]

or, rearranging as an iterative formula for \( h \):

\[ h = \left( \frac{nQ}{b\sqrt{S_0}} \right)^{3/5} (1 + 2h/b)^{2/5} \]

Substitution of numerical values yields iterative formula

\[ h = 0.6136(1 + 0.8h)^{2/5} \]

Iteration (from, e.g., \( h = 0.6136 \)) gives

\[ h_n = 0.7389 \text{ m} \]

Critical depth

\[ h_c = \left( \frac{q^2}{g} \right)^{1/3} = \left( \frac{1.651^2}{9.81} \right)^{1/3} = 0.6525 \text{ m} \]

Answer: normal depth = 0.739 m; critical depth = 0.653 m.

(c) The depth just upstream of the jump is the sequent depth to the normal depth:

\[ h_n = 0.7389 \text{ m} \]

\[ V_n = \frac{Q}{bh_n} = \frac{4.128}{2.5 \times 0.7389} = 2.235 \text{ m s}^{-1} \]

\[ \text{Fr}_n = \frac{V_n}{\sqrt{gh_n}} = \frac{2.235}{\sqrt{9.81 \times 0.7389}} = 0.8301 \]

so that the depth just upstream of the jump (call it \( h_J \)) is

\[ h_J = \frac{h_n}{2} (-1 + \sqrt{1 + 8\text{Fr}_n^2}) = \frac{0.7389}{2} (-1 + \sqrt{1 + 8 \times 0.8301^2}) = 0.5734 \text{ m} \]

We must therefore do a GVF calculation from just downstream of the sluice (where \( h = 0.3 \) m) to just upstream of the hydraulic jump (where \( h = 0.5734 \) m).

GVF equation:

\[ \frac{dh}{dx} = \frac{S_0 - S_f}{1 - \text{Fr}^2} \]

For the direct-step method rewrite the GVF equation “the other way up”:
\[ \frac{dx}{dh} = \frac{1 - \text{Fr}^2}{S_0 - S_f} \quad \text{and} \quad \Delta x \approx \left( \frac{dx}{dh} \right) \Delta h \]

For the working, write the derivative as a function of \( h \); (all lengths in metres).

\[ \text{Fr} = \frac{V}{\sqrt{gh}} = \frac{Q/b}{\sqrt{gh^3}} \quad \Rightarrow \quad \text{Fr}^2 = \frac{(Q/b)^2}{gh^3} = \frac{0.2779}{h^3} \]

\[ S_f = \left( \frac{nQ}{b h^{5/3}} \right)^2 (1 + 2h/b)^{4/3} = 3.926 \times 10^{-4} \left( 1 + 0.8h \right)^{4/3} \]

\[ \Delta h = \frac{0.5734 - 0.3}{2} = 0.1367 \]

**Working formulae:**

\[ \Delta x = \left( \frac{dx}{dh} \right)_{\text{mid}} \Delta h \]

where

\[ \frac{dx}{dh} = \frac{1 - \frac{0.2779}{h^3}}{[20 - 3.926 \times \frac{(1 + 0.8h)^{4/3}}{h^{10/3}}] \times 10^{-4}}, \quad \Delta h = 0.1367 \]

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**Answer:** distance to hydraulic jump = 85.7 m.
Example. (Exam 2014)
An undershot sluice is used to control the flow of water in a long wide channel of slope 0.003
and Manning’s roughness coefficient 0.012 m$^{-1/3}$ s. The flow rate in the channel is 2 m$^3$ s$^{-1}$ per
metre width.

(a) Calculate the normal depth and critical depth in the channel and show that the channel
is hydrodynamically “steep” at this flow rate.

(b) The depth of flow just downstream of the sluice is 0.4 m. Assuming no head losses at
the sluice calculate the depth just upstream of the sluice.

(c) Sketch the depth profile along the channel, indicating clearly any flow transitions
brought about by the sluice and indicating where water depth is increasing or
decreasing.

(d) Use 2 steps in the gradually-varied flow equation to determine how far upstream of the
sluice a hydraulic jump will occur.

\[ S_0 = 0.003 \]
\[ n = 0.012 \text{ m}^{-1/3} \text{ s} \]
\[ q = 2 \text{ m}^2 \text{ s}^{-1} \]

(a) Normal depth

\[ q = Vh = \frac{1}{n} h^{2/3} S_0^{1/2} h \]
where \( R_h = h \) (wide channel)

\[ \Rightarrow \frac{nnq}{\sqrt{S_0}} = h^{5/3} \] (*)

\[ \Rightarrow h_n = \left( \frac{nnq}{\sqrt{S_0}} \right)^{3/5} = \left( \frac{0.012 \times 2}{\sqrt{0.003}} \right)^{3/5} = 0.6095 \text{ m} \]

Critical Depth

\[ h_c = \left( \frac{q^2}{g} \right)^{1/3} = \left( \frac{2^2}{9.81} \right)^{1/3} = 0.7415 \text{ m} \]

The normal depth is smaller than the critical depth. Hence the normal flow is supercritical; i.e.
the channel is steep at this discharge.

Answer: \( h_n = 0.610 \text{ m}; \ h_c = 0.742 \text{ m} \).

(b) Assuming the same total head on either side of the gate:
\[ z_{s1} + \frac{V_1^2}{2g} = z_{s2} + \frac{V_2^2}{2g} \]

\[ \Rightarrow h_1 + \frac{q^2}{2gh_1^2} = h_2 + \frac{q^2}{2gh_2^2} \]

\[ \Rightarrow h_1 + \frac{2^2}{2gh_1^2} = 0.4 + \frac{2^2}{2g \times 0.4^2} \]

\[ \Rightarrow h_1 + \frac{0.2039}{h_1^2} = 1.674 \]

We require the subcritical solution, so rearrange for iteration as

\[ h_1 = 1.674 - \frac{0.2039}{h_1^2} \]

Iteration (e.g. from 1.674) gives \( h_1 = 1.594 \) m

**Answer:** \( h_1 = 1.59 \) m.

(c) The depth increases in each of the GVF regions (S1 and S3) shown below.

(d) Upstream of the sluice there is GVF between the hydraulic jump and the sluice. This starts at the subcritical sequent depth of the hydraulic jump (to be found below) and ends at the depth \( h_1 \) found in part (b).

For the hydraulic jump, on the upstream side (normal flow):

\[ h_A = 0.6095 \text{ m} \]

\[ V_A = \frac{q}{h_A} = \frac{2}{0.6095} = 3.281 \text{ m s}^{-1} \]

\[ \text{Fr}_A = \frac{V}{\sqrt{gh_A}} = \frac{3.281}{\sqrt{9.81 \times 0.6095}} = 1.342 \]

On the downstream side of the hydraulic jump:

\[ h_B = \frac{h_A}{2} (-1 + \sqrt{1 + 8\text{Fr}_A^2}) = \frac{0.6095}{2} (-1 + \sqrt{1 + 8 \times 1.342^2}) = 0.8915 \text{ m} \]
Do a GVF calculation (subcritical, so physically it should start at the fixed downstream control and work upstream, although mathematically it can be done the other way) from just upstream of the sluice gate (where \( h = 1.594 \) m) to just downstream of the hydraulic jump (where \( h = 0.892 \) m). Using two steps the depth increment per step is

\[
\Delta h = \frac{0.892 - 1.594}{2} = -0.351 \text{ m}
\]

GVF equation:

\[
\frac{dh}{dx} = \frac{S_0 - S_f}{1 - \text{Fr}^2}
\]

For the direct-step method rewrite the GVF equation “the other way up”:

\[
\frac{dx}{dh} = \frac{1 - \text{Fr}^2}{S_0 - S_f} \quad \text{and} \quad \Delta x \approx (\frac{dx}{dh}) \Delta h
\]

For the working, write the derivative as a function of \( h \); (all lengths in metres).

\[
\text{Fr} = \frac{V}{\sqrt{gh}} = \frac{q}{\sqrt{gh^3}} \quad \Rightarrow \quad \text{Fr}^2 = \frac{q^2}{gh^3} = \frac{0.4077}{h^3}
\]

\[
S_f = \left(\frac{nq}{h^{5/3}}\right)^2 = \frac{5.76 \times 10^{-4}}{h^{10/3}}
\]

Working formulae:

\[
\Delta x = \left(\frac{dx}{dh}\right)_{\text{mid}} \Delta h
\]

where

\[
\frac{dx}{dh} = \frac{1 - \frac{0.4077}{h^3}}{[30 - \frac{5.76}{h^{10/3}}] \times 10^{-4}} \quad \Delta h = -0.351
\]

<table>
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<tr>
<th>( i )</th>
<th>( h_i )</th>
<th>( x_i )</th>
<th>( h_{\text{mid}} )</th>
<th>( (dx/dh)_{\text{mid}} )</th>
<th>( \Delta x )</th>
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<td>−198.7</td>
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</tbody>
</table>

**Answer:** upstream distance from sluice to hydraulic jump = 199 m.
Section 4.2

Example. (From White, 2006)
A pencil point piercing the surface of a wide rectangular channel flow creates a wedgelike 25° half-angle wave, as in the figure right. If the channel has a Manning’s $n$ of 0.014 m$^{-1/3}$ s and the depth is 350 mm, determine: (a) the Froude number; (b) the critical depth; and (c) the critical slope.

\[ \alpha = 25^\circ \]
\[ n = 0.014 \text{ m}^{-1/3} \text{ s} \]
\[ h = 0.35 \text{ m} \]

(a)
\[
\sin \alpha = \frac{1}{Fr} \\
\Rightarrow Fr = \frac{1}{\sin \alpha} = \frac{1}{\sin 25^\circ} = 2.366
\]

Answer: Froude number = 2.37.

(b)
\[
Fr \equiv \frac{V}{\sqrt{gh}} \\
\Rightarrow V = Fr\sqrt{gh} = 2.366 \times \sqrt{9.81 \times 0.35} = 4.384 \text{ m s}^{-1} \\
\Rightarrow q = Vh = 4.384 \times 0.35 = 1.534 \text{ m}^2 \text{ s}^{-1} \\
\Rightarrow h_c = \left( \frac{q^2}{g} \right)^{1/3} = 0.6213 \text{ m}
\]

Answer: critical depth = 0.621 m.

(c) The critical slope is that at which the normal depth is equal to the critical depth.
Normal flow:

\[ q = Vh \]  
where  
\[ V = \frac{1}{n} R_h^{2/3} S^{1/2} \]  
(Manning)  
\[ R_h = h \]  
(“wide” channel)

\[ q = \frac{1}{n} h^{2/3} S^{1/2} \]

Rearranging, and setting a depth \( h \) equal to the critical depth (\( h_c = 0.6213 \) m),

\[ S = \frac{(nq)^2}{h^{10/3}} = \frac{(0.014 \times 1.534)^2}{0.6213^{10/3}} = 2.254 \times 10^{-3} \]

**Answer:** critical slope = \( 2.25 \times 10^{-3} \).
Sediment Transport; Section 2.1
Note: removed from the present curriculum because of the reduction in teaching hours.

Example. (Exam 2007 – part)

An undershot sluice is placed in a channel with a horizontal bed covered by gravel with a median diameter of 5 cm and density 2650 kg m\(^{-3}\). The flow rate is 4 m\(^3\) s\(^{-1}\) per metre width and initially the depth below the sluice is 0.5 m. Assuming a critical Shields parameter \(\tau_{\text{crit}}^*\) of 0.06 and friction coefficient \(c_f\) of 0.01:

(a) find the depth just upstream of the sluice and show that the bed there is stationary;
(b) show that the bed below the sluice will erode and determine the depth of scour.

\[ d = 0.05 \text{ m} \]
\[ \rho_s = 2650 \text{ kg m}^{-3} \ (s = 2.65) \]
\[ q = 4 \text{ m}^2 \text{ s}^{-1} \]
\[ \tau_{\text{crit}}^* = 0.06 \]
\[ c_f = 0.01 \]

(a) The bed is mobile if and only if the bed shear stress exceeds the critical shear stress for incipient motion. (Alternatively, one could find the velocity at which incipient motion occurs.)

First find the critical shear stress for incipient motion. The local shear stress can be found by finding velocities using open-channel flow theory and using the friction coefficient.

The critical shear stress for incipient motion is

\[ \tau_{\text{crit}} = \tau_{\text{crit}}^* (\rho_s - \rho)gd = 0.06 \times (2650 - 1000) \times 9.81 \times 0.05 = 48.56 \text{ Pa} \]

In the accelerated flow just below the sluice gate,

\[ h = 0.5 \text{ m} \]
\[ V = \frac{q}{h} = \frac{4}{0.5} = 8 \text{ m s}^{-1} \]

The total head relative to the undisturbed bed is thus (since \(z_s = h\) initially):

\[ H = z_s + \frac{V^2}{2g} = 0.5 + \frac{8^2}{2 \times 9.81} = 3.762 \text{ m} \]

Upstream of the sluice one must have the same total head:

\[ H = z_s + \frac{V^2}{2g} = h + \frac{q^2}{2gh^2} \]

but now we look for the subcritical (deep) solution:

\[ h = H - \frac{q^2}{2gh^2} \]
Substituting numerical values, in metre-second units:

\[ h = 3.762 - \frac{0.8155}{h^2} \]

Iterating (from, e.g., \( h = 3.762 \)) gives

\[ h = 3.703 \text{ m} \]

The velocity upstream of the sluice is

\[ V = \frac{q}{h} = \frac{4}{3.703} = 1.080 \text{ m s}^{-1} \]

and the bed shear stress upstream of the sluice is

\[ \tau_b = c_f \left( \frac{1}{2} \rho V^2 \right) = 0.01 \times \frac{1}{2} \times 1000 \times 1.080^2 = 5.832 \text{ N m}^{-2} \]

Since \( \tau_b < \tau_{\text{crit}} \) the bed here is stationary.

**Answer:** depth upstream of sluice = 3.70 m; as demonstrated, the bed here is stationary.

(b) Initially, beneath the sluice,

\[ V = 8 \text{ m} \text{ s}^{-1} \]

\[ \tau_b = c_f \left( \frac{1}{2} \rho V^2 \right) = 0.01 \times \frac{1}{2} \times 1000 \times 8^2 = 320 \text{ N m}^{-2} \]

Since \( \tau_b > \tau_{\text{crit}} \) the bed here is initially mobile.

The bed will continue to erode, the flow depth \( h \) increasing and velocity \( V \) decreasing until

\[ \tau_b = \tau_{\text{crit}} \]

At this point,

\[ c_f \left( \frac{1}{2} \rho V^2 \right) = \tau_{\text{crit}} \]

whence

\[ V = \sqrt{\frac{2 \tau_{\text{crit}}}{c_f \rho}} = \sqrt{\frac{2 \times 0.01 \times 48.56}{1000}} = 3.116 \text{ m s}^{-1} \]

\[ h = \frac{q}{V} = \frac{4}{3.116} = 1.284 \text{ m} \]

This is the depth of flow. Since the downstream water level is set (at 0.5 m above the undisturbed bed) by the gate the depth of the scour hole is

\[ 1.284 - 0.5 = 0.784 \text{ m} \]

**Answer:** depth of scour hole = 0.784 m.