Objectives

(1) Be able to determine the **dimensions** of physical quantities in terms of **fundamental dimensions**.
(2) Understand the **Principle of Dimensional Homogeneity** and its use in checking equations and reducing physical problems.
(3) Be able to carry out a formal dimensional analysis using **Buckingham's Pi Theorem**.
(4) Understand the requirements of **physical modelling** and its limitations.

1. What is dimensional analysis?

2. Dimensions
 2.1 Dimensions and units
 2.2 Primary dimensions
 2.3 Dimensions of derived quantities
 2.4 Working out dimensions
 2.5 Alternative choices for primary dimensions

3. Formal procedure for dimensional analysis
 3.1 Dimensional homogeneity
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 3.3 Applications

4. Physical modelling
 4.1 Method
 4.2 Incomplete similarity ("scale effects")
 4.3 Froude-number scaling

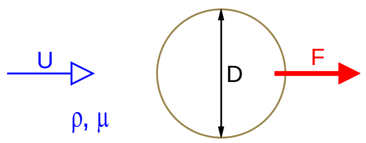
5. Non-dimensional groups in fluid mechanics

1. WHAT IS DIMENSIONAL ANALYSIS?

Dimensional analysis is a means of simplifying a physical problem by appealing to dimensional homogeneity to reduce the number of relevant variables.

It is particularly useful for:
- checking equations;
- presenting and interpreting experimental data;
- attacking problems not amenable to a direct theoretical solution;
- establishing the relative importance of particular physical phenomena;
- physical modelling.

Example.
The drag force, F, on a sphere is a function of approach-flow speed, U, sphere diameter, D, fluid density, ρ, and viscosity, μ. However, instead of having to draw hundreds of graphs portraying its variation with all combinations of these parameters, dimensional analysis will tell us that the problem can be reduced to a dimensionless relationship between just two independent variables:



$$c_D = f(\text{Re})$$

where c_D is the drag coefficient:

$$c_D \equiv \frac{F}{\frac{1}{2}\rho U^2 A} \qquad (A = \frac{\pi D^2}{4})$$

and Re is the Reynolds number:

$$\text{Re} \equiv \frac{\rho U D}{\mu}$$

In this instance dimensional analysis has reduced the number of relevant variables from 5 to 2 and the experimental data to a single graph of c_D against Re.

2. DIMENSIONS

2.1 Dimensions and Units

A *dimension* is the **type** of physical quantity.
A *unit* is a means of assigning a **numerical value** to that quantity.
SI units are preferred in scientific work.

2.2 Primary Dimensions

In fluid mechanics the *primary* or *fundamental* dimensions, together with their SI units, are:

mass	M	(kilogram, kg)
length	L	(metre, m)
time	T	(second, s)
temperature	Θ	(kelvin, K)

In other areas of physics additional dimensions may be necessary. The complete set specified by the SI system consists of the above plus

electric current	I	(ampere, A)
luminous intensity	C	(candela, cd)
amount of substance	n	(mole, mol)

2.3 Dimensions of Derived Quantities

The dimensions of common *derived* mechanical quantities are given in the following table.

	Quantity	Common Symbol(s)	Dimensions
Geometry	Area	A	L^2
	Volume	V	L^3
	Second moment of area	I	L^4
Kinematics	Velocity	U	LT^{-1}
	Acceleration	a	LT^{-2}
	Angle	θ	1 (i.e. dimensionless)
	Angular velocity	ω	T^{-1}
	Quantity of flow	Q	$\mathrm{L}^3\mathrm{T}^{-1}$
	Mass flow rate	$\dot{m}$	MT^{-1}
Dynamics	Force	F	MLT^{-2}
	Moment, torque	T	$\mathrm{ML}^2\mathrm{T}^{-2}$
	Energy, work, heat	E, W	$\mathrm{ML}^2\mathrm{T}^{-2}$
	Power	P	$\mathrm{ML}^2\mathrm{T}^{-3}$
	Pressure, stress	p, τ	$\mathrm{ML}^{-1}\mathrm{T}^{-2}$
Fluid properties	Density	ρ	ML^{-3}
	Viscosity	μ	$\mathrm{ML}^{-1}\mathrm{T}^{-1}$
	Kinematic viscosity	ν	$\mathrm{L}^2\mathrm{T}^{-1}$
	Surface tension	σ	MT^{-2}
	Thermal conductivity	k	$\mathrm{MLT}^{-3}\Theta^{-1}$

	Specific heat	c_p, c_v	$\mathrm{L^2T^{-2}\Theta^{-1}}$
	Bulk modulus	K	$\mathrm{ML^{-1}T^{-2}}$

2.4 Working Out Dimensions

In the following, [ ] means "dimensions of".

Example.
Use the definition

$$\tau = \mu \frac{\mathrm{d}u}{\mathrm{d}y}$$

to determine the dimensions of viscosity.

Solution.
From the definition,

$$\mu = \frac{\tau}{\mathrm{d}u/\mathrm{d}y} \quad = \frac{\text{force/area}}{\text{velocity/length}}$$

Hence,

$$[\mu] = \frac{\mathrm{MLT^{-2}/L^2}}{\mathrm{LT^{-1}/L}} \quad = \mathrm{ML^{-1}T^{-1}}$$

Alternatively, dimensions may be deduced indirectly from any known formula involving that quantity.

Example.
Since $\mathrm{Re} \equiv \rho U L/\mu$ is known to be dimensionless, the dimensions of μ must be the same as those of $\rho U L$; i.e.

$$[\mu] = [\rho][U][L] \quad = (\mathrm{ML^{-3}})(\mathrm{LT^{-1}})(\mathrm{L}) \quad = \mathrm{ML^{-1}T^{-1}}$$

2.5 Alternative Choices For Primary Dimensions

The choice of primary dimensions is not unique. It is not uncommon – and it may sometimes be more convenient – to choose force F as a primary dimension rather than mass, and have a {FLT} rather than {MLT} system.

Example.
Find the dimensions of viscosity μ in the {FLT} rather than {MLT} systems.

Solution.
From the definition,

$$\mu = \frac{\tau}{\mathrm{d}u/\mathrm{d}y} \qquad = \frac{\text{force/area}}{\text{velocity/length}}$$

Hence,

$$[\mu] = \frac{\mathrm{F/L^2}}{\mathrm{LT^{-1}/L}} \qquad = \mathrm{FL^{-2}T}$$

3. FORMAL PROCEDURE FOR DIMENSIONAL ANALYSIS

3.1 Dimensional Homogeneity

The Principle of Dimensional Homogeneity

All ***additive*** *terms in a physical equation must have the same dimensions*.

Examples:

$$s = ut + \frac{1}{2}at^2$$ all terms have the dimensions of length (L)

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = H$$ all terms have the dimensions of length (L)

Dimensional homogeneity is a useful tool for **checking** formulae. For this reason it is useful when analysing a physical problem to retain algebraic symbols for as long as possible, only substituting numbers right at the end. However, dimensional analysis cannot determine numerical factors; e.g. it cannot distinguish between ½at^2 and at^2 in the first formula above.

Dimensional homogeneity is the basis of the formal dimensional analysis that follows.

3.2 Buckingham's Pi Theorem

Experienced practitioners can do dimensional analysis by inspection. However, the formal tool which they are unconsciously using is *Buckingham's Pi Theorem*[1]:

Buckingham's Pi Theorem

(1) If a problem involves
 n relevant variables
 m **independent** dimensions
then it can be reduced to a relationship between
 $n - m$ non-dimensional parameters $\Pi_1, \ldots, \Pi_{n-m}$.

(2) To construct these non-dimensional Π groups:
 (i) Choose m dimensionally-distinct *scaling variables* (aka *repeating variables*).
 (ii) For each of the $n - m$ remaining variables construct a non-dimensional Π of the form
 $$\Pi = (\text{variable})(\text{scale}_1)^a(\text{scale}_2)^b(\text{scale}_3)^c \cdots$$
 where a, b, c, ... are chosen so as to make each Π non-dimensional.

Note. In order to ensure dimensional independence in {MLT} systems it is common – but not obligatory – to choose the scaling variables as: a purely geometric quantity (e.g. a length), a kinematic (time-, but not mass-containing) quantity (e.g. frequency, velocity or acceleration) and a dynamic (mass-, or force-containing) quantity (e.g. density).

[1] Buckingham, E., 1914. The use of Π comes from its use as the mathematical symbol for a product.

3.3 Applications

Example.
Obtain an expression in non-dimensional form for the pressure gradient in a horizontal pipe of circular cross-section. Show how this relates to the expression for frictional head loss.

Solution.
Step 1. Identify the relevant variables.

$$\mathrm{d}p/\mathrm{d}x,\ \rho,\ V,\ D,\ k_s,\ \mu$$

Step 2. Write down dimensions.

$$\frac{\mathrm{d}p}{\mathrm{d}x} \qquad \frac{[\text{force/area}]}{\text{length}} = \frac{\mathrm{MLT^{-2} \times L^{-2}}}{\mathrm{L}} = \mathrm{ML^{-2}T^{-2}}$$

ρ	$\mathrm{ML^{-3}}$
V	$\mathrm{LT^{-1}}$
D	L
k_s	L
μ	$\mathrm{ML^{-1}T^{-1}}$

Step 3. Establish the number of independent dimensions and non-dimensional groups.

Number of relevant variables: $n = 6$
Number of independent dimensions: $m = 3$ (M, L and T)
Number of non-dimensional groups (Πs): $n - m = 3$

Step 4. Choose m (= 3) dimensionally-independent scaling variables.
e.g. geometric (D), kinematic/time-dependent (V), dynamic/mass-dependent (ρ).

Step 5. Create the Πs by non-dimensionalising the remaining variables: $\mathrm{d}p/\mathrm{d}x$, k_s and μ.

$$\Pi_1 = \frac{\mathrm{d}p}{\mathrm{d}x} D^a V^b \rho^c$$

Considering the dimensions of both sides:

$$\begin{aligned} \mathrm{M^0L^0T^0} &= (\mathrm{ML^{-2}T^{-2}})(\mathrm{L})^a(\mathrm{LT^{-1}})^b(\mathrm{ML^{-3}})^c \\ &= \mathrm{M}^{1+c}\mathrm{L}^{-2+a+b-3c}\mathrm{T}^{-2-b} \end{aligned}$$

Equate powers of primary dimensions. Since M only appears in $[\rho]$ and T only appears in $[V]$ it is easiest to deal with these first:

M: $0 = 1 + c \quad \Rightarrow \quad c = -1$
T: $0 = -2 - b \quad \Rightarrow \quad b = -2$
L: $0 = -2 + a + b - 3c \quad \Rightarrow \quad a = 2 - b + 3c \quad = 1$

Hence,

$$\Pi_1 = \frac{\mathrm{d}p}{\mathrm{d}x} D^1 V^{-2} \rho^{-1} \quad = \frac{D\dfrac{\mathrm{d}p}{\mathrm{d}x}}{\rho V^2}$$

(OK – ratio of two pressures)

k_s can be non-dimensionalised by inspection, since it already has the same dimensions (L) as one of the scaling variables:

$$\Pi_2 = \frac{k_s}{D}$$

Finally,

$$\Pi_3 = \mu D^a V^b \rho^c$$

Considering the dimensions of both sides:

$$\begin{aligned} \mathrm{M^0L^0T^0} &= (\mathrm{ML^{-1}T^{-1}})(\mathrm{L})^a(\mathrm{LT^{-1}})^b(\mathrm{ML^{-3}})^c \\ &= \mathrm{M}^{1+c}\mathrm{L}^{-1+a+b-3c}\mathrm{T}^{-1-b} \end{aligned}$$

Again, as M only appears in $[\rho]$ and T only appears in $[V]$ then deal with these first:

M: $\quad 0 = 1 + c \quad \Rightarrow \quad c = -1$

T: $\quad 0 = -1 - b \quad \Rightarrow \quad b = -1$

L: $\quad 0 = -1 + a + b - 3c \quad \Rightarrow \quad a = 1 - b + 3c \quad = -1$

Hence,

$$\Pi_3 = \mu D^{-1}V^{-1}\rho^{-1} \quad = \frac{\mu}{\rho V D}$$ (OK – reciprocal of Reynolds number)

Step 6. Set out the non-dimensional relationship.

$$\Pi_1 = f(\Pi_2, \Pi_3)$$

or

$$\frac{D\dfrac{\mathrm{d}p}{\mathrm{d}x}}{\rho V^2} = f(\frac{k_s}{D}, \frac{\mu}{\rho V D}) \qquad (*)$$

Step 7. Rearrange (if required) for convenience.

We may replace any Π by a power of that Π, or by a product with the other Πs, provided that we retain the same number of independent dimensionless groups. Here, we recognise Π_3 as the reciprocal of the Reynolds number, so it is more natural to use $\Pi_3' = (\Pi_3)^{-1} = \mathrm{Re}$ as the third non-dimensional group. We can also write the pressure gradient in terms of head loss: $\mathrm{d}p/\mathrm{d}x = \rho g(h_f/L)$. With these two modifications the non-dimensional relationship (*) then becomes

$$\frac{g h_f D}{L V^2} = f(\frac{k_s}{D}, \mathrm{Re})$$

or

$$h_f = \frac{L}{D} \times \frac{V^2}{g} \times f(\frac{k_s}{D}, \mathrm{Re})$$

Since numerical factors (here, $1/2$) can be absorbed into the non-specified function, this can easily be identified with the Darcy-Weisbach equation

$$h_f = \lambda \frac{L}{D}\frac{V^2}{2g}$$

where λ is a function of relative roughness k_s/D and Reynolds number Re, a function given (Topic 2) by the Colebrook-White equation.

Example.
The drag force on a body in a fluid flow is a function of the body size (expressed via a characteristic length, L) and the fluid velocity, V, density, ρ, and viscosity, μ. Perform a dimensional analysis to reduce this to a single functional dependence

$$c_D = f(\mathrm{Re})$$

where c_D is a drag coefficient and Re is the Reynolds number.

What additional non-dimensional groups might appear in practice?

Notes.

(1) Dimensional analysis simply says that there **is** a relationship; it doesn't say what the relationship is. For the specific relationship one must appeal to other theory, simulation, or experimental data.

(2) If there is only one Π group … then it can't be a function of anything else … so it must be a constant.

(3) If Π_1, Π_2, Π_3, ... are suitable non-dimensional groups then we are liberty to replace some or all of them by any powers or products with the other Πs, provided that we retain the same number of independent non-dimensional groups; e.g. Π_1^{-1}, Π_1/Π_3^2 etc..

(4) It is very common in fluid mechanics to find (often after the rearrangement mentioned in (3)) certain combinations which can be recognised as familiar key parameters; e.g. Reynolds number ($\mathrm{Re} = \rho U L/\mu$) or Froude number ($\mathrm{Fr} = U/\sqrt{gL}$).

(5) Often the hardest part of the dimensional analysis is determining which <u>are</u> the relevant variables. For example, surface tension is always present in free-surface flows, but can be neglected if the Weber number $\mathrm{We} = \rho U^2 L/\sigma$ is large. Similarly, all fluids are compressible, but compressibility effects on the flow can be ignored if the Mach number ($\mathrm{Ma} = U/c$) is small; i.e. velocity is much less than the speed of sound.

(6) Although certain primary dimensions (e.g. M, L, T) appear when the variables are listed, they may do not do so *independently*, in this case, there will be fewer independent dimensions.

As an example of (6), the following example illustrates a case where M and T always appear in the combination MT^{-2}, giving only one independent dimension.

Example.

The tip deflection, δ, of a cantilever beam is a function of tip load, W, beam length, l, second moment of area, I, and Young's modulus, E. Perform a dimensional analysis of this problem.

Step 1. Identify the relevant variables.

$\delta, W, l, I, E.$

Step 2. Write down dimensions.

δ	L
W	$\mathrm{MLT^{-2}}$
l	L
I	$\mathrm{L^4}$
E	$\mathrm{ML^{-1}T^{-2}}$

Step 3. Establish the number of independent dimensions and non-dimensional groups.

Number of relevant variables: $n = 5$
Number of independent dimensions: $m = 2$ (L and $\mathrm{MT^{-2}}$ - *note*)
Number of non-dimensional groups (Πs): $n - m = 3$

Step 4. Choose m (= 2) dimensionally-independent scaling variables.
e.g. geometric (l), kinematic/time-dependent (E)

Step 5. Create the Πs by non-dimensionalising the remaining variables: δ, I and W. These give (after some algebra, omitted here):

$$\Pi_1 = \frac{\delta}{l}$$

$$\Pi_2 = \frac{I}{l^4}$$

$$\Pi_3 = \frac{W}{El^2}$$

Step 6. Set out the non-dimensional relationship.

$$\Pi_1 = f(\Pi_2, \Pi_3)$$

or

$$\frac{\delta}{l} = f(\frac{I}{l^4}, \frac{W}{El^2})$$

Note 1. This is as far as dimensional analysis will get us. Detailed theory shows that, for small elastic deflections,

$$\delta = \frac{1}{3}\frac{Wl^3}{EI}$$

or

$$\frac{\delta}{l} = \frac{1}{3}\left(\frac{W}{El^2}\right) \times \left(\frac{I}{l^4}\right)^{-1}$$

Note 2. Although three primary dimensions (M, L, T) appear here, they only do so in two independent groups: (L and MT^{-2}), so that the number of independent dimensions $m = 2$. This would have been more obvious in the alternative {FLT} system, where the variables have the following dimensions:

δ	L
W	F
l	L
I	L^4
E	FL^{-2}

Here, only F and L appear.

4. PHYSICAL MODELLING

4.1 Method

If a dimensional analysis indicates that a problem is described by a functional relationship between non-dimensional parameters Π_1, Π_2, Π_3, ... then complete *similarity* requires that these parameters be the same at both full ("*prototype*") scale and *model* scale. i.e.

$$(\Pi_1)_m = (\Pi_1)_p$$

$$(\Pi_2)_m = (\Pi_2)_p$$

etc.

Example.
A prototype gate valve which will control the flow in a conduit conveying paraffin is to be studied in a model. List the significant variables on which the pressure drop across the valve would depend. Perform dimensional analysis to obtain the relevant non-dimensional groups.

A 1/5-scale model is built to determine the pressure drop across the valve with water as the working fluid.

(a) For a particular opening, when the velocity of paraffin in the prototype is 3.0 m s^{-1} what should be the velocity of water in the model for dynamic similarity?

(b) What is the ratio of the quantities of flow in prototype and model?

(c) Find the pressure drop in the prototype if it is 60 kPa in the model.

(The density and viscosity of paraffin are 800 kg m^{-3} and 0.002 kg m^{-1} s^{-1} respectively. Take the kinematic viscosity of water as 1.0×10^{-6} m^2 s^{-1}).

Solution.
The pressure drop Δp is expected to depend upon the gate opening h, the overall depth d, the velocity V, density ρ and viscosity μ.

List the relevant variables:

Δp, h, d, V, ρ, μ

Write down dimensions:

Δp	$\mathrm{ML^{-1}T^{-2}}$
h	L
d	L
V	$\mathrm{LT^{-1}}$
ρ	$\mathrm{ML^{-3}}$
μ	$\mathrm{ML^{-1}T^{-1}}$

Number of relevant variables: $n = 6$

Number of independent dimensions: $m = 3$ (M, L and T)
Number of non-dimensional groups (Πs): $n - m = 3$
Choose m (= 3) scaling variables:
geometric (d); kinematic/time-dependent (V); dynamic/mass-dependent (ρ).

Form dimensionless groups by non-dimensionalising the remaining variables: Δp, h and μ.

For Δp:

$$\Pi_1 = \Delta p\, d^a V^b \rho^c$$

Considering the dimensions of both sides:

$$\begin{aligned} \mathrm{M^0L^0T^0} &= (\mathrm{ML^{-1}T^{-2}})(\mathrm{L})^a(\mathrm{LT^{-1}})^b(\mathrm{ML^{-3}})^c \\ &= \mathrm{M}^{1+c}\mathrm{L}^{-1+a+b-3c}\mathrm{T}^{-2-b} \end{aligned}$$

Equate powers of primary dimensions:

M: $0 = 1 + c \quad \Rightarrow \quad c = -1$
T: $0 = -2 - b \quad \Rightarrow \quad b = -2$
L: $0 = -1 + a + b - 3c \quad \Rightarrow \quad a = 1 - b + 3c \quad = 0$

Hence,

$$\Pi_1 = \Delta p\, V^{-2}\rho^{-1} \quad = \frac{\Delta p}{\rho V^2}$$

h can be done by inspection, since it has the same dimension as the scale d:

$$\Pi_2 = \frac{h}{d}$$

For μ:

$$\Pi_3 = \mu\, d^a V^b \rho^c$$

Considering the dimensions of both sides:

$$\begin{aligned} \mathrm{M^0L^0T^0} &= (\mathrm{ML^{-1}T^{-1}})(\mathrm{L})^a(\mathrm{LT^{-1}})^b(\mathrm{ML^{-3}})^c \\ &= \mathrm{M}^{1+c}\mathrm{L}^{-1+a+b-3c}\mathrm{T}^{-1-b} \end{aligned}$$

Equate powers of primary dimensions:

M: $0 = 1 + c \quad \Rightarrow \quad c = -1$
T: $0 = -1 - b \quad \Rightarrow \quad b = -1$
L: $0 = -1 + a + b - 3c \quad \Rightarrow \quad a = 1 - b + 3c \quad = -1$

Hence,

$$\Pi_3 = \mu d^{-1} V^{-1} \rho^{-1} \quad = \frac{\mu}{\rho V d}$$

Recognition of the Reynolds number suggests that we replace Π_3 by

$$\Pi_3' = (\Pi_3)^{-1} \quad = \frac{\rho V d}{\mu}$$

Hence, dimensional analysis yields

$$\Pi_1 = f(\Pi_2, \Pi_3')$$

i.e.

$$\frac{\Delta p}{\rho V^2} = f(\frac{h}{d}, \frac{\rho V d}{\mu})$$

(a) Dynamic similarity requires that all non-dimensional groups be the same in model and prototype; i.e.

$$\Pi_1 = \left(\frac{\Delta p}{\rho V^2}\right)_p = \left(\frac{\Delta p}{\rho V^2}\right)_m$$

$$\Pi_2 = \left(\frac{h}{d}\right)_p = \left(\frac{h}{d}\right)_m \qquad \text{(automatic if similar shape; i.e. “geometric similarity”)}$$

$$\Pi_3' = \left(\frac{\rho V d}{\mu}\right)_p = \left(\frac{\rho V d}{\mu}\right)_m$$

From the last, we have a velocity ratio

$$\frac{V_p}{V_m} = \frac{(\mu/\rho)_p}{(\mu/\rho)_m}\frac{d_m}{d_p} \quad = \frac{0.002/800}{1.0 \times 10^{-6}} \times \frac{1}{5} \quad = 0.5$$

Hence,

$$V_m = \frac{V_p}{0.5} \quad = \frac{3.0}{0.5} \quad = 6.0 \text{ m s}^{-1}$$

(b) The ratio of the quantities of flow is

$$\frac{Q_p}{Q_m} = \frac{(\text{velocity} \times \text{area})_p}{(\text{velocity} \times \text{area})_m} \quad = \frac{V_p}{V_m}\left(\frac{d_p}{d_m}\right)^2 \quad = 0.5 \times 5^2 \quad = 12.5$$

(c) Finally, for the pressure drop,

$$\Pi_1 = \left(\frac{\Delta p}{\rho V^2}\right)_p = \left(\frac{\Delta p}{\rho V^2}\right)_m \quad \Rightarrow \quad \frac{(\Delta p)_p}{(\Delta p)_m} = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2 \quad = \frac{800}{1000} \times 0.5^2 \quad = 0.2$$

Hence,

$$\Delta p_p = 0.2 \times \Delta p_m \quad = 0.2 \times (60 \text{ kPa}) \quad = 12.0 \text{ kPa}$$

4.2 Incomplete Similarity ("Scale Effects")

For a multi-parameter problem it is often not possible to achieve full similarity. In particular, it is rare to be able to achieve full Reynolds-number scaling when other dimensionless parameters are also involved. For hydraulic modelling of flows with a free surface the most important requirement is *Froude-number scaling* (Section 4.3)

It is common to distinguish three levels of similarity.

Geometric similarity – the ratio of all corresponding lengths in model and prototype are the same (i.e. they have the same shape).

Kinematic similarity – the ratio of all corresponding lengths and times (and hence the ratios of all corresponding velocities) in model and prototype are the same.

Dynamic similarity – the ratio of all forces in model and prototype are the same;
e.g. $\text{Re} = (\text{inertial force}) / (\text{viscous force})$ is the same in both. ("Inertial force" means "$\text{mass} \times \text{acceleration}$" – i.e., the sum of all forces.)

Geometric similarity is almost always assumed. However, in some applications – notably river modelling – it is necessary to distort vertical scales to prevent undue influence of, for example, surface tension or bed roughness.

Achieving full similarity is particularly a problem with the Reynolds number $\text{Re} = UL/\nu$.
- Using the same working fluid would require a velocity ratio inversely proportional to the length-scale ratio and hence impractically large velocities in the scale model.
- A velocity scale fixed by, for example, the Froude number (see Section 4.3) means that the only way to maintain the same Reynolds number is to adjust the kinematic viscosity (substantially).

In practice, Reynolds-number similarity is unimportant if flows in **both** model and prototype are fully turbulent; then momentum transport by viscous stresses is much less than that by turbulent eddies and so the precise value of molecular viscosity μ is unimportant. In some cases this may mean deliberately triggering transition to turbulence in boundary layers (for example by the use of tripping wires or roughness strips).

Surface effects

Full geometric similarity requires that not only the main dimensions of objects but also the surface roughness and, for mobile beds, the sediment size be in proportion. This would put impossible requirements on surface finish or grain size. In practice, it is sufficient that the surface be aerodynamically rough: $u_\tau k_s/\nu \geq 5$, where $u_\tau = \sqrt{\tau_w/\rho}$ is the friction velocity and k_s a typical height of surface irregularities. This imposes a minimum velocity in model tests.

Other Fluid Phenomena

When scaled down in size, fluid phenomena which were negligible at full scale may become important in laboratory models. A common example is surface tension.

4.3 Froude-Number Scaling

The most important parameter to preserve in hydraulic modelling of free-surface flows driven by gravity is the Froude number, $\text{Fr} = U/\sqrt{gL}$. Preserving this parameter between model (m) and prototype (p) dictates the scaling of other variables in terms of the length scale ratio.

Velocity

$$(\text{Fr})_m = (\text{Fr})_p$$

$$\left(\frac{U}{\sqrt{gL}}\right)_m = \left(\frac{U}{\sqrt{gL}}\right)_p \qquad \Rightarrow \qquad \frac{U_m}{U_p} = \left(\frac{L_m}{L_p}\right)^{1/2}$$

i.e. the velocity ratio is the square root of the length-scale ratio.

Quantity of flow

$$Q \sim \text{velocity} \times \text{area} \qquad \Rightarrow \qquad \frac{Q_m}{Q_p} = \left(\frac{L_m}{L_p}\right)^{5/2}$$

Force

$$F \sim \text{pressure} \times \text{area} \qquad \Rightarrow \qquad \frac{F_m}{F_p} = \left(\frac{L_m}{L_p}\right)^{3}$$

This arises since the pressure ratio is equal to the length-scale ratio – this can be seen from either hydrostatics (pressure $\propto$ height) or from the dynamic pressure (proportional to $(\text{velocity})^2$ which, from the Froude number, is proportional to length).

Time

$$t \sim \text{length/velocity} \qquad \Rightarrow \qquad \frac{t_m}{t_p} = \left(\frac{L_m}{L_p}\right)^{1/2}$$

Hence the quantity of flow scales as the length-scale ratio to the 5/2 power, whilst the time-scale ratio is the square root of the length-scale ratio. For example, at 1:100 geometric scale, a full-scale tidal period of 12.4 hours becomes 1.24 hours.

Example.
The force exerted on a bridge pier in a river is to be tested in a 1:10 scale model using water as the working fluid. In the prototype the depth of water is 2.0 m, the velocity of flow is 1.5 m s^{-1} and the width of the river is 20 m.

(a) List the variables affecting the force on the pier and perform dimensional analysis. Can you satisfy all the conditions for complete similarity? What is the most important parameter to choose for dynamic similarity?

(b) What are the depth, velocity and quantity of flow in the model?

(c) If the hydrodynamic force on the model bridge pier is 5 N, what would it be on the prototype?

5. NON-DIMENSIONAL GROUPS IN FLUID MECHANICS

Dynamic similarity requires that the ratio of all forces be the same. The ratio of different forces produces many of the key non-dimensional parameters in fluid mechanics.

(Note that "inertial force" means "mass × acceleration" – i.e. the total force. Each non-dimensional group then involves the ratio of a particular force to the total force. This reflects the fraction of the total that this particular force is responsible for, so you can see whether its effect is likely to be small or large.)

Reynolds number $\mathrm{Re} = \dfrac{\rho U L}{\mu} = \dfrac{\text{inertial force}}{\text{viscous force}}$ (viscous flows)

Froude number $\mathrm{Fr} = \dfrac{U}{\sqrt{gL}} = \left(\dfrac{\text{inertial force}}{\text{gravitational force}}\right)^{1/2}$ (free-surface flows)

Weber number $\mathrm{We} = \dfrac{\rho U^2 L}{\sigma} = \dfrac{\text{inertial force}}{\text{surface tension}}$ (near-surface flows)

Rossby number $\mathrm{Ro} = \dfrac{U}{\Omega L} = \dfrac{\text{inertial force}}{\text{Coriolis force}}$ (rotating flows)

Mach number $\mathrm{Ma} = \dfrac{U}{c} = \left(\dfrac{\text{inertial force}}{\text{compressibility force}}\right)^{1/2}$ (compressible flows)

These groups occur regularly when dimensional analysis is applied to fluid-dynamical problems. They can be derived by considering forces on a small volume of fluid. They can also be derived by non-dimensionalising the differential equations of fluid flow (see White, 2021), or the online notes for the 4$^{\text{th}}$-year Computational Hydraulics unit.