3. APPROXIMATIONS AND SIMPLIFIED EQUATIONS SPRING 2024

Fluid dynamics is governed by equations for mass, momentum and energy. The momentum equation for a viscous fluid is called the *Navier-Stokes equation*; it is based upon:

- continuum mechanics;
- the momentum principle;
- shear stress proportional to velocity gradient.

A fluid for which the last is true is called a *Newtonian* fluid. This is the case for most fluids in engineering. However, there are important *non-Newtonian* fluids; e.g. mud, cement, blood, paint, polymer solutions. CFD is very useful for these, as their governing equations are usually impossible to solve analytically.

The full equations are time-dependent, 3-dimensional, viscous, compressible, non-linear and highly coupled. However, in most cases it is possible to simplify analysis by adopting a reduced equation set. Some common approximations are listed below.

Reduction of dimension:

- steady-state;
- two-dimensional.

Neglect of some fluid property:

- incompressible;
- inviscid.

Simplified forces:

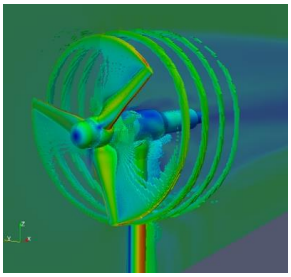
- hydrostatic;
- Boussinesq approximation for density.

Approximations based upon averaging:

- depth-averaging (shallow-water equations);
- Reynolds-averaging (turbulent flows).

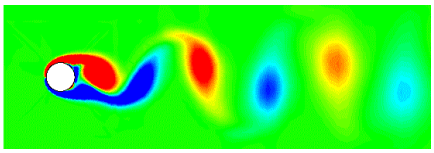
The consequences of these approximations are examined in the following sections.

3.1 Steady-State *vs* Time-Dependent Flow



Many flows are naturally time-dependent. Examples include waves, tides, turbines, reciprocating pumps and internal combustion engines.

Other flows have stationary boundaries but become time-dependent because of an instability. An important example is vortex shedding around cylindrical objects. Depending on the Reynolds number, the instability may or may not progress to fully-developed turbulence.



Some computational solution procedures rely on a time-stepping method to march to steady state; examples are transonic flow and open-channel flow (where the mathematical nature of the governing equations is different for Mach or Froude numbers less than or greater than 1).

Thus, there are three major reasons for using the full time-dependent equations:

- time-dependent problem;
- time-dependent instability;
- time-marching to steady state.

Time-dependent methods will be addressed in Section 6.

3.2 Two- Dimensional *vs* Three-Dimensional Flow

Geometry and boundary conditions may suggest that the flow is two-dimensional. Two-dimensional calculations require considerably less computer resources: time and memory.

“Two-dimensional” may include “axisymmetric”. This is easier to achieve in the laboratory than cartesian 2-dimensionality, because duct flows have side-wall boundary layers.

Even if the geometry is two-dimensional, instabilities may still lead to three-dimensional flow. Also, turbulence is always three-dimensional, even if the mean flow is two-dimensional.

3.3 Incompressible *vs* Compressible Flow

A *flow* (not a *fluid*, note) is said to be *incompressible* if flow-induced pressure or temperature changes do not cause significant density changes. Compressibility is important in high-speed flow or where there is significant heat input.

Liquid flows are usually treated as incompressible, but gas flows can also be regarded as incompressible at speeds much less than the speed of sound; (a common rule of thumb being Mach number < 0.3).

Density variations within fluids can occur for other reasons, notably from salinity (oceans) and temperature (atmosphere). These lead to buoyancy forces. Because the density variations are not flow-induced, these flows can still be treated as incompressible; i.e. “incompressible” does not necessarily mean “uniform density”.

The key differences in CFD between compressible and incompressible flow concern:
(1) whether there is a need to solve a separate energy equation;
(2) how pressure is determined.

Compressible Flow

First Law of Thermodynamics:

$$\text{change of internal energy} = \text{heat input} + \text{work done on fluid}$$

A transport equation has to be solved for an energy-related variable (e.g. internal energy e or enthalpy $h = e + p/\rho$) in order to obtain the absolute temperature T. For an ideal gas,

$$e = c_v T \qquad \text{or} \qquad h = c_p T \tag{1}$$

c_v and c_p are specific heat capacities at constant volume and constant pressure respectively.

Mass conservation provides a transport equation for density, whilst pressure is derived from an *equation of state*; e.g. the ideal-gas law:

$$p = \rho R T \tag{2}$$

For compressible flow it is necessary to solve an energy equation.

In *density-based methods* for compressible CFD:
- mass equation → ρ;
- energy equation → T;
- equation of state → p.

Incompressible Flow

In incompressible flow, pressure changes (by definition) cause negligible density changes. Temperature is not involved and so a separate thermodynamic energy equation is not necessary. The Mechanical Energy Principle:

$$\text{change of kinetic energy} = \text{work done on fluid}$$

is equivalent to, and readily derived from, the momentum equation. In the inviscid case it is often expressed as Bernoulli's equation (see the Examples).

Incompressibility implies that density is constant along a streamline:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \tag{3}$$

but may vary between streamlines (e.g. due to temperature or salinity differences). Conservation of mass is then replaced by conservation of volume:

$$\sum_{\text{faces}} (\text{volume flux}) = 0 \qquad \text{or} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

Pressure is not derived from an equation of state but from the requirement that solutions of the

momentum equation be mass-consistent (Section 5).

In incompressible flow it is *not* necessary to solve a separate energy equation. In *pressure-based methods* for incompressible CFD: • incompressibility → volume is conserved as well as mass; • requiring solutions of momentum equation to be mass-consistent → equation for p.

3.4 Inviscid *vs* Viscous Flow

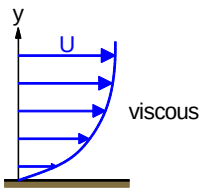
If viscosity is neglected, the *Navier-Stokes equations* become the *Euler equations*.

Consider streamwise momentum in a developing boundary layer:

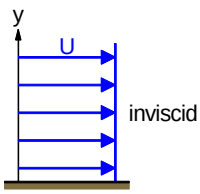
$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$

mass × acceleration = pressure force + viscous force

Dropping the viscous term reduces the order of the highest derivative from 2 to 1 and hence one less boundary condition is required.

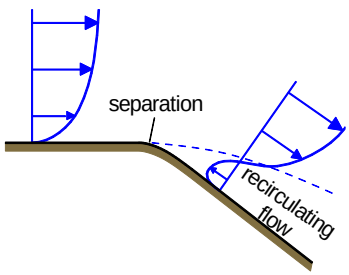


- Viscous (*real*) flows require a no-slip (zero-relative-velocity) condition at rigid walls – the *dynamic* boundary condition.



- Inviscid (*ideal*) flows require only the velocity component normal to the wall to be zero – the *kinematic* boundary condition. The wall shear stress is zero.

Although its magnitude is small, and consequently its direct influence via the shear stress is tiny, viscosity can have a global influence out of all proportion to its size. The most important effect is *flow separation*, where the viscous boundary layer required to satisfy the non-slip condition is first slowed and then reversed by an adverse pressure gradient. Boundary-layer separation has two important consequences:



- major disturbance to the flow;
- a large increase in pressure drag.

<u>Velocity Potential, ϕ</u>

In inviscid flow it may be shown[1] that the velocity components can be written as the gradient of a single scalar variable, the *velocity potential* ϕ:

[1] Since pressure acts perpendicularly to a surface and cannot impart rotation, an inviscid fluid can be regarded as irrotational ($\nabla \wedge \mathbf{u} = 0$), and so the velocity field can be written as the gradient of a scalar function.

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \qquad w = \frac{\partial \phi}{\partial z} \qquad \text{(concisely written: } \mathbf{u} = \nabla\phi) \tag{5}$$

Substituting these into the continuity equation for incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(concisely written: } \nabla \bullet \mathbf{u} = 0)$$

gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

which is often written

$$\nabla^2 \phi = 0 \qquad (\textit{Laplace's equation}) \tag{6}$$

The entire flow is completely determined by a single scalar field ϕ satisfying Laplace's equation. Since Laplace's equation occurs in many branches of physics (waves, electrostatics, heat conduction, gravitation, optics, ...) many good solvers already exist.

Velocity components u, v and w are obtained by differentiating ϕ. Pressure is then recovered from Bernoulli's equation:

$$p + \rho g z + \frac{1}{2}\rho U^2 = \text{constant (along a streamline)} \tag{7}$$

where U is the magnitude of velocity.

The potential-flow assumption often gives an adequate description of velocity and pressure fields for real fluids, except very close to solid surfaces where viscous forces are significant. It is useful, for example, in calculating lift on aerofoils or in water-wave theory (Hydraulics 3). However, in ignoring viscosity it implies that there are neither tangential stresses on boundaries nor flow separation, which leads to the erroneous conclusion (*D'Alembert's Paradox*) that an object moving through a fluid experiences no drag.

3.5 Hydrostatic *vs* Non-Hydrostatic Flow

The equation for the vertical component of momentum can be written:

$$\rho \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z} - \rho g + \text{viscous forces} \tag{8}$$

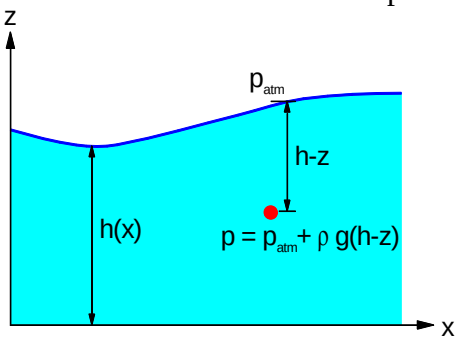
For large horizontal scales the vertical acceleration $\mathrm{D}w/\mathrm{D}t$ is much less than g and the viscous forces are small. The balance of terms is then the same as in a stationary fluid:

$$\frac{\partial p}{\partial z} \approx -\rho g \qquad \text{i.e. pressure forces balance weight}$$

With this hydrostatic approximation, in constant-density flows with a free surface the pressure is determined everywhere by the height of the free surface $h(x, y)$:

$$p = p_{\text{atm}} + \rho g (h - z) \tag{9}$$

This results in huge computational savings because the position of the surface automatically determines the pressure field without the need to solve a separate pressure equation.



The hydrostatic approximation is widely used in conjunction with the depth-averaged shallow-water equations (see Section 3.7 below and the Shallow-Flows part of this course.

3.6 Boussinesq Approximation for Density[2]

Density variations may arise at low speeds because of changes in temperature or humidity (atmosphere), or salinity (oceans) which give rise to *buoyancy forces*. The effect of these density changes can be significant even if the fractional change in density is small.

The *Boussinesq approximation* retains density variations in the gravity term (giving buoyancy forces) but disregards them in the inertial (mass × acceleration) term. For the vertical momentum equation:

$$\rho \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z} - \rho g + \cdots \quad = -\frac{\partial p}{\partial z} - \rho_0 g - (\rho - \rho_0) g + \cdots \tag{10}$$

the Boussinesq approximation is simply to replace ρ on the LHS by the constant density ρ_0. The approximation is valid if *relative* density variations are small; i.e. $\Delta\rho/\rho_0 \ll 1$. This condition is usually satisfied in the atmosphere and oceans.

On the RHS of the momentum equation, the part of the weight resulting from the constant reference density ρ_0 is usually subsumed into a modified pressure $p^* = p + \rho_0 g z$, so that

$$\rho_0 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p^*}{\partial z} - (\rho - \rho_0) g + \cdots \tag{11}$$

The relative change in density is typically proportional to the change in some scalar θ (e.g. temperature or salinity):

$$\frac{\rho - \rho_0}{\rho_0} = -\alpha(\theta - \theta_0) \tag{12}$$

where α is the coefficient of expansion. (The sign adopted here is that for temperature, where an increase in temperature leads to a reduction in density; the opposite sign would be used for salinity-driven density changes.) The vertical momentum equation can then be written

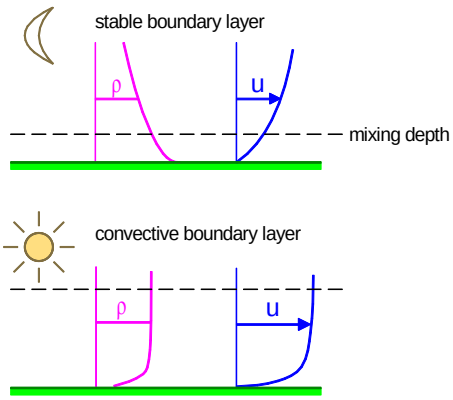
$$\rho_0 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p^*}{\partial z} + \underbrace{\rho_0 \alpha(\theta - \theta_0) g}_{\text{buoyancy force}} + \cdots \tag{13}$$

[2] Several very different approximations are also referred to as *the* Boussinesq approximation in different contexts – e.g. shallow-water equations or eddy-viscosity turbulence models.

Temperature variations in the atmosphere, brought about by surface (or cloud-top) heating or cooling, are responsible for significant changes in airflow and turbulence.

- On a cold night the atmosphere is *stable*. Cool, dense air collects near the surface and vertical motions are suppressed; the boundary-layer depth is 100 m or less.

- On a warm day the atmosphere is *unstable*. Surface heating produces warm; light air near the ground and convection occurs; the boundary layer may be 2 km deep.



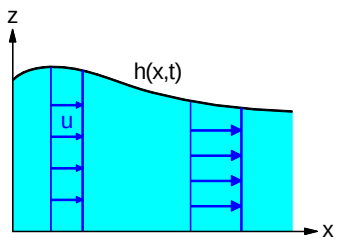
Vertical dispersion of pollution is very different in the two cases.

3.7 Depth-Averaged (Shallow-Water) Equations

This approximation is used for flow of a constant-density fluid with a free surface, where the depth of fluid is small compared with typical horizontal scales. In this "hydraulic" approximation, the fluid can be regarded as quasi-2d with:

- horizontal velocity components u, v;
- depth of water, h.

The depth h may vary due to changes in the levels of the free-surface, the bed, or both.



By applying mass and momentum principles to a vertical column of constant-density fluid of variable depth h, the depth-integrated equations governing the motion can be written (for the one-dimensional case and in conservative form) as:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{14}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial (u^2 h)}{\partial x} = -\frac{\partial (\frac{1}{2} g h^2)}{\partial x} + \frac{1}{\rho}(\tau_{\text{surface}} - \tau_{\text{bed}}) \tag{15}$$

The $\frac{1}{2} g h^2$ term comes from ($1/\rho$ times) the hydrostatic pressure force per unit width on a water column of height h; i.e.

$$\text{average hydrostatic pressure} \times \text{area} = \left(\frac{1}{2}\rho g h\right) \times (h \times 1)$$

The final term is the net effect of surface stress (due to wind) and bed stress (due to friction). These equations are derived in the Examples here and in the Shallow-Flows part of this course.

The resulting *shallow-water equations* are mathematically similar to those for a compressible gas. There are direct analogies between

- discontinuities: hydraulic jumps (shallow flow) and shocks (compressible flow);
- critical flow through a venturi (shallow) or gas flow through a throat (compressible).

In both cases there is a characteristic wave speed ($c = \sqrt{gh}$ in the hydraulic case; $c = \sqrt{\gamma p/\rho}$ in compressible flow). Whether this is greater or smaller than the flow velocity determines whether disturbances can propagate upstream and hence the nature of the flow. The ratio of

flow speed to wave speed is known as:

Froude number: $\mathrm{Fr} = \dfrac{u}{\sqrt{gh}}$ in shallow flow

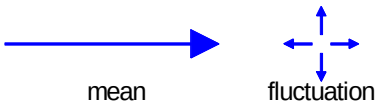
Mach number: $\mathrm{Ma} = \dfrac{u}{c}$ in compressible flow

3.8 Reynolds-Averaged Equations (Turbulent Flow)

The majority of flows encountered in engineering are turbulent. Most, however, can be regarded as time-dependent, three-dimensional *fluctuations* superimposed on a much simpler *mean* flow. Usually, we are only interested in the latter.

The process of *Reynolds-averaging*[3] starts by decomposing each flow variable into mean and turbulent parts:

$$u = \underset{\text{mean}}{\bar{u}} + \underset{\text{fluctuation}}{u'}$$



The "mean" may be a time average (the usual case in the laboratory) or an ensemble average (a probability mean over a hypothetical large number of identical experiments).

When the Navier-Stokes equations are averaged, the result is (see Section 7):

- the same equation for mass, and similar equation for *mean* momentum, except for …
- turbulent fluxes, $-\rho\overline{u'v'}$ etc. (called the *Reynolds stresses*) which provide a net transport of momentum.

For example, the viscous shear stress

$$\tau_{\text{visc}} = \mu \frac{\partial \bar{u}}{\partial y}$$

is supplemented by an additional (and usually much larger) turbulent stress:

$$\tau_{\text{turb}} = -\rho\overline{u'v'} \tag{16}$$

To solve the mean-flow equations, a *turbulence model* is required to supply these turbulent stresses. Popular models (see Section 8) exploit an analogy between viscous and turbulent transport and employ an *eddy viscosity* μ_t supplementing the molecular viscosity. Thus,

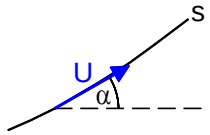
$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'} \quad \rightarrow \quad (\mu + \mu_t)\frac{\partial \bar{u}}{\partial y} \tag{17}$$

[3] Osborne Reynolds, first Professor of Engineering at Owens College, later to become the Victoria University of Manchester. His actual experimental apparatus, including that for the famous pipe-flow experiments, is in the basement of the George Begg building at the University of Manchester. A modern replica is in the foyer.

Examples

Q1.
Discuss the circumstances under which a fluid flow can be approximated as:
(a) incompressible;
(b) inviscid.

Q2.
By resolving forces along a streamline, the steady-state momentum equation for an inviscid fluid can be written



$$\rho U \frac{\partial U}{\partial s} = -\frac{\partial p}{\partial s} - \rho g \sin\alpha$$

where U is the velocity magnitude, s is the distance along a streamline and α is the angle between local velocity and the horizontal. Assuming incompressible flow, derive Bernoulli's equation.

Q3.
For incompressible flow in a rotating reference frame the force per unit volume, $\mathbf{f}$, is the sum of pressure, gravitational, Coriolis and viscous forces:

$$\mathbf{f} = -\nabla p - \rho g \mathbf{e}_z - 2\rho \mathbf{\Omega} \wedge \mathbf{u} + \mu \nabla^2 \mathbf{u}$$

where $\mathbf{e}_z$ is a unit vector in the z direction and $\mathbf{\Omega}$ is the angular velocity of the rotating frame.

(a) If the density is uniform, show that pressure and gravitational forces can be combined in a *piezometric* pressure (which should be defined).

(b) If the density varies, describe the "Boussinesq" approximation in this context and give an application in which it is used.

(c) Show how the momentum equation (with Boussinesq approximation for density) can be non-dimensionalised in terms of densimetric Froude number, Rossby number and Reynolds number:

$$\mathrm{Fr} = \frac{U_0}{\sqrt{(\Delta\rho/\rho_0) g L_0}} \quad , \qquad \mathrm{Ro} = \frac{U_0}{\Omega L_0} \quad , \qquad \mathrm{Re} = \frac{\rho_0 U_0 L_0}{\mu}$$

where ρ_0, L_0, U_0 are characteristic density, length and velocity scales, respectively, and $\Delta\rho$ is a typical magnitude of density variation.

Q4. (Exam 2023 – part)
The vector momentum equation for a viscous fluid of variable density is

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p - \rho g \mathbf{e}_z + \mu \nabla^2 \mathbf{u} \qquad (*)$$

where t is time, ρ is density, $\mathbf{u} = (u, v, w)$ is velocity, p is pressure, μ is dynamic viscosity, g is the acceleration due to gravity and $\mathbf{e}_z$ is a unit vector in the z direction.

(a) Define the operator $\mathrm{D}/\mathrm{D}t$ mathematically and explain its physical significance.

(b) Define the operator ∇^2 mathematically.

(c) Show that, for a *constant* density ρ_0, the pressure and gravitational terms can be combined as a single gradient term involving the *piezometric* pressure.

(d) If density variations occur as a result of temperature changes, then

$$\frac{\rho - \rho_0}{\rho_0} = -\alpha(\theta - \theta_0)$$

where θ is temperature and α is the coefficient of thermal expansion. Describe the *Boussinesq approximation* in this context, rewrite equation (*) with this approximation, and state the conditions under which the approximation is valid.

(e) Show that, with the Boussinesq approximation, equation (*) can be non-dimensionalised as

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \frac{1}{\mathrm{Fr}^2}\theta \mathbf{e}_z + \frac{1}{\mathrm{Re}}\nabla^2 \mathbf{u}$$

where all variables are now non-dimensional, and Re and Fr are, respectively, the Reynolds number and densimetric Froude number (both to be defined).

Q5. (Exam 2020 – part)

(a) State the conditions under which a fluid flow can be described by a velocity potential, ϕ, and explain why this is computationally useful when such conditions are met.

(b) The velocity potential for a 2-d flow is

$$\phi = x^3 - 3xy^2$$

Find the velocity components (u, v).

(c) For the velocity field in part (b), find the corresponding stream function ψ (you may adopt either sign convention) and suggest a geometry in which such a flow may arise.

Q6.

(a) In flow with a free surface, by taking a control volume as a column of (time-varying) height $h(x, y, t)$ and horizontal cross-section $\Delta x \times \Delta y$, assuming that density is constant, τ_{13} and τ_{23} are the only significant stress components, the horizontal velocity field may be replaced by the depth-averaged velocity (u, v) and the pressure is hydrostatic, derive the shallow-water equations for continuity and x-momentum in the form

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2)}{\partial x} + \frac{\partial (hvu)}{\partial y} = -gh\frac{\partial z_s}{\partial x} + \frac{\tau_{13}(\text{surface}) - \tau_{13}(\text{bed})}{\rho}$$

(b) Provide an alternative derivation by integrating the continuity and horizontal momentum equations for incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} + \frac{\partial (\rho vu)}{\partial y} + \frac{\partial (\rho wu)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{13}}{\partial z}$$

over a depth $h = z_s - z_b$.

For part (b) you will need the boundary condition that the top and bottom surfaces $z = z_s(x, y)$ and $z = z_b(x, y)$ are material surfaces:

$$\frac{\mathrm{D}}{\mathrm{D}t}(z - z_s) = 0 \qquad \text{or} \qquad w - \frac{\partial z_s}{\partial t} - u\frac{\partial z_s}{\partial x} - v\frac{\partial z_s}{\partial y} = 0 \qquad \text{on } z = z_s$$

and similarly for z_b, together with Leibniz' Theorem for differentiating an integral:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x')\,\mathrm{d}x' = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x'}\,\mathrm{d}x' + f(b)\frac{\mathrm{d}b}{\mathrm{d}x} - f(a)\frac{\mathrm{d}a}{\mathrm{d}x}$$

Note: this is easily extended to consider additional forces such as Coriolis forces and other stress-like terms (e.g. "*horizontal diffusion*"). *This will be covered in the* "*Shallow-Flows*" *part of the course.*