3. Approximations and Simplified Equations

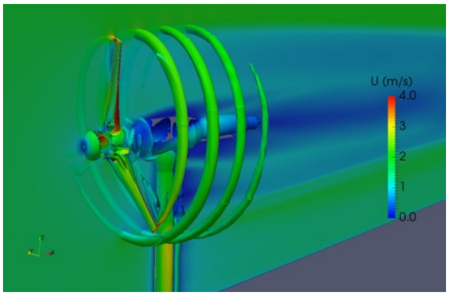
Common Approximations

- Reduction of **dimension**:
  - steady-state
  - 2-dimensional
- Neglect of some **fluid property**:
  - incompressible
  - inviscid
- Simplified **forces**:
  - hydrostatic
  - Boussinesq approximation for density
- **Averaging**:
  - depth-averaging (shallow-water flows)
  - Reynolds averaging (turbulent flows)

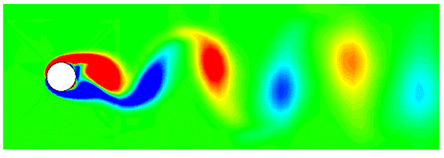
Time-Dependent vs Steady-State

- Use time-dependent calculations for:
  - **time-dependent** problems (waves, pumps, …)
  - flows with a natural **instability** (vortex shedding, ...)
  - time-marching to **steady state** (high-Mach or high-Froude flows)
- Time-dependent calculations:
  - "parabolic" or "hyperbolic" equations
  - 1st-order in time; solved by **forward-marching** in time
- Steady-state calculations:
  - "elliptic" equations
  - 2nd-order in space: **implicit**, **iterative** solution methods

Marine Current Turbine



Vortex Shedding From a Cylinder



2- or 3-Dimensional

- Determined by geometry and boundary conditions
- Significant computational implications
- 2-d flows difficult to achieve in the laboratory
- Axisymmetric flows also “2-d”
- Flow instabilities can still lead to 3-dimensionality

Incompressible Flow

- **Definition:** **flow-induced** pressure and temperature changes don't cause significant density changes
- Usually requires velocity << speed of sound (Ma << 1)
- Does **not** necessarily mean "uniform density"
  - Environmental flows driven by density differences in atmosphere or ocean can still be regarded as incompressible

Compressible vs Incompressible CFD

Compressible flow:

- density changes due to large pressure and/or temperature changes;
- requires an equation for internal energy *e* (or enthalpy *h*):

  change in energy = heat input + work done on fluid
- mass equation → density ρ
  energy equation → temperature T
  equation of state → pressure p (e.g. ideal gas law, $p = \rho RT$)

Incompressible flow:

- density constant along a streamline; volume conserved
- mechanical energy equation:

  change in kinetic energy = work done on fluid

  can be derived from momentum: **no separate energy equation**
- a pressure equation arises from the requirement that solutions of the momentum equation also be mass-consistent

Compressible vs Incompressible CFD

Compressible flow:

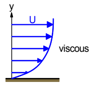
- requires an energy equation
- pressure determined by equation of state

Incompressible flow:

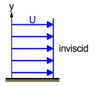
- does **not** require a separate energy equation
- pressure equation arises from mass-consistency

Viscous vs Inviscid

- **Viscous** (Navier-Stokes) equations:
  - dynamic ("no-slip") boundary condition: (u, v, w) = 0



- **Inviscid** (Euler) equations:
  - no 2nd-order derivatives (one less b.c.)
  - kinematic ("slip-wall") boundary condition: u_n=0



- Inviscid approximation OK only if boundary layer is:
  - thin (high Re)
  - attached (no flow separation)
- Inviscid approximation implies no drag, no heat transfer, no sediment transport, …

Potential Flow

- **Approximation:** inviscid, incompressible
- Velocity derived from a **velocity potential** ϕ:

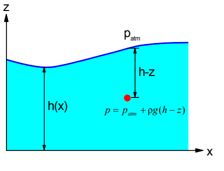
$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$
$$\longrightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Laplace's Equation

$$\mathbf{u} = \nabla \phi$$
$$\nabla \bullet \mathbf{u} = 0$$
$$\longrightarrow \nabla^2 \phi = 0$$

- **Consequences:**
  - entire flow field determined by a single scalar
  - very common equation; plenty of good solvers around
  - ignores boundary-layer effects: no drag or flow separation

Hydrostatic Approximation



- **Approximation:**
  - pressure forces balance weight:

$$\frac{\partial p}{\partial z} = -\rho g \qquad \Delta p = -\rho g \Delta z$$

- **Validity:**
  - always true in stationary fluid
  - good approximation if vertical acceleration << g
- **Consequence:**
  - pressure is determined everywhere from the depth below the free-surface:

$$p = p_{atm} + \rho g(h - z)$$

Boussinesq Approximation for Density

- **Application:** variable-density environmental flows:
  - atmosphere (temperature);
  - oceans (salinity).

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - \rho g + \cdots \qquad = -\frac{\partial p}{\partial z} - \rho_0 g - (\rho - \rho_0) g + \cdots$$

- **Approximation:**
  - retain density changes in buoyancy force
  - neglect density changes in inertial term (mass × acceleration)

$$\rho_0 \frac{Dw}{Dt} = -\frac{\partial p^*}{\partial z} - (\rho - \rho_0) g \qquad\qquad p^* = p + \rho_0 g z$$

- **Comments:**
  - sometimes needed in theoretical work (to linearise equations);
  - usually unnecessary in general-purpose CFD (included in iteration).

Density-Determining Scalar

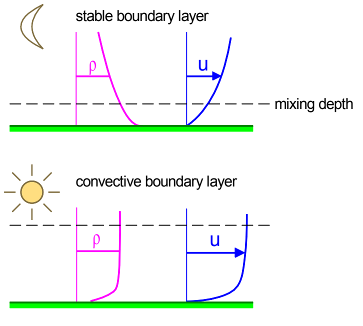
Vertical momentum equation: $\rho_0 \frac{Dw}{Dt} = -\frac{\partial p^*}{\partial z} - (\rho - \rho_0) g + \cdots$

Density-determining scalar θ (e.g. temperature or salinity):

$$\frac{\rho - \rho_0}{\rho_0} = -\alpha (\theta - \theta_0)$$

$$\rho_0 \frac{Dw}{Dt} = -\frac{\partial p^*}{\partial z} + \underbrace{\rho_0 \alpha (\theta - \theta_0) g}_{buoyancy\, force} + \cdots$$

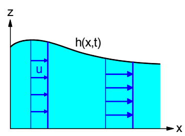
Atmospheric Boundary Layer



Fresh-Water Outfall

Shallow-Water Equations

- **Application:** open-channel hydraulics
- **Approximation:** depth-averaged
  - horizontal velocities *u*, *v*
  - water depth *h*



$$(mass) \quad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

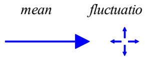
$$(momentum) \quad \frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}(u^2 h) = -\frac{\partial}{\partial x}(\tfrac{1}{2} g h^2) + \frac{1}{\rho}(\tau_{surface} - \tau_{bed})$$

- **Compressible-flow analogy:**
  - discontinuity: hydraulic jump ↔ shock
  - wave speed: $c = \sqrt{gh}$ ↔ $c = \sqrt{\gamma p / \rho}$
  - ratio of current to wave speed: Froude number Fr ↔ Mach number Ma

Reynolds Averaging

Application: turbulent flows

$$u = \underset{mean}{\overline{u}} + \underset{fluctuation}{u'}$$



Averaging produces an additional effective stress in the **mean** flow equations:

$\tau_{turb} = -\rho \overline{u'v'}$ **Reynolds stress**

Common modelling practice:

$\tau_{turb} = \mu_t \frac{\partial \overline{u}}{\partial y}$ μ_t = **eddy viscosity**

Turbulent Jet