10. Advanced Turbulence Modelling

Part 1. Models

Part 2. Implementation

Turbulence Modelling

Purpose:

- model turbulent fluxes $\overline{u_i u_j}$ and $\overline{u_i \phi}$

in order to:

- close the mean flow equations
- quantify mixing

Types of Turbulence Model

Reynolds-Averaged Navier-Stokes (RANS) Models

- **Eddy-viscosity models (EVM):**
  - (deviatoric) stress proportional to mean strain
- **Non-linear eddy-viscosity models (NLEVM):**
  - stress is a non-linear function of mean strain and vorticity
- **Differential stress models (DSM) / Reynolds-stress transport models (RSTM):**
  - solve transport equations for all Reynolds stresses

Models That Compute Fluctuating Quantities

- **Large-eddy simulation (LES):**
  - time-dependent calculation; model subgrid-scale motions
- **Direct numerical simulation (DNS):**
  - time-dependent calculation; resolve all scales of motion

Eddy-Viscosity Models

$$-\rho\overline{u_i u_j} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\rho k\delta_{ij}$$

$$-\rho\overline{uv} = \mu_t\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)$$

$$-\rho\overline{u^2} = 2\mu_t\frac{\partial U}{\partial x} - \frac{2}{3}\rho k$$

Lumped in with pressure:

$$-p\delta_{ij} + \tau_{ij}$$
$$= -(p + \tfrac{2}{3}\rho k)\delta_{ij} + \tau_{ij}^{(deviatoric)}$$

- This is a **model**!
- μ is a property of the **fluid**; μ_t is a property of the **flow**
- μ_t varies with **position**
- **$\mu_t >> \mu$** throughout much of the flow

k-ε Models

Eddy viscosity: $\nu_t = C_\mu \frac{k^2}{\varepsilon}$

Turbulent transport equations:

$$\rho\frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial}{\partial x_i}\left(\Gamma^{(k)}\frac{\partial k}{\partial x_i}\right) + \rho(P^{(k)} - \varepsilon)$$

$$\rho\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = \frac{\partial}{\partial x_i}\left(\Gamma^{(\varepsilon)}\frac{\partial \varepsilon}{\partial x_i}\right) + \rho(C_{\varepsilon1}P^{(k)} - C_{\varepsilon2}\varepsilon)\frac{\varepsilon}{k}$$

rate of change | *diffusion* | *production* | *dissipation*

$$\Gamma^{(k)} = \mu + \frac{\mu_t}{\sigma^{(k)}} \qquad P^{(k)} \equiv -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}$$

$$\Gamma^{(\varepsilon)} = \mu + \frac{\mu_t}{\sigma^{(\varepsilon)}}$$

k-ω Models

Eddy viscosity: $\nu_t = \frac{k}{\omega}$

Turbulent transport equations:

$$\rho \frac{Dk}{Dt} = \frac{\partial}{\partial x_i}(\Gamma^{(k)} \frac{\partial k}{\partial x_i}) + \rho(P^{(k)} - \beta^* \omega k)$$

$$\rho \frac{D\omega}{Dt} = \frac{\partial}{\partial x_i}(\Gamma^{(\omega)} \frac{\partial \omega}{\partial x_i}) + \rho(\frac{\alpha}{\nu_t} P^{(k)} - \beta \omega^2)$$

Eddy-Viscosity Models - Assessment

For

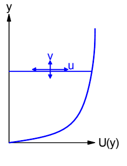
- Easy to implement in viscous solvers
- Extra viscosity aids stability
- Theoretical justification in simple flows

Against

- Lack of turbulence physics; (particularly **anisotropy** and **history** effects)
- Based on a single scalar μ_t; at most one stress component can be predicted accurately

Eddy-Viscosity Models in Simple Shear

$$-\rho \overline{u_i u_j} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3}\rho k \delta_{ij}$$



Shear stress: $-\rho \overline{uv} = \mu_t \frac{\partial U}{\partial y}$

Normal stresses: $\overline{u^2} = \overline{v^2} = \overline{w^2} = \frac{2}{3}k$

Experiment: $\overline{u^2} : \overline{v^2} : \overline{w^2} = 1.0 : 0.4 : 0.6$

anisotropy

Non-Linear Eddy-Viscosity Models

Linear EVM: $\overline{u_i u_j} = -\nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + \frac{2}{3} k \delta_{ij}$ $\qquad \nu_t = C_\mu \frac{k^2}{\varepsilon}$

$$\overline{u_i u_j} - \frac{2}{3} k \delta_{ij} = -2\nu_t S_{ij}$$

strain: $S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$

$$\frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij} = -2C_\mu \frac{k}{\varepsilon} S_{ij}$$

vorticity: $\Omega_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$

$$a_{ij} = -2C_\mu s_{ij}$$

anisotropy: $a_{ij} = \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij}$

$$\mathbf{a} = -2C_\mu \mathbf{s}$$

$s_{ij} = \frac{k}{\varepsilon} S_{ij}, \quad \omega_{ij} = \frac{k}{\varepsilon}\Omega_{ij}$

Non-linear EVM: $\mathbf{a} = -2C_\mu \mathbf{s} + \mathbf{NL}(\mathbf{s}, \boldsymbol{\omega})$

Quadratic NLEVM

$$\mathbf{a} = -2C_\mu \mathbf{s} + \beta_1(\mathbf{s}^2 - \tfrac{1}{3}\{\mathbf{s}^2\}\mathbf{I}) + \beta_2(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}) + \beta_3(\boldsymbol{\omega}^2 - \tfrac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I})$$

Quadratic terms admit **anisotropy** in simple shear:

$$\frac{\overline{u^2}}{k} = \frac{2}{3} + (\beta_1 + 6\beta_2 - \beta_3)\frac{\sigma^2}{12}$$

$$\frac{\overline{v^2}}{k} = \frac{2}{3} + (\beta_1 - 6\beta_2 - \beta_3)\frac{\sigma^2}{12}$$

$$\frac{\overline{w^2}}{k} = \frac{2}{3} - (\beta_1 - \beta_3)\frac{\sigma^2}{6}$$

$$\sigma = \frac{k}{\varepsilon}\frac{\partial U}{\partial y}$$

General Non-linear Eddy-Viscosity Model

$$\mathbf{a} = \sum_{\alpha=1}^{10} C_\alpha \mathbf{T}_\alpha(\mathbf{s}, \boldsymbol{\omega})$$

- **10 bases**
  - symmetric
  - traceless
- **Quintic**

Linear: $\mathbf{T}_1 = \mathbf{s}$

Quadratic: $\mathbf{T}_2 = \mathbf{s}^2 - \frac{1}{3}\{\mathbf{s}^2\}\mathbf{I}$

$\mathbf{T}_3 = \boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}$

$\mathbf{T}_4 = \boldsymbol{\omega}^2 - \frac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I}$

Vanish in 2d

Cubic: $\mathbf{T}_5 = \boldsymbol{\omega}^2\mathbf{s} + \mathbf{s}\boldsymbol{\omega}^2 - \{\boldsymbol{\omega}^2\}\mathbf{s} - \frac{2}{3}\{\boldsymbol{\omega}\mathbf{s}\boldsymbol{\omega}\}\mathbf{I}$

$\mathbf{T}_6 = \boldsymbol{\omega}\mathbf{s}^2 - \mathbf{s}^2\boldsymbol{\omega}$

Quartic: $\mathbf{T}_7 = \boldsymbol{\omega}^2\mathbf{s}^2 + \mathbf{s}^2\boldsymbol{\omega}^2 - \frac{2}{3}\{\mathbf{s}^2\boldsymbol{\omega}^2\}\mathbf{I} - \{\boldsymbol{\omega}^2\}(\mathbf{s}^2 - \frac{1}{3}\{\mathbf{s}^2\}\mathbf{I})$

$\mathbf{T}_8 = \mathbf{s}^2\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}\mathbf{s}^2 - \frac{1}{2}\{\mathbf{s}^2\}(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega})$

$\mathbf{T}_9 = \boldsymbol{\omega}\mathbf{s}\boldsymbol{\omega}^2 - \boldsymbol{\omega}^2\mathbf{s}\boldsymbol{\omega} - \frac{1}{2}\{\boldsymbol{\omega}^2\}(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega})$

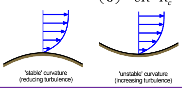
Quintic: $\mathbf{T}_{10} = \boldsymbol{\omega}\mathbf{s}^2\boldsymbol{\omega}^2 - \boldsymbol{\omega}^2\mathbf{s}^2\boldsymbol{\omega}$

Cubic Eddy-Viscosity Model

$$\mathbf{a} = -2C_\mu f_\mu \mathbf{s}$$
$$+\beta_1(\mathbf{s}^2 - \tfrac{1}{3}\{\mathbf{s}^2\}\mathbf{I}) + \beta_2(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}) + \beta_3(\boldsymbol{\omega}^2 - \tfrac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I})$$
$$-\gamma_1\{\mathbf{s}^2\}\mathbf{s} - \gamma_2\{\boldsymbol{\omega}^2\}\mathbf{s} - \gamma_3(\boldsymbol{\omega}^2\mathbf{s} + \mathbf{s}\boldsymbol{\omega}^2 - \{\boldsymbol{\omega}^2\}\mathbf{s} - \tfrac{2}{3}\{\boldsymbol{\omega}\mathbf{s}\boldsymbol{\omega}\}\mathbf{I}) - \gamma_4(\boldsymbol{\omega}\mathbf{s}^2 - \mathbf{s}^2\boldsymbol{\omega})$$

cubic terms ↔ curvature

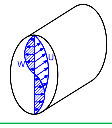
$$\{\mathbf{s}^2\} + \{\boldsymbol{\omega}^2\} \equiv 2(s_{12}^2 - \omega_{12}^2) = -2\left(\frac{k}{\varepsilon}\right)^2 \frac{\partial U_s}{\partial R}\frac{U_s}{R_c}$$



quadratic terms ↔ anisotropy

$$\frac{\overline{u^2}}{k} = \frac{2}{3} + (\beta_1 + 6\beta_2 - \beta_3)\frac{\sigma^2}{12}$$
$$\frac{\overline{v^2}}{k} = \frac{2}{3} + (\beta_1 - 6\beta_2 - \beta_3)\frac{\sigma^2}{12}$$
$$\frac{\overline{w^2}}{k} = \frac{2}{3} - (\beta_1 - \beta_3)\frac{\sigma^2}{6}$$
$$\sigma = \frac{k}{\varepsilon}\frac{\partial U}{\partial y}$$

cubic term ↔ swirl



Properties of the Non-Linear Relationship

In **2-d incompressible flow**:

$$\mathbf{s} = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{12} & -s_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega} = \begin{pmatrix} 0 & \omega_{12} & 0 \\ -\omega_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{s}^2 = (s_{11}^2 + s_{12}^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\omega}^2 = -\omega_{12}^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega} = 2\omega_{12}s_{12}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 2\omega_{12}s_{11}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Properties of the Non-Linear Relationship

$$\mathbf{a} = -2C_\mu f_\mu \mathbf{s} + \beta_1(\mathbf{s}^2 - \tfrac{1}{3}\{\mathbf{s}^2\}\mathbf{I}) + \beta_2(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}) + \beta_3(\boldsymbol{\omega}^2 - \tfrac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I}) - \gamma_1\{\mathbf{s}^2\}\mathbf{s} - \gamma_2\{\boldsymbol{\omega}^2\}\mathbf{s} - \gamma_3(\boldsymbol{\omega}^2\mathbf{s} + \mathbf{s}\boldsymbol{\omega}^2 - \{\boldsymbol{\omega}^2\}\mathbf{s} - \tfrac{2}{3}\{\boldsymbol{\omega}\mathbf{s}\boldsymbol{\omega}\}\mathbf{I}) - \gamma_4(\boldsymbol{\omega}\mathbf{s}^2 - \mathbf{s}^2\boldsymbol{\omega})$$

1. In **2-d incompressible** flow:

$$\mathbf{s}^2 = (s_{11}^2 + s_{12}^2)\mathbf{I}_2 = \tfrac{1}{2}\{\mathbf{s}^2\}\mathbf{I}_2$$
$$\boldsymbol{\omega}^2 = -\omega_{12}^2\mathbf{I}_2 = \tfrac{1}{2}\{\boldsymbol{\omega}^2\}\mathbf{I}_2$$

$$\mathbf{s}^2 = (s_{11}^2 + s_{12}^2)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \boldsymbol{\omega}^2 = -\omega_{12}^2\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

2. (a) In **any incompressible** flow:

$$\frac{P^{(k)}}{\varepsilon} = -a_{ij}s_{ij} = -\{\mathbf{as}\}$$

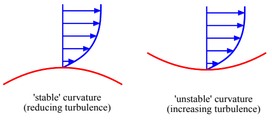
(b) In **2-d incompressible** flow, the quadratic terms do not contribute to the production of turbulent kinetic energy

3. In **2-d incompressible** flow, the γ_3- and γ_4-related cubic terms vanish

4. In simple shear the quadratic terms yield **anisotropy**

$$\frac{\overline{u^2}}{k} = \frac{2}{3} + (\beta_1 + 6\beta_2 - \beta_3)\frac{\sigma^2}{12}$$
$$\frac{\overline{v^2}}{k} = \frac{2}{3} + (\beta_1 - 6\beta_2 - \beta_3)\frac{\sigma^2}{12}$$
$$\frac{\overline{w^2}}{k} = \frac{2}{3} - (\beta_1 - \beta_3)\frac{\sigma^2}{6}$$

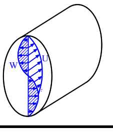
5. The γ_1 and γ_2 terms yield sensitivity to **curvature**



$$\frac{\partial U}{\partial y} = \frac{\partial U_s}{\partial R}, \quad \frac{\partial V}{\partial x} = -\frac{U_s}{R_c}$$

$$\{\mathbf{s}^2\} + \{\boldsymbol{\omega}^2\} \equiv -2\left(\frac{k}{\varepsilon}\right)^2 \frac{\partial U_s}{\partial R}\frac{U_s}{R_c}$$

6. In 3-d flows, the γ_4 term evokes sensitivity to **swirl**



Reynolds-Stress Transport Equations

Fluctuating momentum equation: $\dfrac{\partial u_i}{\partial t} = \cdots$

Form $\quad u_j \times \dfrac{\partial u_i}{\partial t} + u_i \times \dfrac{\partial u_j}{\partial t} = \cdots \quad$ and average

Reynolds-Stress Transport Equations

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + U_k \frac{\partial}{\partial x_k}(\overline{u_i u_j}) = \frac{\partial}{\partial x_k}[\nu \frac{\partial}{\partial x_k}(\overline{u_i u_j}) - \frac{1}{\rho}\overline{p(u_i \delta_{jk} + u_j \delta_{ik})} - \overline{u_i u_j u_k}]$$

$$- (\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k})$$

$$+ \overline{u_i f_j + u_j f_i}$$

$$+ \overline{\frac{p}{\rho}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})}$$

$$- 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$$

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}(\overline{u_i u_j}) = \frac{\partial d_{ijk}}{\partial x_k} + \rho(P_{ij} + F_{ij} + \Phi_{ij} - \varepsilon_{ij})$$

Differential Stress Models

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}(\overline{u_i u_j}) = \frac{\partial d_{ijk}}{\partial x_k} + \rho(P_{ij} + F_{ij} + \Phi_{ij} - \varepsilon_{ij})$$

RATE OF CHANGE & ADVECTION
Exact

DIFFUSION
Modelled

$$d_{ijk} = (\mu\delta_{kl} + C_s \frac{\rho k \overline{u_k u_l}}{\varepsilon})\frac{\partial}{\partial x_l}(\overline{u_i u_j})$$

PRODUCTION by mean flow
Exact

PRODUCTION by body forces
Exact

PRESSURE-STRAIN
Modelled

$$\Phi_{ij} = \Phi_{ij}^{(1)} + \Phi_{ij}^{(2)} + \Phi_{ij}^{(w)}$$

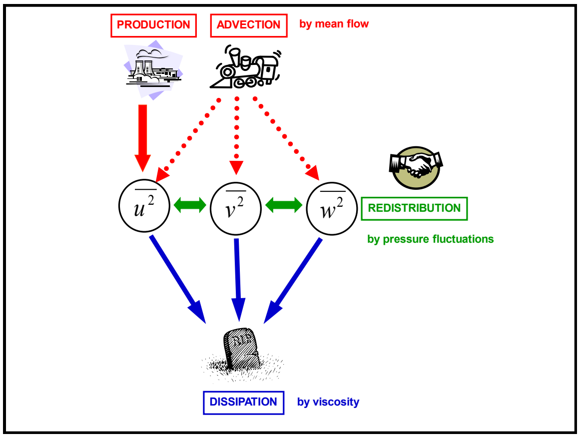
$$\Phi_{11}^{(1)} = -C_1 \frac{\varepsilon}{k}(\overline{u^2} - \tfrac{2}{3}k), \qquad \Phi_{12}^{(1)} = -C_1 \frac{\varepsilon}{k}\overline{uv}$$

$$\Phi_{11}^{(2)} = -C_2 (P_{11} - \tfrac{2}{3}P^{(k)}), \qquad \Phi_{12}^{(2)} = -C_2 P_{12}$$

DISSIPATION by viscosity
Modelled

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \tfrac{2}{3}\varepsilon$$

$$\varepsilon_{23} = \varepsilon_{31} = \varepsilon_{12} = 0$$



Differential Stress Models
Assessment

For

- Good turbulence physics
- Advection and production terms are exact

Against

- Significant modelling required
- Computationally demanding
- Numerical instability

Differential Stress Models
Classic References

- Basic DSM (Launder, Reece and Rodi, 1975; Gibson and Launder, 1978)
- Speziale, Sarkar and Gatski (1991):
  - non-linear Φ_{ij}; no wall reflection
- Craft (1996):
  - low-Re; wall-geometry-independent
- Jakirlić and Hanjalić (1995):
  - low-Re; anisotropic dissipation
- Wilcox (1988):
  - low-Re; ω-based

Part 1. Models

Part 2. Implementation

Components of a Turbulence Model

1. A means of **specifying turbulent stresses**:
   - constitutive relation (eddy-viscosity models)

     or
   - transport equations for stresses (differential stress models)
2. Additional **scalar-transport equations**

Considerations for the Mean-Flow Equations

- The turbulent flux is only partly diffusive:

$$-\rho\overline{uv} = \boxed{\mu_t(\frac{\partial U}{\partial y}} + \boxed{\frac{\partial V}{\partial x}) + (non-linear\ \ terms)}$$

diffusive part **non-diffusive part**

- **Effective viscosities** can be used to stabilise differential stress models

Considerations for the Turbulence Equations

- Turbulence equations are usually source-dominated
- Some variables (e.g. k, ε) must be ≥ 0:
  - bounded advection scheme
  - special treatment of the source term

Source-Term Discretisation

$$a_P\phi_P - \sum_F a_F\phi_F = b_P + s_P\phi_P$$

Stability $\longleftrightarrow$ $s_P \le 0$

Positive ϕ $\longleftrightarrow$ $b_P \ge 0$

If $b_P < 0$ then write as:

$$b_P + s_P\phi_P \to (\frac{b_P}{\phi_P} + s_P)\phi_P$$

i.e.

$$s_P \to s_P + \frac{b_P}{\phi_P^*}$$

$$b_P \to 0$$

Wall Boundary Conditions

Near walls:

- No-slip condition applies
- Large flow gradients
- Preferential damping of wall-normal fluctuations
- Viscous and turbulent stresses comparable

Use either:

- Fine grids and **low-Re turbulence models**
- Coarser grids and **wall functions**

Low-Re Turbulence Models

- Resolve flow right to the boundary:

$$y^+ \equiv \frac{u_\tau y}{\nu} \le 1, \qquad u_\tau \equiv \sqrt{\tau_w/\rho}$$

- Include effects of molecular viscosity:

$$\nu_t = C_\mu f_\mu \frac{k^2}{\tilde{\varepsilon}} \qquad C_{\varepsilon 1} \to C_{\varepsilon 1} f_1 \qquad C_{\varepsilon 2} \to C_{\varepsilon 2} f_2$$

$$f_\mu,\ f_1,\ f_2 \quad \text{are functions of} \quad \frac{u_\tau y}{\nu},\ \frac{k^{1/2} y}{\nu} \text{ or } \frac{k^2}{\nu\varepsilon}$$

- Try to ensure correct asymptotic behaviour as $y \to 0$:

$$k \propto y^2, \qquad \varepsilon \sim \frac{2\nu k}{y^2} \sim \text{constant}, \qquad \nu_t \propto y^3 \qquad (y \to 0)$$

Example

(a) By expanding the fluctuating velocities in the form

$$
\begin{aligned}
u &= a_1 + b_1 y + c_1 y^2 + \cdots \\
v &= a_2 + b_2 y + c_2 y^2 + \cdots \\
w &= a_3 + b_3 y + c_3 y^2 + \cdots
\end{aligned}
$$

show that

$$\overline{u^2} = \overline{b_1^2} y^2 + \cdots$$

and derive similar expressions for

$$\overline{v^2}, \quad \overline{w^2}, \quad \overline{uv}, \quad k, \quad \nu_t$$

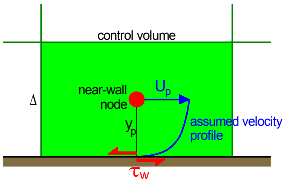
(b) Use the turbulent kinetic energy equation and the near-wall behaviour of *k* from above to show that the near-wall behaviour of ε is

$$\varepsilon \sim \frac{2\nu k}{y^2} \sim constant \qquad (y \to 0)$$

Wall Functions (High-Re Approach)

- Bridge (don't resolve) the viscosity-affected region, using theoretical boundary-layer profiles
- OK in equilibrium turbulence; dodgy near separation/reattachment
- Optimal near-wall spacing: $30 < y^+ < 150$

Wall Functions - Requirements



Variable	Required from wall function
U, V, W	Wall shear stress
$k, \overline{u_i u_j}$	Cell-averaged production and dissipation
ε	Value at the near-wall node

Wall Functions – Assumed Profiles

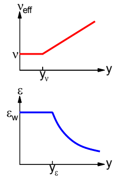
Basis:

$$\tau_w = \rho \nu_{eff} \frac{\partial U}{\partial y}$$

$$\nu_{eff} = \nu + \nu_t = \nu + \max\{0, \kappa u_0 (y - y_\nu)\}$$

$$\varepsilon = \begin{cases} \varepsilon_w & (y \le y_\varepsilon) \\ \dfrac{u_0^3}{\kappa (y - y_d)} & (y > y_\varepsilon) \end{cases}$$

$$u_0 = C_\mu^{1/4} k_P^{1/2}$$



Generates a mean-velocity profile ...

$$\frac{U}{u_0} = \frac{\tau_w}{\rho u_0^2} \times \begin{cases} y^+, & y^+ \le y_\nu^+ \\ y_\nu^+ + \dfrac{1}{\kappa} \ln\{1 + \kappa (y^+ - y_\nu^+)\}, & y^+ \ge y_\nu^+ \end{cases}$$

$$y^+ \equiv \frac{y u_0}{\nu}$$

... which is inverted for the wall stress:

$$\tau_w = \frac{\rho \nu U_P}{y_P} \times \begin{cases} 1, & y_P^+ \le y_\nu^+ \\ \dfrac{y_P^+}{y_\nu^+ + \frac{1}{\kappa} \ln\{1 + \kappa (y^+ - y_\nu^+)\}}, & y_P^+ \ge y_\nu^+ \end{cases}$$

Wall Functions - Implementation

Wall shear stress applied via an **effective wall viscosity**:

$$\tau_w = \rho \nu_{eff,wall} \frac{U_p}{y_p}$$

$$\nu_{eff,wall} = \nu \times \begin{cases} 1, & y_P^+ \le y_\nu^+ \\ \dfrac{y_P^+}{y_\nu^+ + \frac{1}{\kappa} \ln\{1 + \kappa (y_P^+ - y_\nu^+)\}}, & y_P^+ \ge y_\nu^+ \end{cases}$$

Cell-averaged production and dissipation:

$$P_{av}^{(k)} \equiv \frac{1}{\Delta} \int_0^\Delta P^{(k)} \; \mathrm{d}y \quad = \frac{(\tau_w / \rho)^2}{\kappa u_0 \Delta} \left\{ \ln[1 + \kappa (\Delta^+ - y_\nu^+)] - \frac{\kappa (\Delta^+ - y_\nu^+)}{1 + \kappa (\Delta^+ - y_\nu^+)} \right\}$$

$$\varepsilon_{av} = \frac{1}{\Delta} \int_0^\Delta \varepsilon \; \mathrm{d}y \quad = \frac{u_0^3}{\kappa \Delta} \left[\ln\left(\frac{\Delta - y_d}{y_\varepsilon - y_d}\right) + \frac{y_\varepsilon}{y_\varepsilon - y_d} \right]$$

Effective Viscosity To Stabilise DSMs

Differential stress models:

no turbulent viscosity → numerical instability

Effective viscosity approach – add and subtract a gradient term:

$$\overline{u_\alpha u_\beta} = \overline{\left(\overline{u_\alpha u_\beta} + \nu_{\alpha\beta} \frac{\partial U_\alpha}{\partial x_\beta}\right)} - \overline{\nu_{\alpha\beta}} \frac{\partial U_\alpha}{\partial x_\beta}$$

Simplest: $\nu_{\alpha\beta} = \nu_t = C_\mu \dfrac{k^2}{\varepsilon}$

Better:

$$\overline{u^2} = -\nu_{11} \frac{\partial U}{\partial x} + \ldots \qquad \nu_{11} = 2 \left(\frac{1 - \frac{2}{3} C_2}{C_1} \right) \frac{k \overline{u^2}}{\varepsilon}$$

$$\overline{uv} = -\nu_{12} \frac{\partial U}{\partial y} + \ldots \qquad \nu_{12} = \left(\frac{1 - C_2}{C_1} \right) \frac{k \overline{v^2}}{\varepsilon}$$