Computational Fluid Dynamics: The Finite-Volume Method

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1. Introduction
What is ... Computational Fluid Dynamics?

The use of computers and numerical methods to solve problems involving fluid flow
Aerodynamics
Wind Loading
Turbine Technology
Vortex Shedding
Dispersion of Pollution
Ventilation
Particle-Laden Plumes (Sea Outfalls)

(ex302)
\( \rho_a = 1040 \text{ kg/m}^3 \)

(ex322)
\( \rho_a = 1020 \text{ kg/m}^3 \)

(ex336)
\( \rho_a = 1020 \text{ kg/m}^3 \)
\( U_a = 0.026 \text{ m/s} \)

Free surface

Solid bed
Sediment Scour

River bend

Bridge pier
Discretisation

Field variables:

Equations:

$$\frac{df}{dx} \approx \frac{\Delta f}{\Delta x} = \frac{f_2 - f_1}{x_2 - x_1}$$
Basic Principles of CFD

1. **Discretise space:**
   replace field variables \((\rho, u, v, w, p, \ldots)\) by values at a finite number of nodes

2. **Discretise equations:**
   - continuum equations \(\rightarrow\) algebraic equations

3. **Solve:**
   large system of simultaneous equations
Stages of a CFD Analysis

- **Pre-processing:**
  - problem formulation (geometry, equations, boundary conditions)
  - computational mesh

- **Solving:**
  - discretisation
  - solving

- **Post-processing:**
  - analysis
  - visualisation
Fluid-Flow Equations

- **Mass:** change of mass = 0
- **Momentum:** change of momentum = force × time
- **Energy:** change of energy = work + heat
- **(Other constituents)**

In fluid mechanics, these are normally expressed in **rate** form
Form of Equations

- **Integral** (control-volume)
- **Differential**
Integral (Control-Volume) Approach

Consider the budget of any transported physical quantity in any control volume

\[
\frac{\text{CHANGE}}{\text{time}} + \frac{\text{OUT} - \text{IN}}{\text{time}} = \frac{\text{CREATED}}{\text{time}}
\]

\[
\left(\text{TIME DERIVATIVE of amount in } V\right) + \left(\text{NET FLUX through boundary of } V\right) = \left(\text{SOURCE inside } V\right)
\]

\[
\left(\text{TIME DERIVATIVE of amount in } V\right) + \left(\text{ADVECTION + DIFFUSION through boundary of } V\right) = \left(\text{SOURCE inside } V\right)
\]

→ **Finite-volume** method for CFD
Differential Equations For Fluid Flow

• Derived by considering the rate of change at a point; i.e. using infinitesimal control volumes

• Discretisation gives a finite-difference method for CFD

• Several types:
  – fixed-point (“Eulerian”): conservative
  – moving with the flow (“Lagrangian”): non-conservative
  – derived variables; e.g. potential flow
Main Methods for CFD

- **Finite-difference:**
  - discretise differential equations
  
  \[
  0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}
  \]

- **Finite-volume:**
  - discretise control-volume equations
  
  \[
  0 = \text{net mass outflow} = (\rho u A)_e - (\rho u A)_w + (\rho v A)_n - (\rho v A)_s
  \]

- **Finite-element:**
  - represent solution as a weighted sum of basis functions
  
  \[
  u(\mathbf{x}) = \sum u_\alpha S_\alpha(\mathbf{x})
  \]
Advantages of the Finite-Volume Method in CFD

• Rigorously enforces conservation

• **Flexible** in terms of:
  - geometry
  - fluid phenomena

• Directly relatable to physical quantities
Examples
Example Q1

Water (density 1000 kg m\(^{-3}\)) flows at 2 m s\(^{-1}\) through a circular pipe of diameter 10 cm. What is the mass flux \(C\) across the surfaces \(S_1\) and \(S_2\)?
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$S_1$: mass flux ($C$) = $\rho u A$

$$= 1000 \times 2 \times \frac{\pi \times 0.1^2}{4}$$

$$= 15.7 \text{ kg s}^{-1}$$

$S_2$: the same!  

$C = \rho (u \cos \theta)A$

$C = \rho u (A \cos \theta)$

In general: $C = \rho u \cdot A$
A water jet strikes normal to a fixed plate as shown. Compute the force $F$ required to hold the plate fixed.
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\[ F = 8 \text{ m/s} \]
\[ D = 10 \text{ cm} \]

**force (on fluid) = (momentum flux)_{out} - (momentum flux)_{in}**

\[ -F = 0 - (\rho U A)U \]

\[ F = \rho U^2 A \]

\[ = 1000 \times 8^2 \times \frac{\pi \times 0.1^2}{4} \]

\[ = 503 \text{ N} \]
An explosion releases 2 kg of a toxic gas into a room of dimensions 30 m × 8 m × 5 m. Assuming the room air to be well-mixed and to be vented at a speed of 0.5 m s\(^{-1}\) through an aperture of area 6 m\(^2\), calculate:

(a) the initial concentration of gas in ppm by mass;

(b) the time taken to reach a safe concentration of 1 ppm.

(Take the density of air as 1.2 kg m\(^{-3}\).)
An explosion releases 2 kg of a toxic gas into a room of dimensions $30 \, \text{m} \times 8 \, \text{m} \times 5 \, \text{m}$. Assuming the room air to be well-mixed and to be vented at a speed of $0.5 \, \text{m} \, \text{s}^{-1}$ through an aperture of area $6 \, \text{m}^2$, calculate:

(a) the initial concentration of gas in ppm by mass;

\[ V = 30 \times 8 \times 5 = 1200 \, \text{m}^3 \]

mass of fluid $\times$ concentration $= \text{mass of toxin}$

\[ (\rho V)\phi_0 = 2 \, \text{kg} \]

\[ \phi_0 = \frac{2}{1.2 \times 1200} = 1.389 \times 10^{-3} \]

\[ 1389 \, \text{ppm} \]
An explosion releases 2 kg of a toxic gas into a room of dimensions 30 m × 8 m × 5 m. Assuming the room air to be well-mixed and to be vented at a speed of 0.5 m s\(^{-1}\) through an aperture of area 6 m\(^2\), calculate:

(b) the time taken to reach a safe concentration of 1 ppm.

\[
\phi_0 = 1389 \text{ ppm}
\]
\[
V = 1200 \text{ m}^3
\]
\[
A = 6 \text{ m}^2
\]
\[
U = 0.5 \text{ m s}^{-1}
\]

\[
\text{Volume V}
\]
\[
\text{Concentration } \phi
\]
\[
\text{area A}
\]
\[
\text{U}
\]

Change in amount of toxin = amount in − amount out

Rate of change of amount of toxin = rate of entering − rate of leaving

\[
\frac{d}{dt}(\rho V \phi) = 0 - (\rho u A) \phi
\]

\[
\frac{d\phi}{dt} = -\left(\frac{u A}{V}\right) \phi, \quad \phi = \phi_0 \text{ at } t = 0
\]

\[
\frac{d\phi}{dt} = -\lambda \phi \quad \lambda = \frac{u A}{V} = 0.0025 \text{ s}^{-1}
\]

\[
\phi = \phi_0 e^{-\lambda t}
\]

\[
t = \frac{1}{\lambda} \ln \frac{\phi_0}{\phi} = \frac{1}{0.0025} \ln(1389) = 2895 \text{ s}
\]

\[\approx 48 \text{ min}\]