## 2. FLUID-FLOW EQUATIONS SPRING 2024

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### 2.1 Introduction

Fluid dynamics is governed by conservation equations for:

- mass;
- momentum;
- energy;
- (for a non-homogenous fluid) any other constituents.

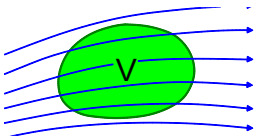
Equations for these can be expressed mathematically as:

- *integral* (*control-volume*) equations;
- *differential* equations.

This course focuses on the control-volume approach (the basis of the finite-volume method) because it relates naturally to physical quantities, is intrinsically conservative and is easier to apply in modern, unstructured-mesh CFD with complex geometries. However, the equivalent differential equations are easier to write down, manipulate and, in a few cases, solve analytically.

The aims of this section are:

(i) to derive differential equations for fluid flow;
(ii) to demonstrate equivalence of integral and differential forms;
(iii) to show that, although there are many different physical quantities, all satisfy a single generic equation: the *scalar-transport* or *advection-diffusion* equation:




$$\begin{pmatrix}\text{TIME DERIVATIVE}\\ \text{of amount in } V\end{pmatrix} + \begin{pmatrix}\text{ADVECTIVE + DIFFUSIVE FLUX}\\ \text{through boundary of } V\end{pmatrix} = \begin{pmatrix}\text{SOURCE}\\ \text{in } V\end{pmatrix} \qquad (1)$$

The finite-volume method is a natural discretisation of this.

## 2.2 Conservative Differential Equations

### 2.2.1 Mass Conservation (Continuity)

Mass is neither created nor destroyed, so:

*rate of change of mass in cell = net inward mass flux*

With the more conventional flux direction (positive outward):

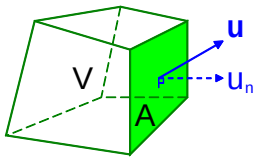
time derivative of mass in cell + net outward mass flux = 0

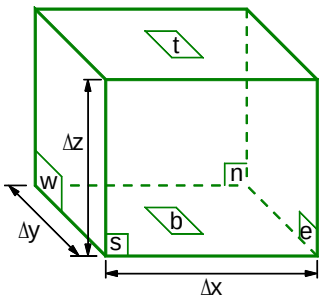
$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{mass}) \qquad + \sum_{\text{faces}} (\text{mass flux}) \qquad = 0 \tag{2}$$

For a cell volume $V$ and a typical face area $A$:

mass of fluid in the cell: $\rho V$

mass flux through one face: $C = \rho \mathbf{u} \bullet \mathbf{A} = \rho u_n A$




A differential equation for mass conservation can be derived by considering the small cartesian control volume shown left

If density and velocity are averages over cell volume or cell face, respectively:

$$\underbrace{\frac{\mathrm{d}(\rho V)}{\mathrm{d}t}}_{\text{time derivative of mass}} + \underbrace{(\rho u A)_e - (\rho u A)_w + (\rho v A)_n - (\rho v A)_s + (\rho w A)_t - (\rho w A)_b}_{\text{net outward mass flux}} = 0$$

Writing volume $V = \Delta x \Delta y \Delta z$ and areas $A_w = A_e = \Delta y \Delta z$ etc:

$$\frac{\mathrm{d}(\rho \Delta x \Delta y \Delta z)}{\mathrm{d}t} + [(\rho u)_e - (\rho u)_w]\Delta y \Delta z + [(\rho v)_n - (\rho v)_s]\Delta z \Delta x + [(\rho w)_t - (\rho w)_b]\Delta x \Delta y = 0$$

Dividing by the volume, $\Delta x \Delta y \Delta z$:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{(\rho u)_e - (\rho u)_w}{\Delta x} + \frac{(\rho v)_n - (\rho v)_s}{\Delta y} + \frac{(\rho w)_t - (\rho w)_b}{\Delta z} = 0$$

i.e. $$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\Delta(\rho u)}{\Delta x} + \frac{\Delta(\rho v)}{\Delta y} + \frac{\Delta(\rho w)}{\Delta z} = 0$$

Taking the limit as $\Delta x, \Delta y, \Delta z \to 0$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{3}$$

This analysis is analogous to the finite-volume procedure, but there the control volume does not shrink to a point (*finite*-volume, not *infinitesimal*-volume) and cells can be any shape.

(*** *Advanced / optional* ***)

For an arbitrary volume $V$ with closed surface $\partial V$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_V \rho\,\mathrm{d}V}_{\text{mass in cell}} + \underbrace{\oint_{\partial V} \rho\mathbf{u}\bullet\mathrm{d}\mathbf{A}}_{\text{net mass flux through boundary}} = 0 \tag{4}$$

For a fixed control volume, take $\mathrm{d}/\mathrm{d}t$ under the integral sign and apply the divergence theorem to turn the surface integral into a volume integral:

$$\int_V \left\{\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\mathbf{u})\right\}\mathrm{d}V = 0$$

Since $V$ is arbitrary, the integrand must be identically zero. Hence,

$$\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\mathbf{u}) = 0 \tag{5}$$

<u>Incompressible Flow</u>

For incompressible flow, volume as well as mass is conserved, so that:

$$\underbrace{(uA)_e - (uA)_w \quad + \quad (vA)_n - (vA)_s \quad + \quad (wA)_t - (wA)_b}_{\text{net outward VOLUME flux}} = 0$$

Substituting for face areas, dividing by volume and proceeding to the limit as above produces

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6}$$

This is usually taken as the continuity equation in incompressible flow.

### 2.2.2 Momentum

*Newton's Second Law*: *rate of change of momentum = force*

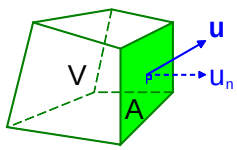
time derivative of momentum in cell + net outward momentum flux = force

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{mass}\times\mathbf{u}) + \sum_{\text{faces}}(\text{mass flux}\times\mathbf{u}) = \mathbf{F} \tag{7}$$

For a cell volume $V$ and a typical face area $A$:

momentum of fluid in the cell = mass × **u** = $(\rho V)\mathbf{u}$

momentum flux through a face = mass flux × **u** = $(\rho u_n A)\mathbf{u}$




Momentum and force are vectors, giving (in principle) 3 equations.

Fluid Forces

There are two main types:

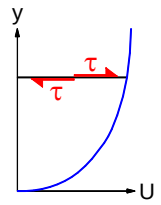
- *surface forces* (proportional to area; act on control-volume faces)
- *body forces* (proportional to volume)

(i) *Surface forces* are usually expressed in terms of *stress*:

$$\text{stress} = \frac{\text{force}}{\text{area}} \qquad \text{or} \qquad \text{force} = \text{stress} \times \text{area}$$

The main surface forces arise from:

- *pressure* $p$: acts normal to a surface;
- *viscous stresses* $\tau$: frictional forces arising from relative motion;
- *reaction forces* from solid boundaries.

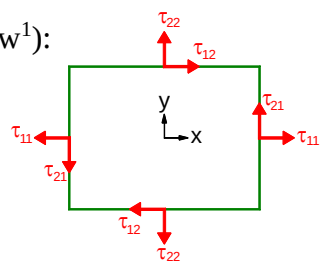



For a simple shear flow there is only one non-zero stress component:

$$\tau \equiv \tau_{12} = \mu \frac{\partial u}{\partial y}$$

but, in general, $\tau_{ij}$ is a symmetric tensor with components (in incompressible flow[1]):

$$\tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

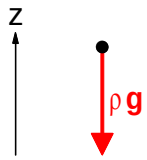



(ii) *Body forces* are often expressed as forces per unit volume, or *force densities*.

The main body forces in this course are:

- *gravity*: the force per unit volume is

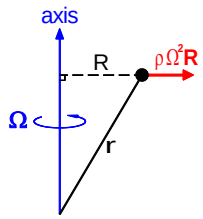
$$\rho \mathbf{g} = -\rho g \mathbf{e}_z$$



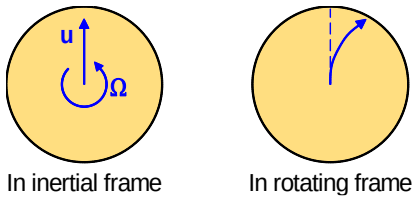

(For constant-density fluids, pressure and weight can be combined as a piezometric pressure $p^* = p + \rho g z$; gravity then no longer appears explicitly in the flow equations.)

- *centrifugal* and *Coriolis* forces (*apparent* forces in a *rotating reference frame*):

*centrifugal force*:
$\rho \Omega^2 \mathbf{R}$

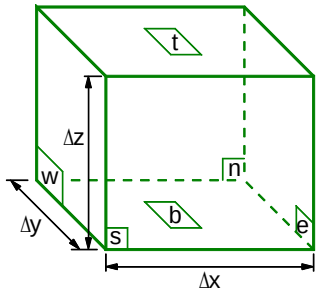



*Coriolis force*:[2]
$-2\rho \mathbf{\Omega} \wedge \mathbf{u}$




[1] There is a slightly extended expression in compressible flow; see the recommended textbooks.
[2] The symbol $\wedge$ here means vector product; $\mathbf{\Omega}$ is the angular velocity vector; its direction is the rotation axis.

Because the centrifugal force can be written as the gradient of some quantity (in this case, $\frac{1}{2}\rho\Omega^2 R^2$) it can, like gravity, be absorbed into a modified pressure; see the Examples.




<u>Differential Equation For Momentum</u>

Consider a fixed cartesian control volume with sides $\Delta x$, $\Delta y$, $\Delta z$.

Follow the same process as for mass conservation.

For the $x$-component of momentum:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}(\rho V u)}_{\substack{\text{time derivative}\\ \text{of momentum}}} + \underbrace{(\rho u A)_e u_e - (\rho u A)_w u_w + (\rho v A)_n u_n - (\rho v A)_s u_s + (\rho w A)_t u_t - (\rho w A)_b u_b}_{\text{net outward momentum flux}}$$

$$= \underbrace{(p_w A_w - p_e A_e)}_{\text{pressure force in } x \text{ direction}} + \text{viscous and other forces}$$

Substituting cell dimensions:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho \Delta x \Delta y \Delta z\, u) + [(\rho u u)_e - (\rho u u)_w]\Delta y \Delta z + [(\rho v u)_n - (\rho v u)_s]\Delta z \Delta x + [(\rho w u)_t - (\rho w u)_b]\Delta x \Delta y$$
$$= (p_w - p_e)\Delta y \Delta z + \text{viscous and other forces}$$

Dividing by volume $\Delta x \Delta y \Delta z$ (and transposing $p_e$ and $p_w$):

$$\frac{\mathrm{d}(\rho u)}{\mathrm{d}t} + \frac{(\rho u u)_e - (\rho u u)_w}{\Delta x} + \frac{(\rho v u)_n - (\rho v u)_s}{\Delta y} + \frac{(\rho w u)_t - (\rho w u)_b}{\Delta z}$$
$$= -\frac{(p_e - p_w)}{\Delta x} + \text{viscous and other forces}$$

In the limit as $\Delta x, \Delta y, \Delta z \to 0$:

$$\boxed{\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} + \frac{\partial(\rho w u)}{\partial z} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u + \text{other forces} \qquad (8)}$$

*Notes*.

(1) The viscous term is given without proof here (but see the optional notes below). $\nabla^2$ is the *Laplacian* operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

(2) The pressure force per unit volume in any direction is minus the pressure gradient in that direction.

(3) The $y$ and $z$-momentum equations can be obtained by inspection / pattern-matching.

(*** *Advanced / optional* ***)

With surface forces determined by stress tensor $\sigma_{ij}$ and body forces determined by force density $f_i$, the control-volume equation for the $i$ component of momentum may be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_V \rho u_i \,\mathrm{d}V}_{\text{momentum in cell}} + \underbrace{\oint_{\partial V} \rho u_i u_j \,\mathrm{d}A_j}_{\text{net momentum flux}} = \underbrace{\oint_{\partial V} \sigma_{ij} \,\mathrm{d}A_j}_{\text{surface forces}} + \underbrace{\int_V f_i \,\mathrm{d}V}_{\text{body forces}} \tag{9}$$

For fixed $V$, take $\mathrm{d}/\mathrm{d}t$ inside integrals and apply the divergence theorem to surface integrals:

$$\int_V \left\{ \frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} - \frac{\partial \sigma_{ij}}{\partial x_j} - f_i \right\} \mathrm{d}V = 0$$

As $V$ is arbitrary, the integrand vanishes identically. Hence, for arbitrary forces:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \tag{10}$$

The stress tensor has pressure and viscous parts:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{11}$$

$$\Rightarrow \quad \frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + f_i \tag{12}$$

For a Newtonian fluid, the viscous stress tensor (including compressible part) is given by

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)$$

If the fluid is incompressible and viscosity is uniform then the viscous term simplifies:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i + f_i$$

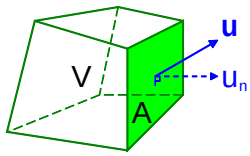
### 2.2.3 General Scalar

A similar equation may be derived for any physical quantity that is advected and diffused in a fluid flow. Examples include salt, sediment and chemical pollutants. For each such quantity an equation is solved for the *concentration* (amount per unit mass of fluid) $\phi$.

Diffusion causes net transport from regions of high concentration to regions of low concentration. For many scalars this rate of transport is proportional to area and to concentration gradient and may be quantified by *Fick's diffusion law*:

$$\text{rate of diffusion} = -\text{diffusivity} \times \text{gradient} \times \text{area}$$
$$= -\Gamma \frac{\partial \phi}{\partial n} A$$

This is often referred to as *gradient diffusion*. An example is heat conduction.




For an arbitrary control volume:

| | | |
|---|---|---|
| amount in cell: | $\rho V \phi$ | (mass × concentration) |
| advective flux: | $(\rho u_n A)\phi$ | (mass flux × concentration) |
| diffusive flux: | $-\Gamma \frac{\partial \phi}{\partial n} A$ | (– diffusivity × gradient × area) |
| source: | $S = sV$ | (source density × volume) |

Balancing the time derivative of the amount in the cell, the net flux through the boundary and rate of production yields the general *scalar-transport* (or *advection-diffusion*) *equation*:

time derivative + net outward flux = source

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{mass} \times \phi) + \sum_{\text{faces}} \left( \text{mass flux} \times \phi - \Gamma \frac{\partial \phi}{\partial n} A \right) = S \qquad (13)$$

(Conservative) differential equation:

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial}{\partial x}\left(\rho u \phi - \Gamma \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho v \phi - \Gamma \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho w \phi - \Gamma \frac{\partial \phi}{\partial z}\right) = s \qquad (14)$$

(**** Advanced / optional ****)
The integral equation may be expressed more mathematically as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\phi \, \mathrm{d}V + \oint_{\partial V} (\rho \mathbf{u} \phi - \Gamma \nabla \phi) \bullet \mathrm{d}\mathbf{A} = \int_V s \, \mathrm{d}V \qquad (15)$$

For a fixed control volume, taking the time derivative under the integral sign and using the divergence theorem gives a corresponding conservative differential equation:

$$\frac{\partial(\rho\phi)}{\partial t} + \nabla \bullet (\rho \mathbf{u} \phi - \Gamma \nabla \phi) = s \qquad (16)$$

### 2.2.4 Momentum Components as Transported Scalars

In the momentum equation, if the viscous force $\tau A = \mu(\partial u/\partial n)A$ is transferred to the LHS it looks like a diffusive flux. For example, for the $x$-component:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{mass} \times u) + \sum_{\text{faces}} \left(\text{mass flux} \times u - \mu \frac{\partial u}{\partial n} A\right) = \text{other forces}$$

Compare this with the generic scalar-transport equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{mass} \times \phi) + \sum_{\text{faces}} \left(\text{mass flux} \times \phi - \Gamma \frac{\partial \phi}{\partial n} A\right) = S$$

Each component of momentum satisfies its own scalar-transport equation, with

| | | |
|---|---|---|
| concentration, $\phi$ | $\leftarrow$ | velocity component ($u$, $v$ or $w$) |
| diffusivity, $\Gamma$ | $\leftarrow$ | viscosity $\mu$ |
| source, $S$ | $\leftarrow$ | other forces |

Consequently, *only one generic scalar-transport equation need be considered*.

In Section 5 we shall see, however, that the momentum components differ from *passive* scalars (those not affecting the flow), because:

- equations are nonlinear (mass flux involves the velocity component being solved for);
- equations are coupled (mass flux involves the other velocity components as well);
- the velocity field must also be mass-consistent.

### 2.2.5 Non-Gradient Diffusion

The analysis above assumes that all non-advective flux is simple gradient diffusion:

$$-\Gamma \frac{\partial \phi}{\partial n} A$$

Actually, the real situation is more complex. For example, in the $u$-momentum equation the full expression for the 1-component of viscous stress through the 2-face is

$$\tau_{12} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

The $\partial u/\partial y$ part is gradient diffusion of $u$, but the $\partial v/\partial x$ term is not. In general, non-advective fluxes $F'$ that can't be represented by gradient diffusion are discretised conservatively (i.e. worked out for cell faces, not particular cells), then transferred to the RHS as a source term:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{mass} \times \phi) + \sum_{\text{faces}} \left(\text{mass flux} \times \phi \quad - \mu \frac{\partial \phi}{\partial n} A \quad + F'\right) = S$$

### 2.2.6 Moving Control Volumes

Control-volume equations are also applicable to moving control volumes, provided the normal velocity component in the mass flux is that *relative to the mesh*; i.e.

$$u_n = (\mathbf{u} - \mathbf{u}_{\text{mesh}}) \bullet \mathbf{n}$$

## 2.3 Non-Conservative Differential Equations

*Conservative* differential equations are so-called because they can be integrated directly to give an equivalent *integral* form involving the net change in a flux, with the flux leaving one cell equal to that entering an adjacent cell. To do so, all terms involving derivatives of dependent variables must have differential operators "on the outside". In one dimension:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = g(x) \qquad\qquad f(x_2) - f(x_1) = \int_{x_1}^{x_2} g(x)\ \mathrm{d}x$$

(differential form) (integral form)

$$\text{flux}_{\text{out}} - \text{flux}_{\text{in}} = \text{source}$$

(*** *Advanced / optional* ***)
The three-dimensional version uses partial derivatives and the divergence theorem to change the differentials to surface flux integrals.

As an example of how the *same* equation can appear in conservative and non-conservative forms, consider a simple 1-d example:

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = g(x) \qquad \text{(conservative form – can be integrated directly)}$$

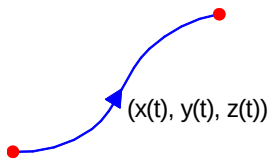
$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \qquad \text{(non-conservative form, obtained by applying the chain rule)}$$

<u>Material Derivatives</u>

The time rate of change of some property in a fluid element moving with the flow is called the *material* (or *substantive*) derivative. It is denoted by $\mathrm{D}\phi/\mathrm{D}t$ and defined below.

Every field variable $\phi$ is a function of both time and position; i.e.

$$\phi = \phi(t, x, y, z)$$




As one follows a path through space, $\phi$ changes with time because:

- $\phi$ changes with time $t$ at each point; and
- $\phi$ changes with position $(x, y, z)$ as one moves along the path.

Thus, the *total* time derivative following an *arbitrary* path $(x(t), y(t), z(t))$ is

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial\phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

The *material derivative* is the time derivative along the particular path *following the flow* ($\mathrm{d}x/\mathrm{d}t = u$, etc.):

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} \equiv \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z} \qquad \text{or} \qquad \frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad (17)$$

In particular, the material derivative of velocity ($\mathrm{D}u/\mathrm{D}t$) is the *acceleration* (Hydraulics 1).

For general scalar $\phi$, the sum of time-dependent and advection terms (total rate of change) is

$$\frac{\partial(\rho\phi)}{\partial t}+\frac{\partial(\rho u\phi)}{\partial x}+\frac{\partial(\rho v\phi)}{\partial y}+\frac{\partial(\rho w\phi)}{\partial z}$$

$$=\left[\frac{\partial\rho}{\partial t}\phi+\rho\frac{\partial\phi}{\partial t}\right]+\left[\frac{\partial(\rho u)}{\partial x}\phi+\rho u\frac{\partial\phi}{\partial x}\right]+\left[\frac{\partial(\rho v)}{\partial y}\phi+\rho v\frac{\partial\phi}{\partial y}\right]+\left[\frac{\partial(\rho w)}{\partial z}\phi+\rho w\frac{\partial\phi}{\partial z}\right]$$

(by the product rule)

$$=\underbrace{\left[\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right]}_{=0\ \text{(by continuity)}}\ \phi\quad+\quad\rho\ \underbrace{\left[\frac{\partial\phi}{\partial t}+u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}+w\frac{\partial\phi}{\partial z}\right]}_{=\mathrm{D}\phi/\mathrm{D}t\ \text{(by definition)}}$$

$$=\rho\frac{\mathrm{D}\phi}{\mathrm{D}t} \tag{18}$$

Using the material derivative, a scalar-transport equation can thus be written in a much more compact, but non-conservative, form. In particular, the momentum equation becomes

$$\underbrace{\rho\frac{\mathrm{D}u}{\mathrm{D}t}}_{\text{mass}\times\text{acceleration}}=-\frac{\partial p}{\partial x}+\mu\nabla^2u\ +\ \text{other forces} \tag{19}$$

This form is much simpler to write and is used both for convenience and to derive theoretical results in special cases (see the Examples). However, in the finite-volume method it is the longer, conservative form which is actually discretised.

(*** *Advanced / optional* ***)

The derivation of (18) above is greatly shortened by use of the summation convention:

$$\frac{\partial(\rho\phi)}{\partial t}+\frac{\partial(\rho u_i\phi)}{\partial x_\iota}=\left[\frac{\partial\rho}{\partial t}\phi+\rho\frac{\partial\phi}{\partial t}\right]+\left[\frac{\partial(\rho u_i)}{\partial x_i}\phi+\rho u_i\frac{\partial\phi}{\partial x_i}\right]$$

$$=\underbrace{\left[\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u_i)}{\partial x_i}\right]}_{=0\ \text{(by continuity)}}\ \phi\quad+\quad\rho\ \underbrace{\left[\frac{\partial\phi}{\partial t}+u_i\frac{\partial\phi}{\partial x_i}\right]}_{=\frac{\mathrm{D}\phi}{\mathrm{D}t}\ \text{(by definition)}}$$

$$=\rho\frac{\mathrm{D}\phi}{\mathrm{D}t}$$

Alternatively, conservative differential equations may derived from *fixed* control volumes (*Eulerian* approach) and their non-conservative counterparts from control volumes moving with the flow (*Lagrangian* approach).

## 2.4 Non-Dimensionalisation

Although it is possible to work entirely in dimensional quantities, there are good theoretical reasons for working in non-dimensional variables. These include the following.

- All *dynamically-similar* problems (same Re, Fr etc.) can be solved with a single computation.
- The number of relevant parameters (and hence the number of graphs needed to report results) is reduced.
- It indicates the relative size of different terms in the governing equations and, in particular, which might conveniently be neglected.
- Computational variables are of a similar order of magnitude (ideally, of order unity), yielding better numerical accuracy.

### 2.4.1 Non-Dimensionalising the Governing Equations

For incompressible flow the governing equations are:

*continuity*: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{20}$$

*momentum*: $$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u \tag{21}$$

(and similar in $y$, $z$ directions)

Adopting reference scales $U_0$, $L_0$ and $\rho_0$ for velocity, length and density, respectively, and derived scales $L_0/U_0$ for time and $\rho_0 U_0^2$ for pressure, each fluid quantity can be written as a product of a dimensional scale and a non-dimensional variable (indicated by $*$):

$$\mathbf{x} = L_0 \mathbf{x}^*, \quad t = \frac{L_0}{U_0} t^*, \quad \mathbf{u} = U_0 \mathbf{u}^*, \quad \rho = \rho_0 \rho^*, \quad p = p_{\mathrm{ref}} + (\rho_0 U_0^2) p^*, \quad \text{etc.}$$

Substituting into mass and momentum equations (20) and (21) yields, after simplification:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0 \tag{22}$$

$$\rho^* \frac{\mathrm{D}u^*}{\mathrm{D}t^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}} \nabla^{*2} u^* \qquad \text{where} \qquad \mathrm{Re} = \frac{\rho_0 U_0 L_0}{\mu} \tag{23}$$

From this, it is seen that the key dimensionless group is the Reynolds number Re. If Re is large then viscous forces would be expected to be negligible in much of the flow.

Having derived the non-dimensional equations it is usual to drop the asterisks and simply declare that you are “working in non-dimensional variables”.

*Note*.

- The objective is that non-dimensional quantities (e.g. $p^*$) should be O(1) in magnitude unity, so the scale for a quantity should reflect its *range* of values, not necessarily its absolute value. In incompressible (but not compressible) flow it is *differences* in pressure that are important, not absolute values. Since flow-induced pressures are usually much smaller than the absolute pressure, one usually works with departures from a constant reference pressure $p_{\text{ref}}$ and use $\rho_0 U_0^2$ as a scale magnitude; Hence, we non-dimensionalise as: $p = p_{\text{ref}} + (\rho_0 U_0^2) p^*$.

- Similarly, in Section 3 when we look at small changes in density due to temperature or salinity that give rise to buoyancy forces we shall use an alternative non-dimensionalisation:

$$\rho = \rho_0 \pm \Delta\rho\, \theta^* \tag{24}$$

with $\Delta\rho$ the overall size of density variation and $\theta^*$ typically varying between 0 and 1.

### 2.4.2 Common Dimensionless Groups

If other fluid forces are included then each introduces another non-dimensional group. For example, gravitational forces lead to a Froude number (Fr) and Coriolis forces to a Rossby number (Ro). Some of the most important dimensionless groups are given below. $U$ and $L$ are representative velocity and length scales, respectively.

| | |
|---|---|
| $\text{Re} \equiv \dfrac{\rho U L}{\mu} \equiv \dfrac{UL}{\nu}$ | Reynolds number (viscous flow; $\mu$ = dynamic viscosity) |
| $\text{Fr} \equiv \dfrac{U}{\sqrt{gL}}$ | Froude number (open-channel flow; $g$ = gravity) |
| $\text{Ma} \equiv \dfrac{U}{c}$ | Mach number (compressible flow; $c$ = speed of sound) |
| $\text{Ro} \equiv \dfrac{U}{\Omega L}$ | Rossby number (rotating flows; $\Omega$ = angular velocity of frame) |
| $\text{We} \equiv \dfrac{\rho U^2 L}{\sigma}$ | Weber number (interfacial flows; $\sigma$ = surface tension) |

*Note*.
For flows with buoyancy forces caused by a change in density, rather than open-channel flows, we sometimes use a *densimetric Froude number* instead; this is defined by

$$\text{Fr} \equiv \frac{U}{\sqrt{(\Delta\rho/\rho) g L}}$$

Here, $g$ is replaced in the usual formula for Froude number by $(\Delta\rho/\rho)g$, sometimes called the *reduced gravity*, $g'$; see Section 3.

## Summary

- Fluid dynamics is governed by conservation equations for mass, momentum, energy (and, for a non-homogeneous fluid, the amount of individual constituents).

- The governing equations can be written in equivalent integral (control-volume) or differential forms.

- The *finite-volume* method is a direct discretisation of the control-volume equations.

- Differential forms of the flow equations may be *conservative* (i.e. can be integrated directly to something of the form "$\text{flux}_{\text{out}} - \text{flux}_{\text{in}} = \text{source}$") or *non-conservative*.

- For any conserved quantity and arbitrary control volume:
  time derivative + net outward flux = source

- There are really just two canonical equations to discretise and solve:
  *mass conservation* (*continuity*):

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{mass}) + \sum_{\text{faces}} (\text{mass flux}) = 0$$

  *scalar-transport* (or *advection-diffusion*) equation:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}(\text{mass} \times \phi)}_{\text{time derivative}} + \sum_{\text{faces}} \Big( \underbrace{\text{mass flux} \times \phi}_{\text{advection}} - \underbrace{\Gamma \frac{\partial \phi}{\partial n} A}_{\text{diffusion}} \Big) = \underbrace{S}_{\text{source}}$$

- Each momentum component satisfies its own scalar-transport equation. The concentration $\phi$ is the velocity component $(u, v, w)$, the diffusivity $\Gamma$ is the dynamic viscosity $\mu$, and the source is non-viscous forces. However, these equations differ from those for a passive scalar because they are non-linear, coupled through the advective fluxes and pressure forces, and the velocity field is also required to be mass-consistent.

- Non-dimensionalising the governing equations allows dynamically-similar flows (those with the same values of Reynolds number, etc.) to be solved with a single calculation, reduces the overall number of parameters, indicates whether certain terms in the governing equations are significant or negligible and ensures that the main computational variables are of similar magnitude.

## Examples

Q1.
In 2-d flow, the continuity and $x$-momentum equations can be written in conservative form as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u$$

respectively.

(a) Show that these can be written in the equivalent non-conservative forms:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u$$

where the (2-d) material derivative is given by $\mathrm{D}/\mathrm{D}t = \partial/\partial t + u\,(\partial/\partial x) + v\,(\partial/\partial y)$.

(b) Define carefully what is meant by the statement that a flow is *incompressible*. To what does the continuity equation reduce in incompressible flow?

(c) Write down conservative forms of the 3-d equations for mass and $x$-momentum.

(d) Write down the $z$-momentum equation, including the gravitational force.

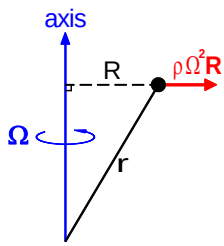
(e) Show that, for constant-density flows, pressure and gravity forces can be combined in the momentum equations via the *piezometric pressure* $p + \rho g z$.

(f) In a rotating reference frame there are additional apparent forces (per unit volume):

centrifugal force: $\rho \Omega^2 \mathbf{R}$

Coriolis force: $-2\rho \mathbf{\Omega} \wedge \mathbf{u}$

where $\mathbf{\Omega}$ is the angular velocity of the reference frame, $\mathbf{u}$ is the fluid velocity in that frame, $\mathbf{r}$ is the position vector (relative to a point on the axis of rotation) and $\mathbf{R}$ is its projection perpendicular to the axis of rotation. $\wedge$ denotes a vector product.




By writing the centrifugal force as the gradient of some quantity, show that it can be subsumed into a modified pressure. Also, find the components of the Coriolis force if rotation is about the $z$ axis.

(*** *Advanced / Optional* ***)

(g) Write the conservative mass and momentum equations in vector notation.

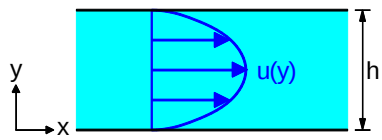
(h) Write the conservative mass and momentum equations in suffix notation using the summation convention.

Q2. (Exact solutions of the Navier-Stokes equation)
The $x$-component of the momentum equation is given by

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u$$

Using this equation, derive the velocity profile in fully-developed, laminar flow for:
(a) pressure-driven flow between stationary parallel planes ("*Plane Poiseuille flow*");
(b) constant-pressure flow between stationary and moving planes ("*Couette flow*").

Assume flow in the $x$ direction, with bounding planes $y = 0$ and $y = h$. The velocity is then $(u(y),0,0)$. In part (a) both walls are stationary. In part (b) the upper wall moves parallel to the lower wall with velocity $U_w$.




(c) (*** *Advanced / optional* ***) In cylindrical polars $(x, r, \phi)$ the Laplacian $\nabla^2$ is more complicated. If axisymmetric, with fully-developed velocity $(u(r),0,0)$, then

$$\nabla^2 u \quad \rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

Derive the velocity profile in a circular pipe with stationary wall at $r = R$ ("*Poiseuille flow*").

Q3. (*** *Advanced / optional* ***)
By applying the divergence theorem, deduce the conservative and non-conservative differential equations corresponding to the general integral scalar-transport equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\phi \;\mathrm{d}V + \oint_{\partial V} (\rho \mathbf{u}\phi - \Gamma\nabla\phi) \bullet \mathrm{d}\mathbf{A} = \int_V s \;\mathrm{d}V$$

Q4.
In each of the following cases, state which one of (i), (ii), (iii) is a valid dimensionless number. Carry out research to find the name and physical significance of these numbers.
($L$ = length; $U$ = velocity; $z$ = height; $p$ = pressure; $\rho$ = density; $\mu$ = dynamic viscosity; $\nu$ = kinematic viscosity; $g$ = gravitational acceleration; $\Omega$ = angular velocity).

| | | | | | | |
|---|---|---|---|---|---|---|
| (a) | (i) | $\dfrac{p - p_{\mathrm{ref}}}{\rho U}$ | (ii) | $\dfrac{p - p_{\mathrm{ref}}}{\frac{1}{2}\rho U^2}$ | (iii) | $\rho U^2 (p - p_{ref})$ |
| (b) | (i) | $\dfrac{\rho U L}{\nu}$ | (ii) | $\dfrac{\rho U L}{\mu}$ | (iii) | $\mu U L$ |
| (c) | (i) | $\dfrac{\left(-\frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^{1/2}}{\left\lvert\frac{\mathrm{d}u}{\mathrm{d}z}\right\rvert}$ | (ii) | $\dfrac{U}{gL}$ | (iii) | $\dfrac{p - p_{ref}}{\rho g}$ |
| (d) | (i) | $\dfrac{U\Omega}{L}$ | (ii) | $\dfrac{\rho g L}{U\Omega}$ | (iii) | $\dfrac{U}{\Omega L}$ |

Q5.
The momentum equation for a viscous fluid in a rotating reference frame is

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{Dt}} = -\nabla p + \mu \nabla^2 \mathbf{u} - 2\rho \mathbf{\Omega} \wedge \mathbf{u} \qquad (*)$$

where $\rho$ is density, $\mathbf{u} = (u, v, w)$ is velocity, $p$ is pressure, $\mu$ is dynamic viscosity and $\mathbf{\Omega}$ is the angular-velocity vector of the reference frame. The symbol $\wedge$ denotes a vector product.

(a) If $\Omega = (0,0,\Omega)$ write down the $x$ and $y$ components of the Coriolis force $(-2\rho\mathbf{\Omega} \wedge \mathbf{u})$.

(b) Hence write down the $x$- and $y$-components of equation (*).

(c) Show how equation (*) can be written in non-dimensional form in terms of a Reynolds number Re and Rossby number Ro (both of which should be defined).

(d) Define the terms *conservative* and *non-conservative* when applied to the differential equations describing fluid flow.

(e) Define (mathematically) the material derivative operator $\mathrm{D}/\mathrm{D}t$. Then, noting that the continuity equation can be written

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

show that the $x$-momentum equation can be written in an equivalent conservative form.

(f) If the $x$-momentum equation were to be regarded as a special case of the general scalar-transport (or advection-diffusion) equation, identify the quantities representing:
(i) concentration;
(ii) diffusivity;
(iii) source.

(g) Explain why the three equations for the components of momentum cannot be treated as independent scalar equations.

Q6.
(a) In a rotating reference frame (with angular velocity vector $\mathbf{\Omega}$) the non-viscous forces on a fluid are, per unit volume,

$$\underset{\text{(I)}}{-\nabla p} + \underset{\text{(II)}}{\rho \mathbf{g}} + \underset{\text{(III)}}{\rho \Omega^2 \mathbf{R}} - \underset{\text{(IV)}}{2\rho \mathbf{\Omega} \wedge \mathbf{u}}$$

where $p$ is pressure, $\mathbf{g} = (0,0,-g)$ is the gravity vector and $\mathbf{R}$ is the vector from the closest point on the axis of rotation to a point. Show that, in a constant-density fluid, force densities (I), (II) and (III) can be combined in terms of a modified pressure.

(b) Consider a closed cylindrical can of radius 5 cm and depth 15 cm. The can is completely filled with fluid of density 1100 kg m$^{-3}$ and is rotating steadily about its axis (which is vertical) at 600 rpm. Where do the maximum and minimum pressures in the can occur, and what is the difference in pressure between them?