Classroom Example 1

Velocities, with correction:

$$u_A = 10 \quad \text{(boundary condition)}$$
$$u_B = 8 + 2(p_1' - p_2')$$
$$u_C = 11 + 2(p_2' - p_3')$$

Apply mass conservation for each cell. Since densities and areas are the same on each face they may be omitted.

Cell 1:

$$u_B - u_A = 0$$
$$\Rightarrow \quad 8 + 2(p_1' - p_2') - 10 = 0$$
$$\Rightarrow \quad 2p_1' - 2p_2' = 2$$

Cell 2:

$$u_C - u_B = 0$$
$$\Rightarrow \quad 11 + 2(p_2' - p_3') - 8 - 2(p_1' - p_2') = 0$$
$$\Rightarrow \quad -2p_1' + 4p_2' - 2p_3' = -3$$

This gives two equations for three unknowns (p_1', p_2' and p_3'). Since pressure is defined only up to a constant (i.e. only pressure *differences* matter) we can, WLOG, take any one of them to be whatever we like. Corresponding to a common outflow boundary condition with pressure fixed we take $p_3' = 0$. (Alternatively, we can just leave them to "float" in value – the unknown pressure will simply cancel when we subtract pressures to compute velocity corrections). Then, simplifying,

$$p_1' - p_2' = 1$$
$$-p_1' + 2p_2' = -\frac{3}{2}$$

These have solution $p_2' = -1/2, p_1' = 1/2$.

Substituting in the velocity-correction formulae:

$$u_B = 8 + 2(p_1' - p_2') \quad = 10$$
$$u_C = 11 + 2(p_2' - p_3') \quad = 10$$

The velocity field has the constant value 10 (as expected).

Classroom Example 2

(a) Velocities – with correction:

$$u_B = 11 + 2(p'_A - p'_B)$$
$$u_D = 14 + 2(p'_C - p'_D)$$
$$v_C = 8 + 3(p'_A - p'_C)$$
$$v_D = 5 + 3(p'_B - p'_D)$$

Apply mass conservation for each cell in turn. Since densities and areas are the same on each face they may be omitted.

<u>Cell A</u>:

$$u_B - 5 + v_C - 15 = 0$$
$$\Rightarrow \quad 11 + 2(p'_A - p'_B) - 5 + 8 + 3(p'_A - p'_C) - 15 = 0$$
$$\Rightarrow \quad 5p'_A - 2p'_B - 3p'_C = 1$$

<u>Cell B</u>:

$$5 - u_B + v_D - 10 = 0$$
$$\Rightarrow \quad 5 - 11 - 2(p'_A - p'_B) + 5 + 3(p'_B - p'_D) - 10 = 0$$
$$\Rightarrow \quad -2p'_A + 5p'_B - 3p'_D = 11$$

<u>Cell C</u>:

$$u_D - 10 + 5 - v_C = 0$$
$$\Rightarrow \quad 14 + 2(p'_B - p'_D) - 10 + 5 - 8 - 3(p'_A - p'_C) = 0$$
$$\Rightarrow \quad -3p'_A + 5p'_C - 2p'_D = -1$$

<u>Cell D</u>:

$$20 - u_D + 10 - v_D = 0$$
$$\Rightarrow \quad 20 - 14 - 2(p'_C - p'_D) + 10 - 5 - 3(p'_B - p'_D) = 0$$
$$\Rightarrow \quad -3p'_B - 2p'_C + 5p'_D = -11$$

Assembling these into a single matrix equation:

$$\begin{pmatrix} 5 & -2 & -3 & 0 \\ -2 & 5 & 0 & -3 \\ -3 & 0 & 5 & -2 \\ 0 & -3 & -2 & 5 \end{pmatrix} \begin{pmatrix} p'_A \\ p'_B \\ p'_C \\ p'_D \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \\ -1 \\ -11 \end{pmatrix}$$

Gaussian elimination as follows.

$$R2 \to 5R2 + 2R1 \quad R3 \to 5R3 + 3R1 \qquad \begin{pmatrix} 5 & -2 & -3 & 0 \\ 0 & 21 & -6 & -15 \\ 0 & -6 & 16 & -10 \\ 0 & -3 & -2 & 5 \end{pmatrix} \begin{pmatrix} p'_A \\ p'_B \\ p'_C \\ p'_D \end{pmatrix} = \begin{pmatrix} 1 \\ 57 \\ -2 \\ -11 \end{pmatrix}$$

$$R2 \to R2/3 \quad R3 \to R3/2 \qquad \begin{pmatrix} 5 & -2 & -3 & 0 \\ 0 & 7 & -2 & -5 \\ 0 & -3 & 8 & -5 \\ 0 & -3 & -2 & 5 \end{pmatrix} \begin{pmatrix} p'_A \\ p'_B \\ p'_C \\ p'_D \end{pmatrix} = \begin{pmatrix} 1 \\ 19 \\ -1 \\ -11 \end{pmatrix}$$

$$R3 \to 7R3 + 3R2 \quad R4 \to 7R4 + 3R2 \qquad \begin{pmatrix} 5 & -2 & -3 & 0 \\ 0 & 7 & -2 & -5 \\ 0 & 0 & 50 & -50 \\ 0 & 0 & -20 & 20 \end{pmatrix} \begin{pmatrix} p'_A \\ p'_B \\ p'_C \\ p'_D \end{pmatrix} = \begin{pmatrix} 1 \\ 19 \\ 50 \\ -20 \end{pmatrix}$$

$$R3 \to R3/50 \quad R4 \to R4/20 \qquad \begin{pmatrix} 5 & -2 & -3 & 0 \\ 0 & 7 & -2 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} p'_A \\ p'_B \\ p'_C \\ p'_D \end{pmatrix} = \begin{pmatrix} 1 \\ 19 \\ 1 \\ -1 \end{pmatrix}$$

$$R4 \to R4 + R3 \qquad \begin{pmatrix} 5 & -2 & -3 & 0 \\ 0 & 7 & -2 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p'_A \\ p'_B \\ p'_C \\ p'_D \end{pmatrix} = \begin{pmatrix} 1 \\ 19 \\ 1 \\ 0 \end{pmatrix}$$

p'_D can be chosen arbitrarily; we could set an arbitrary convenient value, but it is perfectly legitimate to leave it as undetermined.

Back-substituting:

$$p'_C - p'_D = 1$$

$$\Rightarrow \qquad p'_C = 1 + p'_D$$

$$7p'_B - 2p'_C - 5p'_D = 19$$

$$\Rightarrow \qquad p'_B = \frac{19 + 2p'_C + 5p'_D}{7} \qquad = 3 + p'_D$$

$$5p'_A - 2p'_B - 3p'_C = 1$$

$$\Rightarrow \qquad p'_A = \frac{1 + 2p'_B + 3p'_C}{5} \qquad = 2 + p'_D$$

Finally, use these to correct the velocity:

$$u_B = 11 + 2(p'_A - p'_B) \quad = 9$$
$$u_D = 14 + 2(p'_C - p'_D) \quad = 16$$
$$v_C = 8 + 3(p'_A - p'_C) \quad = 11$$
$$v_D = 5 + 3(p'_B - p'_D) \quad = 14$$

with p'_D cancelling in each case.

(b) Although (non-unique) velocity corrections alone could have been computed to make the system mass-consistent, the only way to ensure that it also remains a solution of the momentum equation is to relate velocity and pressure corrections in the form implied by the momentum equation.

Q1.

(a)

$$(p_w - p_e)A$$

(b) Since $V = A\Delta x$, the net force per unit volume is

$$\frac{(p_w - p_e)A}{A\Delta x} = \frac{p_w - p_e}{\Delta x}$$

(c) The average pressure gradient is

$$\frac{p_e - p_w}{\Delta x}$$

which is minus the expression in part (b).

Hence, shrinking the control volume to a point, the pressure force in the x-direction, per unit volume, is $-\partial p/\partial x$.

Q2.

The net pressure force component in the x or y direction *into* a face is given by:

$$\text{pressure} \times \text{projected area into the cell}$$

Hence,

$$F_x = -\sum_{\text{faces}} p\Delta y$$

$$F_y = \sum_{\text{faces}} p\Delta x$$

where Δx and Δy are coordinate increments when the cell is traversed anticlockwise.

Starting from the lowest face (edge in 2d):

$$F_x = -\sum_{\text{faces}} p\Delta y \quad = -[1\times(-1) + 2\times 3 + 4\times 1 + 10\times(-3)] \quad = 21$$

$$F_y = \sum_{\text{faces}} p\Delta x \quad = \quad 1\times 5 + 2\times 1 + 4\times(-4) + 10\times(-2) \quad = -29$$

Answer: net force = (21, –29) units

Note. In general,

$$\mathbf{F} = -\sum_{\text{faces}} p\mathbf{A}$$

where $\mathbf{A}$ is the outward face area vector (Section 9)

Q3.
(a) The net pressure force component in x or y direction on a face is given by:

$$\text{pressure} \times \text{projected area into the cell}$$

Hence, accounting for direction if the cell is traversed anticlockwise:

$$F_x = \sum_{\text{faces}} p(-\Delta y)$$

$$F_y = \sum_{\text{faces}} p\Delta x$$

Thus, starting from the lowest face:

$$F_x = 3 \times 0 + 5 \times (-5) + 2 \times 5 \quad = -15$$

$$F_y = 3 \times 4 + 5 \times (-1) + 2 \times (-3) \quad = 1$$

Answer: force = (–15, 1).

(b) Since "force = mass × acceleration" we need the mass of the cell.

Since it has unit depth and is a triangle in plan, the cell has volume

$$\frac{1}{2} \times \text{base} \times \text{height} \quad = \frac{1}{2} \times 4 \times 5 \quad = 10$$

Since $\rho = 1.0$, the mass is 10 units also. Hence,

$$\mathbf{a} = \frac{1}{m}\mathbf{F} \quad = \frac{1}{10} \times (-15, 1) \quad = (-1.5, 0.1)$$

Answer: acceleration = (–1.5, 0.1)

Q4.

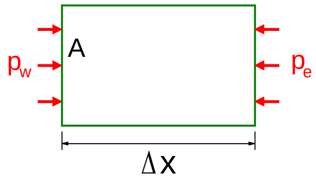
u	5		4		3		2
p	0.6		0.7		1.1		1.6
p_{face}		0.65		0.9		1.35	
$\hat{u}_{\text{node}}$ $(= u + 3\Delta p)$			4.75		4.35		
$\hat{u}_{\text{face}}$				4.55			
$u_{\text{face}} (= \hat{u}_{\text{face}} - 3\Delta p)$				3.35			

Answer: the velocity on cell face *f* is 3.35

Q5.

(a) If the cell length in the x direction is Δx and the cross-sectional area is A then the net pressure force in the x direction is

$$(p_w - p_e)A$$



whilst the volume is

$$V = A\Delta x$$

The net force per unit volume is then

$$\frac{(p_w - p_e)A}{A\Delta x} = \frac{p_w - p_e}{\Delta x} = -\left(\frac{p_e - p_w}{\Delta x}\right)$$

In the limit as $\Delta x \to 0$ this tends to $-\partial p / \partial x$.

(b) The Rhie-Chow algorithm is used:

- to predict the advective velocities (the cell-face velocities in the mass flux);
- to prevent odd-even pressure decoupling that arises from centred differencing of pressure forces and linear interpolation of velocities.

(c)

(i) By linear interpolation (just average the velocities either side) we have

$$u_w = 2.5; \quad u_e = 3.5$$

(ii) By Rhie-Chow interpolation, the complete calculation sequence is:

- calculate pressures at cell *faces*, p_f, by interpolation;
- calculate pseudovelocities $\hat{u}$ at *nodes* by inversion: $\hat{u} = u + \frac{1}{2}\Delta p$;
- calculate pseudovelocities $\hat{u}$ at *faces* by interpolation;
- calculate final advective velocity at *faces* by $u = \hat{u} - \frac{1}{2}\Delta p$.

This is illustrated below.

	i-2		i-1	w	i	e	i+1		i+2
$p =$	2		6		5		7		4
$u =$	1		2		3		4		5
$p_f =$		4		5.5		6		5.5	
$\hat{u} =$			2.75		3.25		3.75		
$\hat{u}_f =$				3		3.5			
$u_f =$				3.5		2.5			

The final face velocities are:

$$u_w = 3.5; \quad u_e = 2.5$$

Comparing the pairs of values for u_w and u_e, it can be seen that the Rhie-Chow algorithm actually changes which of the two velocities is larger. Inspection of the pressure field shows that it is trying to reduce the local peak to the left by increasing u_w and reduce the local peak to the right by decreasing u_e.

(d) Corrected velocities on the west and east faces are:

$$u_w = 3.5 - \frac{1}{2}(p'_i - p'_{i-1})$$

$$u_e = 2.5 - \frac{1}{2}(p'_{i+1} - p'_i)$$

Continuity requires that

$$u_e A - u_w A = 0$$

where A is a cell-face area. Substituting, and dividing by A, gives:

$$2.5 - \frac{1}{2}(p'_{i+1} - p'_i) - 3.5 + \frac{1}{2}(p'_i - p'_{i-1}) = 0$$

$$\Rightarrow \qquad -\frac{1}{2}p'_{i-1} + p'_i - \frac{1}{2}p'_{i+1} = 1$$

This is the required pressure-correction equation.

Q6.
Corrected velocities:

$$u_1 = 4$$
$$u_2 = 3 - 4(p_2' - p_1')$$
$$u_3 = 5 - 4(p_3' - p_2')$$
$$u_4 = 6 - 4(p_4' - p_3')$$

Mass conservation for cells centred on the internal pressure nodes yields pressure-correction equations as follows.

<u>Cell 1</u>

$$u_2 - u_1 = 0$$
$\Rightarrow$ $$3 - 4(p_2' - p_1') - 4 = 0$$
$\Rightarrow$ $$4p_1' - 4p_2' = 1$$

<u>Cell 2</u>

$$u_3 - u_2 = 0$$
$\Rightarrow$ $$5 - 4(p_3' - p_2') - 3 + 4(p_2' - p_1') = 0$$
$\Rightarrow$ $$-4p_1' + 8p_2' - 4p_3' = -2$$

<u>Cell 3</u>

$$u_4 - u_3 = 0$$
$\Rightarrow$ $$6 - 4(p_4' - p_3') - 5 + 4(p_3' - p_2') = 0$$
$\Rightarrow$ $$-4p_2' + 8p_3' - 4p_4' = -1$$

Solving à la tri-diagonal matrix algorithm …
From the first cell:

$$4p_1' = 4p_2' + 1$$

From the second cell:

$$-(4p_2' + 1) + 8p_2' - 4p_3' = -2 \qquad \text{or} \qquad 4p_2' = 4p_3' - 1$$

From the third cell:

$$-(4p_3' - 1) + 8p_3' - 4p_4' = -1 \qquad \text{or} \qquad 4p_3' = 4p_4' - 2$$

Hence,

$$p_3' = p_4' - \frac{1}{2}$$
$$p_2' = p_4' - \frac{3}{4}$$
$$p_1' = p_4' - \frac{1}{2}$$

(p'_4 can be chosen as anything convenient, since it is only pressure differences that matter).

Substituting these into the velocity-correction formulae gives

$$u_1 = u_2 = u_3 = u_4 = 4$$

as anticipated.

Q7.

(a)

(1) Set up and solve the momentum equation with current pressure.

(2) Use the discretised momentum equation to write velocity corrections in terms of pressure corrections.

(3) Set up and solve the mass equation, writing velocity corrections in terms of pressure corrections.

(4) Correct velocity and pressure.

(5) Repeat from (1) until converged.

(b) Co-location and standard interpolation for advective velocities leads to odd-even node decoupling and oscillations in the pressure field.

Remedies:

- staggered velocity-pressure mesh;
- Rhie-Chow interpolation for advective velocities.

(c) Total flow rates:

inflow: $5A + 3A = 8A$

outflow: $6A + 6A = 12A$

where A is the face area of a cell.

For global mass conservation we require "total flow in = total flow out" and hence the outflow velocities must be multiplied by a scaling factor

$$\frac{8}{12} = \frac{2}{3}$$

The outflow velocities then become

$$u_E = 4, \quad u_F = 4$$

(d) Applying mass conservation to scalar cells A – D in turn we require (on dividing by the density ρ and common cell face area A):

$$u_C - 5 + 0 - v_A = 0$$
$$u_D - 3 + v_A - 0 = 0$$
$$4 - u_C + 0 - v_C = 0$$
$$4 - u_D + v_C - 0 = 0$$

Substituting for the velocities in terms of the current velocities and pressure corrections:

$$4 + 2(p'_A - p'_C) - 5 + 0 - (-2) - 3(p'_B - p'_A) = 0$$
$$(-4) + 2(p'_B - p'_D) - 3 + (-2) + 3(p'_B - p'_A) - 0 = 0$$
$$4 - 4 - 2(p'_A - p'_C) + 0 - 1 - 3(p'_D - p'_C) = 0$$
$$4 - (-4) - 2(p'_B - p'_D) + 1 + 3(p'_D - p'_C) - 0 = 0$$

Hence,

$$\begin{pmatrix} 5 & -3 & -2 & 0 \\ -3 & 5 & 0 & -2 \\ -2 & 0 & 5 & -3 \\ 0 & -2 & -3 & 5 \end{pmatrix} \begin{pmatrix} p'_A \\ p'_B \\ p'_C \\ p'_D \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 1 \\ -9 \end{pmatrix}$$

Solve by Gaussian elimination:

$R2 \to 5R2 + 3R1$
$R3 \to 5R3 + 2R1$

$$\begin{pmatrix} 5 & -3 & -2 & 0 \\ 0 & 16 & -6 & -10 \\ 0 & -6 & 21 & -15 \\ 0 & -2 & -3 & 5 \end{pmatrix} \begin{pmatrix} p'_A \\ p'_B \\ p'_C \\ p'_D \end{pmatrix} = \begin{pmatrix} -1 \\ 42 \\ 3 \\ -9 \end{pmatrix}$$

$R3 \to 8R3 + 3R2$
$R4 \to 8R4 + R2$

$$\begin{pmatrix} 5 & -3 & -2 & 0 \\ 0 & 16 & -6 & -10 \\ 0 & 0 & 150 & -150 \\ 0 & 0 & -30 & 30 \end{pmatrix} \begin{pmatrix} p'_A \\ p'_B \\ p'_C \\ p'_D \end{pmatrix} = \begin{pmatrix} -1 \\ 42 \\ 150 \\ -30 \end{pmatrix}$$

$R4 \to 5R4 + R3$

$$\begin{pmatrix} 5 & -3 & -2 & 0 \\ 0 & 16 & -6 & -10 \\ 0 & 0 & 150 & -150 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p'_A \\ p'_B \\ p'_C \\ p'_D \end{pmatrix} = \begin{pmatrix} -1 \\ 42 \\ 150 \\ 0 \end{pmatrix}$$

$R2 \to R2/2$
$R3 \to R3/150$

$$\begin{pmatrix} 5 & -3 & -2 & 0 \\ 0 & 8 & -3 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p'_A \\ p'_B \\ p'_C \\ p'_D \end{pmatrix} = \begin{pmatrix} -1 \\ 21 \\ 1 \\ 0 \end{pmatrix}$$

p'_D can be chosen arbitrarily; (pressure is fixed only up to a constant).

Back-substituting:

$$p'_C - p'_D = 1 \qquad \Rightarrow \qquad p'_C = p'_D + 1$$

$$8p'_B - 3p'_C - 5p'_D = 21 \qquad \Rightarrow \qquad p'_B = p'_D + 3$$

$$5p'_A - 3p'_B - 2p'_C = -1 \qquad \Rightarrow \qquad p'_A = p'_D + 2$$

Finally, update velocity:

$$u_C = 4 + 2(p'_A - p'_C) \quad = 6$$
$$u_D = -4 + 2(p'_B - p'_D) \quad = 2$$
$$v_A = -2 + 3(p'_B - p'_A) \quad = 1$$
$$v_C = 1 + 3(p'_D - p'_C) \quad = -2$$

Q8.
(a) The Rhie-Chow algorithm is used to avoid odd-even pressure decoupling resulting from the naïve use of linear interpolation to get advective cell-face velocities to compute mass fluxes on a collocated mesh.

Mathematically, if the momentum equation gives a pressure-velocity coupling of

$$u = \hat{u} - d\Delta p$$

then the Rhie-Chow algorithm is (e.g. for an east face):

$$u_e = (\overline{u + d\Delta p})_e - \overline{d_e}(p_E - p_P)$$

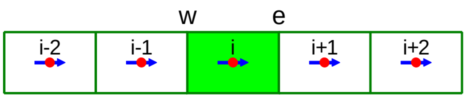
The overbar denotes interpolation to cell faces.

(b) By Rhie-Chow interpolation, the complete calculation sequence is:

- calculate pressures at cell *faces*, p_f, by interpolation;
- calculate pseudovelocities $\hat{u}$ at *nodes* by inversion: $\hat{u} = u + 2\Delta p$;
- calculate pseudovelocities $\hat{u}$ at *faces* by interpolation;
- calculate final advective velocity at *faces* by $u = \hat{u} - 2\Delta p$.

Here, Δp indicates a centred difference.

This is illustrated below.



	i-2		i-1	w	i	e	i+1		i+2
p =	4		5		2		3		0
u =	10		10		10		10		10
p_f =		4.5		3.5		2.5		1.5	
$\hat{u}$ =			8		8		8		
$\hat{u}_f$ =				8		8			
u_f =				14		6			

The final face velocities are:

$$u_w = 14; \quad u_e = 6$$

The Rhie-Chow algorithm is trying to smooth out the local peaks in pressure (here, at nodes $i-1$ and $i+1$).

(c) By mass conservation (in 1-d; dividing by density × area):

$$u_e - u_w = 0$$

In terms of velocity/pressure corrections (at cell faces):

$$6 - 2(p'_{i+1} - p'_i) - 14 + 2(p'_i - p'_{i-1}) = 0$$

Hence,

$$-2p'_{i-1} + 4p'_i - 2p'_{i+1} = 8$$

(d) If $p'_{i+1} = p'_{i-1} = 0$ then the equation reduces to

$$4p'_i = 8$$

or

$$p'_i = 2$$

Then, correcting the velocities:

$$u_e = 6 - 2(p'_{i+1} - p'_i) \quad = 10$$

$$u_w = 14 - 2(p'_i - p'_{i-1}) \quad = 10$$

Correcting the pressure at node i:

$$p_i = 2 + p'_i \quad = 4$$

This then recovers both a continuity-satisfying velocity field ($u = 10$) and a uniform pressure gradient ($p = 5, 4, 3$) across the centre cells.