

Objectives

- (1) Calculate the **friction factor** for a pipe using the **Colebrook-White equation**.
- (2) Undertake **head loss, discharge** and **sizing** calculations for single pipelines.
- (3) Use **head-loss vs discharge** relationships to calculate flow in **pipe networks**.
- (4) Relate **normal depth** to discharge for **uniform flow** in **open channels**.

1. Pipe flow

- 1.1 Introduction
- 1.2 Governing equations for circular pipes
- 1.3 Laminar pipe flow
- 1.4 Turbulent pipe flow
- 1.5 Expressions for the Darcy friction factor, λ
- 1.6 Other losses
- 1.7 Pipeline calculations
- 1.8 Energy and hydraulic grade lines
- 1.9 Simple pipe networks
- 1.10 Complex pipe networks (*optional*)

2. Open-channel flow

- 2.1 Normal flow
- 2.2 Hydraulic radius and the drag law
- 2.3 Friction laws – Chézy and Manning's formulae
- 2.4 Open-channel flow calculations
- 2.5 Conveyance
- 2.6 Optimal shape of cross-section

Appendix

1. PIPE FLOW

1.1 Introduction

The flow of water, oil, air and gas in pipes is of great importance to engineers. In particular, the design of distribution systems depends on the relationship between discharge, Q , diameter, D , and available head, h .

Flow Regimes: Laminar or Turbulent

In 1883, Osborne Reynolds¹ demonstrated the occurrence of two *regimes* of flow – laminar or turbulent – according to the size of a dimensionless parameter later named the Reynolds number. The conventional definition for round pipes is

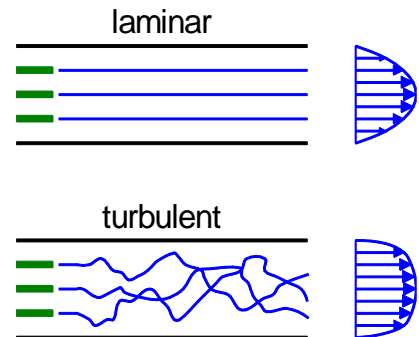
$$Re \equiv \frac{VD}{\nu} \quad (1)$$

where:

V = average, or bulk, velocity ($= Q/A$)

D = diameter

ν = kinematic viscosity ($= \mu/\rho$)



For smooth-walled pipes the *critical* Reynolds number at which *transition* between laminar and turbulent regimes occurs is usually taken as

$$Re_{crit} \approx 2300 \quad (\text{for PIPES only!}) \quad (2)$$

In practice, transition from intermittent to fully-turbulent flow typically occurs over the range $2000 < Re < 4000$.

Development Length

At inflow, the velocity profile is often uniform. A thin boundary layer develops on the pipe wall because of friction. This grows with distance until it fills the cross-section. Beyond this distance the velocity profile becomes *fully-developed* (i.e., doesn't change any further with downstream distance). Typical correlations for this *development length* are (from White, 2021):

$$\frac{L_{dev}}{D} = \begin{cases} 0.06Re & (\text{laminar}) \\ 4.4Re^{1/6} & (\text{turbulent}) \end{cases} \quad (3)$$

The kinematic viscosity of air and water is such that most pipe flows in civil engineering have high Reynolds numbers, are fully turbulent, and have a negligible development length.

¹ at the University of Manchester!

Example.

$\nu_{\text{water}} = 1.0 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$. Calculate the Reynolds numbers for water flow with average velocity 0.5 m s^{-1} in pipes of inside diameter 12 mm and 0.3 m. Estimate the development length in each case.

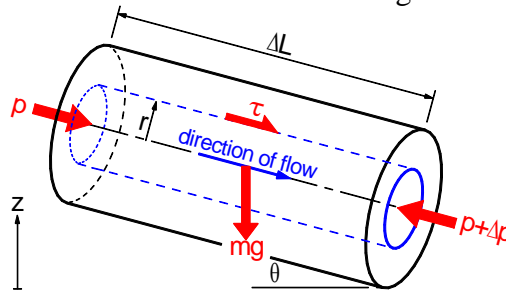
Answer: $Re = 6000$ and 1.5×10^5 ; $L_{\text{develop}} = 0.23 \text{ m}$ and 9.6 m .

1.2 Governing Equations For Circular Pipes

Fully-developed pipe flow is determined by a balance between three forces:

- pressure;
- weight (component along the pipe axis);
- friction.

For a circular pipe of radius R , consider the forces with components along the pipe axis for an internal cylindrical fluid element of radius $r < R$ and length Δl .



Note:

- (1) p is the average pressure over a cross-section; for circular pipes this is equal to the centreline pressure, with equal and opposite hydrostatic variations above and below.
- (2) The arrow drawn for stress indicates its conventional positive direction, corresponding to the stress exerted by the **outer** on the **inner** fluid. In this instance the inner fluid moves faster so that, if V is positive, τ will actually be negative.

Balancing forces along the pipe axis:

$$\underbrace{p(\pi r^2) - (p + \Delta p)(\pi r^2)}_{\text{net pressure force}} + \underbrace{mg \sin \theta}_{\text{weight}} + \underbrace{\tau(2\pi r \Delta l)}_{\text{friction}} = 0$$

From the geometry,

$$m = \rho(\pi r^2 \Delta l), \quad \sin \theta = -\frac{\Delta z}{\Delta l}$$

Hence:

$$-\Delta p(\pi r^2) - \rho \pi r^2 g \Delta z + \tau(2\pi r \Delta l) = 0$$

Dividing by the volume, $\pi r^2 \Delta l$,

$$-\frac{\Delta(p + \rho g z)}{\Delta l} + 2\frac{\tau}{r} = 0$$

Writing $p^* = p + \rho g z$ for the piezometric pressure and rearranging for the shear stress:

$$\tau = \frac{1}{2} r \frac{\Delta p^*}{\Delta l} \quad (4)$$

Since the flow is fully-developed, the shear stress and the gradient of the piezometric pressure are independent of distance. For convenience write G for the streamwise pressure gradient:

$$G = -\frac{\Delta p^*}{\Delta l} = -\frac{dp^*}{dl} \quad (\text{constant}) \quad (5)$$

(The negative sign is included because we expect p^* to drop along the pipe.) Hence, from (4),

$$\tau = -\frac{1}{2} Gr \quad (6)$$

where

$$G = -\frac{dp^*}{dl} = \frac{\text{pressure drop}}{\text{length}} = \frac{\rho g h_f}{L} \quad (7)$$

G is the piezometric pressure gradient and h_f is the head lost by friction over length L .

(6) applies to any fully-developed pipe flow, irrespective of whether it is laminar or turbulent. For **laminar** flow it can be used to derive the velocity profile, because τ can be related to the velocity gradient du/dr (Section 1.3). For **turbulent** flow an analytical velocity profile is not available, but gross parameters such as quantity of flow and head loss may be obtained if the wall shear stress τ_w can be related empirically to the dynamic pressure $1/2\rho V^2$ (Section 1.4).

1.3 Laminar Pipe Flow

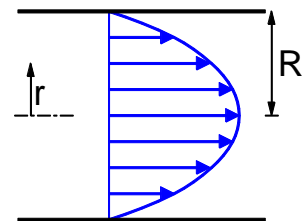
Laminar flow through a circular pipe is called *Poiseuille*² flow or *Hagen*³-*Poiseuille* flow.

In laminar flow the shear stress is related to the velocity gradient:

$$\tau = \mu \frac{du}{dr} \quad (8)$$

Hence, from (6) and (8),

$$\frac{du}{dr} = -\frac{1}{2} \frac{G}{\mu} r$$



Integrating and applying the no-slip condition at the wall ($u = 0$ on $r = R$),

Laminar pipe-flow velocity profile

$$u = \frac{G}{4\mu} (R^2 - r^2) \quad (9)$$

² J.L.M Poiseuille (1799-1869); French physician who was interested in flow in blood vessels.

³ G.L.H. Hagen; German engineer who, in 1839, measured water flow in long brass pipes and reported that there appeared to be two regimes of flow.

Example. Find, from the velocity distribution given above,

- (a) the centreline velocity, U_0 ;
- (b) the average velocity, V ;
- (c) the volumetric flow rate, Q , in terms of head loss and pipe diameter;
- (d) the friction factor, λ , defined by $h_f = \lambda \frac{L}{D} \left(\frac{V^2}{2g} \right)$, as a function of Reynolds number, Re .

$$\text{Answer: (a) } U_0 = \frac{GR^2}{4\mu}; \quad \text{(b) } V = \frac{1}{2}u_0 = \frac{GR^2}{8\mu}; \quad \text{(c) } Q = \frac{\pi}{128} \frac{\rho gh_f D^4}{\mu L}; \quad \text{(d) } \lambda = \frac{64}{Re}$$

Part (d) of this example demonstrates that the friction factor λ is not constant for a given pipe.

1.4 Turbulent Pipe Flow

In turbulent flow one is usually interested in time-averaged quantities. “Velocity” usually implies time-averaged velocity and the shear stress τ is the time-averaged rate of transport of momentum per unit area: it is dominated by turbulent mixing rather than viscous stresses.

In turbulent flow there is no longer an explicit relationship between mean stress τ and mean velocity gradient du/dr because a far greater transfer of momentum arises from the net effect of turbulent eddies than the much smaller viscous forces. Hence, to relate quantity of flow to head loss we require an empirical relation connecting the *wall shear stress* and the *average velocity* in the pipe. As a first step define a *skin friction coefficient*, c_f , by

$$c_f \equiv \frac{\text{wall shear stress}}{\text{dynamic pressure}} \equiv \frac{\tau_w}{\frac{1}{2}\rho V^2} \quad (10)$$

Later, c_f will be absorbed into a *friction factor*, λ , to simplify the expression for head loss.

For the length of pipe shown, the balance of forces along the axis in fully-developed flow is:

$$\underbrace{-\Delta p \times \frac{\pi D^2}{4}}_{\text{net pressure force}} + \underbrace{mg}_{\text{weight}} \sin \theta - \underbrace{\tau_w \times \pi DL}_{\text{wall friction}} = 0$$

From the geometry,

$$m = \rho \left(\frac{\pi D^2}{4} \times L \right), \quad \sin \theta = -\frac{\Delta z}{L}$$

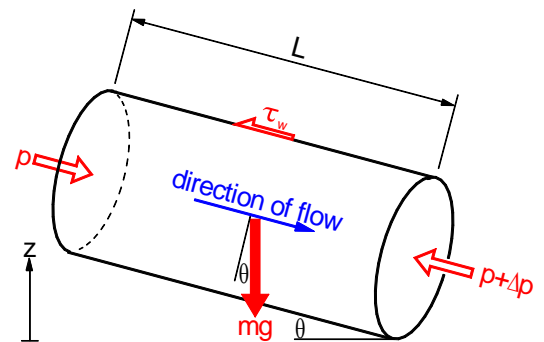
Substituting these gives:

$$-\Delta p \times \frac{\pi D^2}{4} - \rho g \Delta z \times \frac{\pi D^2}{4} = \tau_w \times \pi DL$$

$$\Rightarrow -\Delta(p + \rho g z) \times \frac{\pi D^2}{4} = \tau_w \times \pi DL$$

Dividing by the cross-sectional area ($\pi D^2/4$),

$$-\Delta p^* = 4 \frac{L}{D} \tau_w$$



Write:

$$\tau_w = c_f \left(\frac{1}{2} \rho V^2\right) \quad (\text{definition of skin-friction coefficient})$$

Substituting, and rearranging, gives the **drop** in piezometric pressure:

$$|\Delta p^*| = 4c_f \frac{L}{D} \left(\frac{1}{2} \rho V^2\right)$$

The quantity $4c_f$ is known as the (*Darcy*) *friction factor* and is denoted λ .

Darcy⁴-Weisbach⁵ Equation

$$|\Delta p^*| = \lambda \frac{L}{D} \left(\frac{1}{2} \rho V^2\right) \quad (11)$$

$$\text{pressure loss due to friction} = \lambda \frac{L}{D} \times (\text{dynamic pressure})$$

Dividing by ρg this can equally well be written in terms of **head** rather than **pressure**:

$$h_f = \lambda \frac{L}{D} \left(\frac{V^2}{2g}\right) \quad (12)$$

$$\text{head loss due to friction} = \lambda \frac{L}{D} \times (\text{dynamic head})$$

*** **Very important** ***

There is considerable disagreement about what is meant by “*friction factor*” and what symbol should be used to denote it. What is represented here by λ is also denoted f by some authors and $4f$ by others! **Be very wary of the definition.** You can usually distinguish it by the expression for friction factor in laminar flow: $64/\text{Re}$ with the notation here; $16/\text{Re}$ with the next-most-common alternative.

It remains to specify λ for a turbulent pipe flow. Methods for doing so are discussed in Section 1.5 and lead to the *Colebrook-White equation*. Since λ depends on both the relative roughness of the pipe, k_s/D , and the flow velocity itself (through the Reynolds number $\text{Re} \equiv VD/\nu$) either an iterative solution or a chart-based solution is usually required.

Although the bulk velocity, V , appears in the head-loss equation, the more important quantity is the quantity of flow, Q . These two variables are related, for circular pipes, by

$$Q = VA = V \frac{\pi D^2}{4}$$

where D is the pipe diameter.

⁴ Henri Darcy (1803-1858); French engineer; conducted experiments on pipe flow.

⁵ Julius Weisbach; German professor who, in 1850, published the first modern textbook on hydrodynamics.

At high Reynolds numbers λ tends to a constant (determined by surface roughness) for any particular pipe. In this regime compare:

$$h_f \propto \frac{Q^2}{D^5} \quad (\text{turbulent flow})$$

$$h_f \propto \frac{Q}{D^4} \quad (\text{laminar flow})$$

Note in both cases the very strong dependence (4th or 5th power) of the head loss on the diameter of the pipe.

1.5 Expressions for the Darcy Friction Factor, λ

Laminar Flow (theory)

$$\lambda = \frac{64}{\text{Re}}$$

Turbulent Flow (smooth or rough pipes)

Nikuradse⁶ (1933) used sand grains to roughen pipe surfaces. He defined a *relative roughness* k_s/D , where k_s is the sand-grain size and D is the diameter of the pipe. His experimental curves for friction factor (see, e.g., White's textbook) showed 5 regions:

1. laminar flow ($\text{Re} < \text{Re}_{\text{crit}} \approx 2000$; roughness irrelevant)
2. laminar-to-turbulent transition (approximately $2000 < \text{Re} < 4000$)
3. smooth-wall turbulent flow (λ is a function of Reynolds number only)
4. fully-rough-wall turbulent flow (λ is a function of relative roughness only)
5. intermediate roughness (λ is a function of both Re and k_s/D)

In the smooth- or rough-wall limits, Prandtl⁷ and Von Kármán⁸ gave, respectively:

$$\text{Smooth-wall turbulence:} \quad \frac{1}{\sqrt{\lambda}} = 2.0 \log_{10} \left(\frac{\text{Re} \sqrt{\lambda}}{2.51} \right)$$

$$\text{Rough-wall turbulence:} \quad \frac{1}{\sqrt{\lambda}} = 2.0 \log_{10} \left(\frac{3.7D}{k_s} \right)$$

However, in practice, many commercial pipes lie in the region where both roughness and Reynolds number are important, so that the friction factor is not constant for any particular pipe, but depends on the flow rate. Colebrook and White (1937) combined smooth- and rough-wall turbulence laws into a single formula, the *Colebrook-White equation*.

⁶ Johann Nikuradse (1894-1979); PhD student of Prandtl.

⁷ Ludwig Prandtl (1875-1953); German engineer; introduced boundary-layer theory.

⁸ Theodore von Kármán (1881-1963); Hungarian mathematician and aeronautical engineer; gave his name to the double row of vortices shed from a 2-d bluff body and now known as a Kármán vortex street.

Colebrook-White Equation:

$$\frac{1}{\sqrt{\lambda}} = -2.0 \log_{10} \left(\frac{k_s}{3.7D} + \frac{2.51}{\text{Re} \sqrt{\lambda}} \right) \quad (13)$$

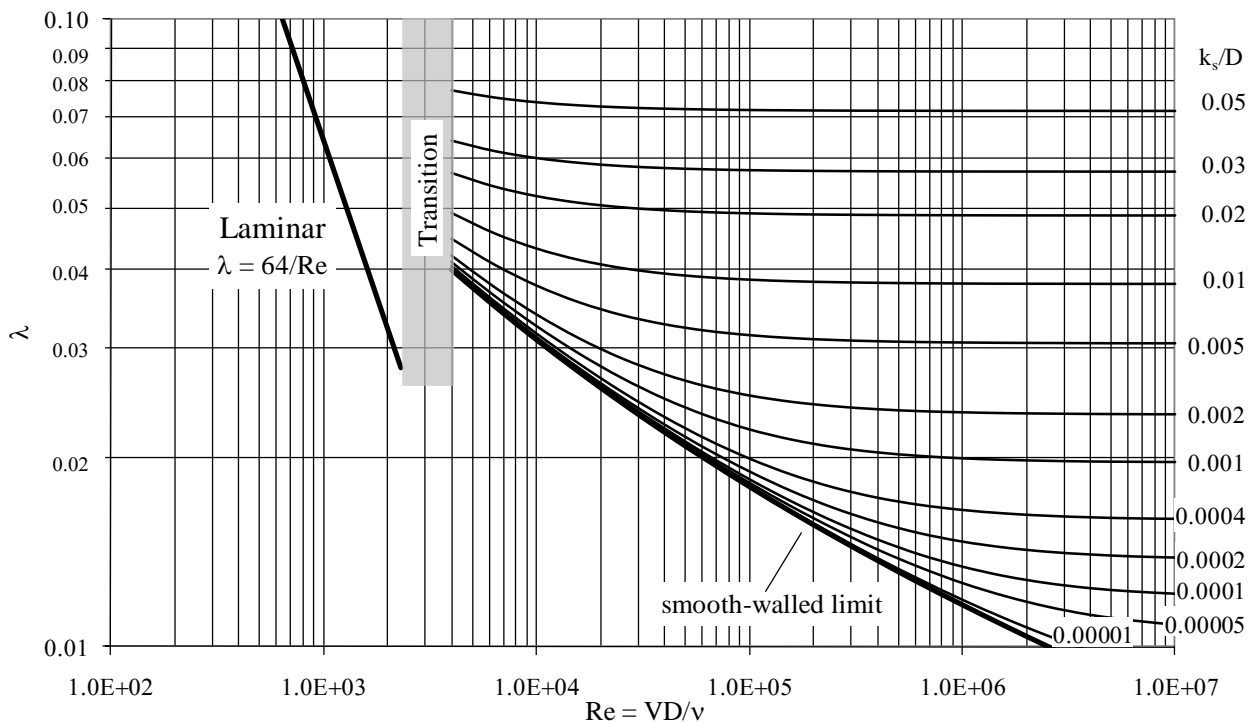
This is the main formula for the friction factor in turbulent flow. The main difficulty is that it is implicit (λ appears on both sides of the equation) and so must be solved iteratively. There are several explicit *approximations* to (13), accurate to within a few percent for realistic ranges of Reynolds number – see the references in Massey and White’s textbooks.

Equivalent Sand Roughness

For commercial pipes the pattern of surface roughness may be very different to that in the artificially-roughened surfaces of Nikuradse. Colebrook (1939) and Moody (1944) gathered data to establish effective roughness for typical pipe materials. Typical values of k_s are given in the Appendix.

Moody Chart

Graphical solutions of (13) exist. The most well known is the *Moody chart* (λ versus Re for various values of relative roughness k_s/D). The curves are just solutions of the Colebrook-White equation. My home-produced version is shown below.



1.6 Other Losses

Pipeline systems are subject to two sorts of losses:

- **wall-friction**, contributing a continuous fall in head over a large distance;
- **minor losses** due to abrupt changes in geometry; e.g. pipe junctions, valves, etc.

Each type of loss can be quantified using a *loss coefficient*, K , the ratio of pressure loss to dynamic pressure (or head loss to dynamic head):

$$\begin{aligned} \text{pressure loss} &= K \left(\frac{1}{2} \rho V^2 \right) \\ \text{head loss} &= K \left(\frac{V^2}{2g} \right) \end{aligned} \tag{14}$$

Typical values of K are given below.

Commercial pipe fittings (approximate)

Fitting	K
Globe valve	10
Gate valve – wide open	0.2
Gate valve – ½ open	5.6
90° elbow	0.9
Side outlet of T-junction	1.8

Entry/exit losses

Configuration	K
Bell-mouthed entry	0
Abrupt entry	0.5
Protruding entry	1.0
Bell-mouthed exit	0.2
Abrupt enlargement	1.0

Minor losses are a “one-off” loss, occurring at a single point. Frictional losses are proportional to the length of pipe, L , and, in the grand scheme of things, usually dominate. For long pipelines, minor losses are often ignored.

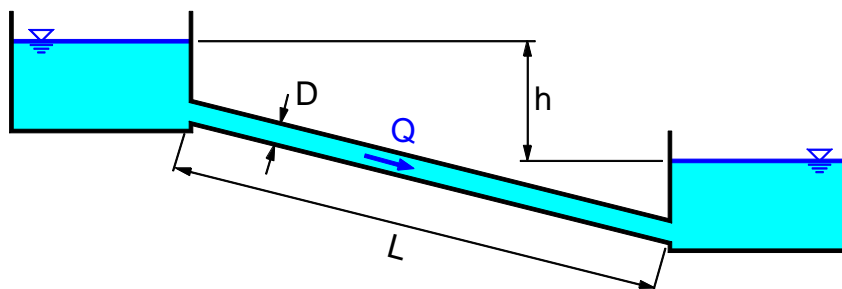
1.7 Pipeline Calculations

The objective is to establish the relationship between available head and quantity of flow.

$$\text{Available head, } H_1 - H_2 = \text{sum of head losses along the pipe} \quad (15)$$

The available head is the overall **drop** in head from start to end of the pipe, usually determined by still-water levels, sometimes supplemented by additional pumping head. Head losses are proportional to the dynamic head, $V^2/2g$. Fluid then flows through the pipe at precisely the right velocity, V , (or discharge, Q) that (15) is satisfied.

Pipe parameters are illustrated below. Although a reservoir is indicated at each end of the pipe, this is simply a diagrammatic way of saying “a point at which the total head is known”.



Typical pipeline problems are: given two of the following parameters, find the third:

head loss:	h
quantity of flow:	Q
diameter:	D

Other parameters: length L , roughness k_s , kinematic viscosity ν and minor loss coefficient K .

Calculations involve:

(1) *Head losses*

E.g. with friction factor, λ , and minor-loss coefficient, K :

$$h = \left(\lambda \frac{L}{D} + K\right) \left(\frac{V^2}{2g}\right) \quad (16)$$

(2) *Expressions for loss coefficients*

E.g. the Colebrook-White equation for the friction factor:

$$\frac{1}{\sqrt{\lambda}} = -2.0 \log_{10} \left(\frac{k_s}{3.7D} + \frac{2.51}{\text{Re} \sqrt{\lambda}} \right) \quad (17)$$

In most problems (16) and/or (17) must be solved iteratively. The exception is the calculation of Q when h and D are known (and minor losses neglected) because in this special case (Type 1 in the examples which follow) the Reynolds number can be expanded to give:

$$\frac{1}{\sqrt{\lambda}} = -2.0 \log_{10} \left(\frac{k_s}{3.7D} + \frac{2.51\nu}{D\sqrt{\lambda V^2}} \right) \quad (18)$$

If minor-loss coefficient $K = 0$ then the combination λV^2 can be found from (16) and hence λ can be found. Knowledge of both λV^2 and λ gives V and thence Q .

Inlet/Outlet Head

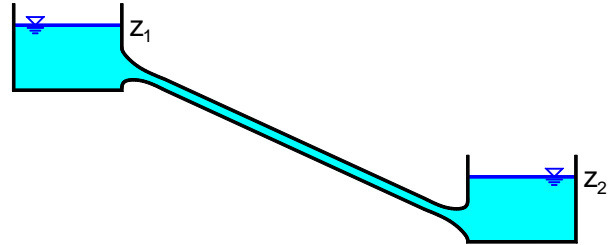
If there is a free surface in still water then both gauge pressure and velocity there are zero and so the total head equals the surface elevation: $H = z$. If, however, the pipe discharges to atmosphere as a free jet then the total head includes the dynamic head, $V^2/2g$.

If the discharge is to another reservoir, then (with a well-rounded exit):

$$H_1 = z_1, \quad H_2 = z_2$$

and the loss in head is just the difference in still-water levels:

$$h = z_1 - z_2$$



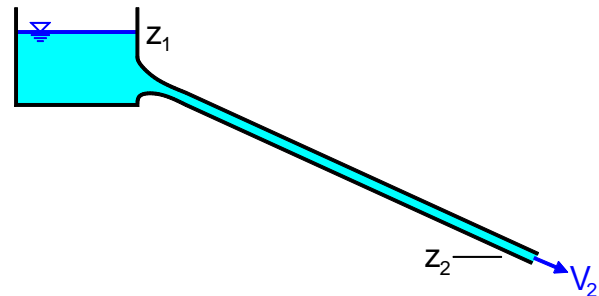
Alternatively, if the discharge is a free jet to atmosphere, then

$$H_1 = z_1, \quad H_2 = z_2 + V_2^2/2g$$

and the loss in head is

$$h = z_1 - z_2 - V_2^2/2g$$

Thus, in terms of piezometric head you could, if you preferred, treat this as an “exit loss” with coefficient 1.0.



The second case also applies if there is an **abrupt** exit into a tank, since flow separation means that the pressure in the jet leaving the pipe is essentially the hydrostatic pressure in the tank (piezometric head z_2) but there is still a dynamic head $V_2^2/2g$, whose energy is ultimately dissipated in the receiving tank. Again, this is equivalent to an exit loss coefficient 1.0. For long pipelines, however, this is usually negligible compared with the frictional losses.

Flow problem: diameter, D , and head difference, h , known; find the quantity of flow, Q .

Example. A pipeline 10 km long, 300 mm diameter and with roughness 0.03 mm conveys water from a reservoir (top water level 850 m AOD) to a water treatment plant (700 m AOD). Assuming that the reservoir remains full, and neglecting minor losses, estimate the quantity of flow. Take $\nu = 1.0 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$.

Solution.

List known parameters:

$$L = 10000 \text{ m}$$

$$D = 0.3 \text{ m}$$

$$h = 150 \text{ m}$$

$$k_s = 3 \times 10^{-5} \text{ m}$$

$$\nu = 1.0 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$$

Since D and h are known, the head-loss equation enables us to find λV^2 :

$$h = \lambda \frac{L}{D} \left(\frac{V^2}{2g} \right)$$

$$\Rightarrow \lambda V^2 = \frac{2gDh}{L} = \frac{2 \times 9.81 \times 0.3 \times 150}{10000} = 0.08829 \text{ m}^2 \text{ s}^{-2}$$

Rewriting the Colebrook-White equation:

$$\begin{aligned} \frac{1}{\sqrt{\lambda}} &= -2.0 \log_{10} \left(\frac{k_s}{3.7D} + \frac{2.51}{\text{Re} \sqrt{\lambda}} \right) \\ &= -2.0 \log_{10} \left(\frac{k_s}{3.7D} + \frac{2.51\nu}{D\sqrt{\lambda V^2}} \right) \\ &= -2.0 \log_{10} \left(\frac{3 \times 10^{-5}}{3.7 \times 0.3} + \frac{2.51 \times 1.0 \times 10^{-6}}{0.3\sqrt{0.08829}} \right) \\ &= 8.516 \end{aligned}$$

Hence,

$$\lambda = \frac{1}{8.516^2} = 0.01379$$

Knowledge of both λV^2 and λ gives

$$V = \sqrt{\frac{\lambda V^2}{\lambda}} = \sqrt{\frac{0.08829}{0.01379}} = 2.530 \text{ m s}^{-1}$$

Finally, the quantity of flow may be computed as velocity \times area:

$$Q = VA = V \left(\frac{\pi D^2}{4} \right) = 2.530 \times \frac{\pi \times 0.3^2}{4} = 0.1788 \text{ m}^3 \text{ s}^{-1}$$

Answer: quantity of flow = $0.179 \text{ m}^3 \text{ s}^{-1}$.

Head-loss problem: diameter, D , and quantity of flow, Q , known; find the head loss, h .

Example. The outflow from a pipeline is 30 L s^{-1} . The pipe diameter is 150 mm, length 500 m and roughness estimated at 0.06 mm. Find the frictional head loss along the pipe.

Solution.

List known parameters:

$$\begin{aligned}Q &= 0.03 \text{ m}^3 \text{ s}^{-1} \\L &= 500 \text{ m} \\D &= 0.15 \text{ m} \\k_s &= 6 \times 10^{-5} \text{ m} \\v &= 1.0 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}\end{aligned}$$

Inspect the head-loss equation:

$$h = \lambda \frac{L}{D} \left(\frac{V^2}{2g} \right)$$

We can get V from Q and D , but to find h we will require the friction factor.

First V :

$$V = \frac{Q}{A} = \frac{Q}{\pi D^2/4} = \frac{0.03}{\pi \times 0.15^2/4} = 1.698 \text{ m s}^{-1}$$

Inspect the Colebrook-White equation:

$$\frac{1}{\sqrt{\lambda}} = -2.0 \log_{10} \left(\frac{k_s}{3.7D} + \frac{2.51}{\text{Re} \sqrt{\lambda}} \right)$$

To use this we require the Reynolds number:

$$\text{Re} = \frac{VD}{v} = \frac{1.698 \times 0.15}{1.0 \times 10^{-6}} = 254700$$

Substituting values for k_s , D and Re in the Colebrook-White equation and rearranging for λ :

$$\lambda = \frac{1}{\left[2.0 \times \log_{10} \left(1.081 \times 10^{-4} + \frac{9.854 \times 10^{-6}}{\sqrt{\lambda}} \right) \right]^2}$$

Iterating from an initial guess, with successive values substituted into the RHS:

Initial guess: $\lambda = 0.01$

Successive iterations: $\lambda = 0.01841, 0.01784, 0.01787, 0.01787, \dots$

λ can then be substituted in the head-loss equation to derive h :

$$h = \lambda \frac{L}{D} \left(\frac{V^2}{2g} \right) = 0.01787 \times \frac{500}{0.15} \times \frac{1.698^2}{2 \times 9.81} = 8.753 \text{ m}$$

Answer: head loss = 8.75 m.

Sizing problem: quantity of flow, Q , and head difference, h , known; find the required diameter, D .

Example. A flow of $0.4 \text{ m}^3 \text{ s}^{-1}$ is to be conveyed from a headworks at 1050 m AOD to a treatment plant at 1000 m AOD. The length of the pipeline is 5 km. Estimate the required diameter, assuming that $k_s = 0.03 \text{ mm}$.

Solution.

List known parameters:

$$\begin{aligned} Q &= 0.4 \text{ m}^3 \text{ s}^{-1} \\ h &= 50 \text{ m} \\ L &= 5000 \text{ m} \\ k_s &= 3 \times 10^{-5} \text{ m} \\ \nu &= 1.0 \times 10^{-6} \text{ m}^2 \text{ s}^{-1} \end{aligned}$$

Before iterating, try to write D in terms of λ . From the head-loss equation:

$$h = \lambda \frac{L}{D} \left(\frac{V^2}{2g} \right) = \frac{\lambda L}{2gD} \left(\frac{Q}{A} \right)^2 = \frac{\lambda L}{2gD} \left(\frac{Q}{\pi D^2/4} \right)^2 = \frac{8LQ^2\lambda}{\pi^2 g D^5}$$

$$\Rightarrow D = \left(\frac{8LQ^2}{\pi^2 g h} \lambda \right)^{1/5}$$

Substituting values of Q , L and h gives a working expression (with D in metres):

$$D = (1.322 \lambda)^{1/5} \quad (*)$$

The Colebrook-White equation for λ is:

$$\frac{1}{\sqrt{\lambda}} = -2.0 \log_{10} \left(\frac{k_s}{3.7D} + \frac{2.51}{\text{Re} \sqrt{\lambda}} \right)$$

The Reynolds number can be written in terms of the diameter D :

$$\text{Re} = \frac{VD}{\nu} = \left(\frac{Q}{\pi D^2/4} \right) \frac{D}{\nu} = \frac{4Q}{\pi \nu D} = \frac{5.093 \times 10^5}{D}$$

Substituting this expression for Re we obtain an iterative formula for λ :

$$\lambda = \frac{1}{\left[2.0 \times \log_{10} \left(\frac{8.108 \times 10^{-6}}{D} + \frac{4.928 \times 10^{-6} D}{\sqrt{\lambda}} \right) \right]^2} \quad (**)$$

Iterate (*) and (**) in turn, until convergence.

Guess:	$\lambda = 0.01$	\Rightarrow	$D = 0.4210 \text{ m}$
Iteration 1:	$\lambda = 0.01293$	\Rightarrow	$D = 0.4432 \text{ m}$
Iteration 2:	$\lambda = 0.01276$	\Rightarrow	$D = 0.4420 \text{ m}$
Iteration 3:	$\lambda = 0.01277$	\Rightarrow	$D = 0.4421 \text{ m}$
Iteration 4:	$\lambda = 0.01277$	\Rightarrow	$D = 0.4421 \text{ m}$

Answer: required diameter = 0.442 m.

In practice, commercial pipes are only made with certain standard diameters and the next available larger diameter should be chosen.

Combined Pipe Friction and Minor Losses

In many circumstances, “minor” losses (including exit losses) actually contribute a significant proportion of the total head loss and must be included in the head-loss equation

$$h = \left(\lambda \frac{L}{D} + K \right) \frac{V^2}{2g}$$

An iterative solution in conjunction with the Colebrook-White equation is then inevitable, irrespective of the type of problem.

Example.

A reservoir is to be used to supply water to a factory 5 km away. The water level in the reservoir is 60 m above the factory. The pipe lining has roughness 0.5 mm. Minor losses due to valves and pipe fittings can be accommodated by a loss coefficient $K = 80$. Calculate the minimum diameter of pipe required to convey a discharge of $0.3 \text{ m}^3 \text{ s}^{-1}$.

Answer: 0.443 m

1.8 Energy and Hydraulic Grade Lines

Energy grade lines and hydraulic grade lines are graphical means of portraying the energy changes along a pipeline.

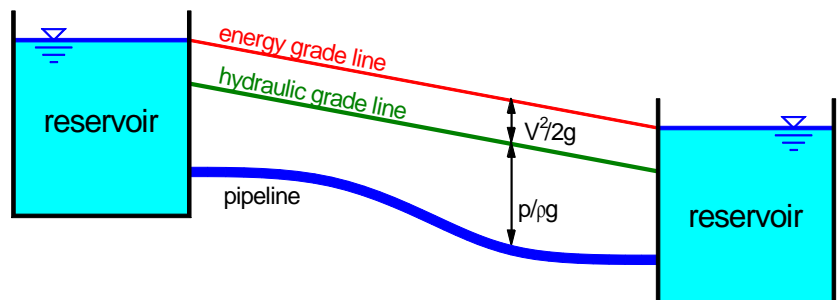
Three elevations may be drawn:

pipe centreline	z	geometric height
hydraulic grade line (HGL)	$\frac{p}{\rho g} + z$	piezometric head
energy grade line (EGL)	$\frac{p}{\rho g} + z + \frac{V^2}{2g}$	total head

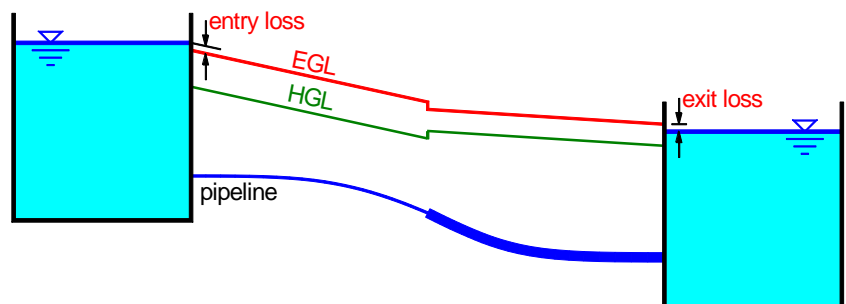
p is the gauge pressure (i.e. difference between the pressure and atmospheric pressure).

Illustrations

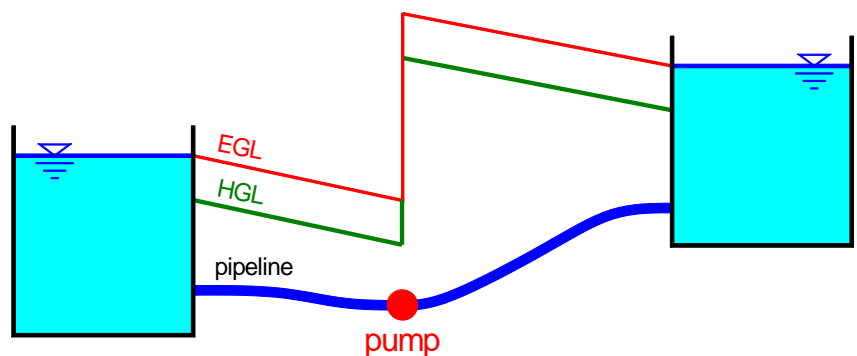
Pipe friction only



Pipe friction with minor losses (exaggerated), including change in pipe diameter.



Pumped system



Energy Grade Line (EGL)

- Shows the change in **total head** along the pipeline.
- Starts and ends at still-water levels.
- Steady downward slope reflects pipe friction.
Slope change if pipe radius changes; (frictional losses less at lower velocity).
Small discontinuities correspond to minor losses.
Large discontinuities correspond to turbines (loss of head) or pumps (gain of head).
- The EGL represents the maximum height to which water may be delivered at atmospheric pressure.

Hydraulic Grade Line (HGL)

- Shows the change in **piezometric head** along the pipeline.
- For pipe flow the HGL lies a distance $p/\rho g$ above the pipe centreline. Thus, the difference between pipe elevation and hydraulic grade line gives the static pressure, p . If the HGL drops below pipe elevation this means negative gauge pressures (i.e. less than atmospheric). This is generally undesirable since:
 - extraneous matter may be sucked into the pipe through any leaks;
 - for very negative gauge pressures, dissolved gases may come out of solution and cause cavitation damage.

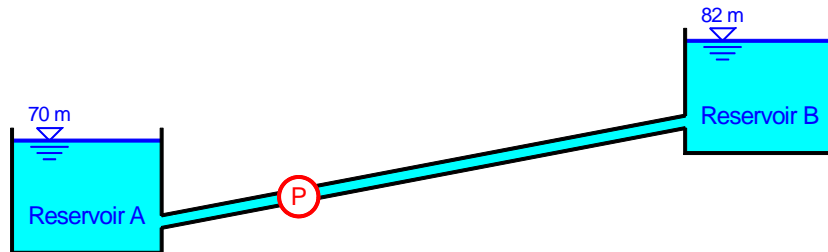
An HGL more than $p_{\text{atm}}/\rho g$ (≈ 10 m of water) below the pipeline is impossible.

- The HGL is the height to which the liquid would rise in a piezometer tube.
- For open-channel flows (as opposed to pipes), pressure is atmospheric (i.e. $p = 0$) at the surface. The HGL is then simply the height of the free surface.

The EGL is always higher than the HGL by an amount equal to the dynamic head $V^2/2g$. For uniform pipes (constant V), the two grade lines are parallel.

Example. (Exam 2016)

The two reservoirs illustrated are used for water storage and supply. The water levels in the reservoirs are constant and equal to 70 m AOD in the lower reservoir (Reservoir A) and 82 m AOD in the upper reservoir (Reservoir B). The reservoirs are connected by a 1.2 km long pipe with diameter $D = 200$ mm and wall roughness $k_s = 0.2$ mm. A pump is installed in the pipe as illustrated in the figure.

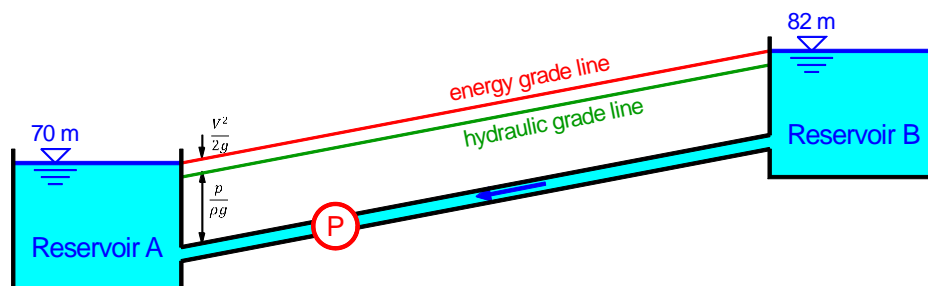


Neglecting minor losses,

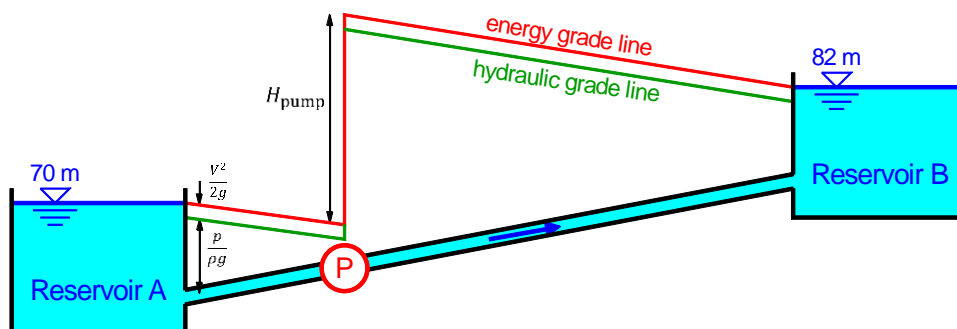
- sketch the qualitative behaviour of the energy and hydraulic grade lines between Reservoir A and Reservoir B if the system operates under gravity alone (i.e. without the pump);
- sketch the qualitative behaviour of the energy and hydraulic grade lines between Reservoir A and Reservoir B when the pump is operating and the flow direction is from Reservoir A to Reservoir B;
- find the pump head required to deliver a discharge of $0.025 \text{ m}^3 \text{ s}^{-1}$ to reservoir B.

Solution.

(a)



(b)



(c)

$$L = 1200 \text{ m}$$

$$D = 0.2 \text{ m}$$

$$k_s = 2 \times 10^{-4} \text{ m}$$

$$Q = 0.025 \text{ m}^3 \text{ s}^{-1}$$

$$(\nu = 1.0 \times 10^{-6} \text{ m}^2 \text{ s}^{-1})$$

From the flow rate, the bulk velocity is

$$V = \frac{\text{flow rate}}{\text{area}} = \frac{Q}{\pi D^2/4} = \frac{4 \times 0.025}{\pi \times 0.2^2} = 0.7958 \text{ m s}^{-1}$$

and the Reynolds number is

$$\text{Re} = \frac{VD}{\nu} = \frac{0.7958 \times 0.2}{1.0 \times 10^{-6}} = 159200$$

The pump head is required to provide the static lift ($h_s = 82 - 70 = 12 \text{ m}$) and overcome frictional losses (h_f).

Inspect the head-loss equation:

$$h_f = \lambda \frac{L V^2}{D 2g}$$

h_f will be known if we can find λ .

Rearranging the Colebrook-White equation:

$$\lambda = \frac{1}{\left[2.0 \log_{10} \left(\frac{k_s}{3.7D} + \frac{2.51}{\text{Re}\sqrt{\lambda}} \right)\right]^2} = \frac{1}{\left[2.0 \log_{10} \left(\frac{2 \times 10^{-4}}{3.7 \times 0.2} + \frac{2.51}{159200\sqrt{\lambda}} \right)\right]^2}$$

Hence,

$$\lambda = \frac{1}{\left[2.0 \log_{10} \left(2.703 \times 10^{-4} + \frac{1.577 \times 10^{-5}}{\sqrt{\lambda}} \right)\right]^2}$$

Iterating (from, e.g., 0.01) gives $\lambda = 0.02135$.

Hence,

$$h_f = \lambda \frac{L V^2}{D 2g} = 0.02135 \times \frac{1200}{0.2} \times \frac{0.7958^2}{2 \times 9.81} = 4.135 \text{ m}$$

The pump head required is

$$H_{\text{pump}} = h_s + h_f = 12 + 4.135 = 16.14 \text{ m}$$

Answer: 16.1 m.

1.9 Simple Pipe Networks

For all pipe networks the following basic principles apply:

- (1) continuity at junctions (total flow in = total flow out);
- (2) the head is uniquely defined at any point;
- (3) each pipe satisfies its individual *resistance law* (i.e. *head-loss vs discharge relation*):

$$h = \alpha Q^2$$

The last of these comes from the proportionality between head loss and dynamic head, i.e.

$$h = \left(\lambda \frac{L}{D} + K \right) \frac{V^2}{2g}, \quad \text{where} \quad V = \frac{Q}{\pi D^2/4}$$

λ is the friction factor and K is the sum of minor loss coefficients.

For hand calculations, α is often taken as a constant for each pipe. (A computer program would be able to take into account its slight variation with flow rate).

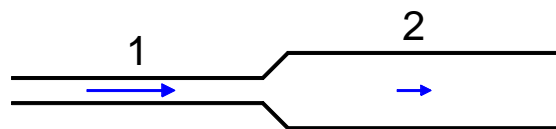
There is a useful analogy with electrical networks:

head, H	\leftrightarrow	potential, V
discharge, Q	\leftrightarrow	current, I
pipe	\leftrightarrow	resistor

However, the hydraulic equivalent of Ohm's law is usually non-linear:

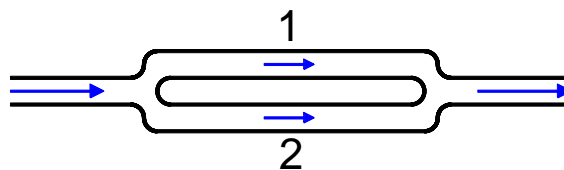
head loss $\Delta H \propto Q^2$	\leftrightarrow	potential difference $\Delta V \propto I$
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1.9.1 Pipes in Series



$Q_1 = Q_2$	same flow
$\Delta H = \Delta H_1 + \Delta H_2$	add the head changes

1.9.2 Pipes in Parallel



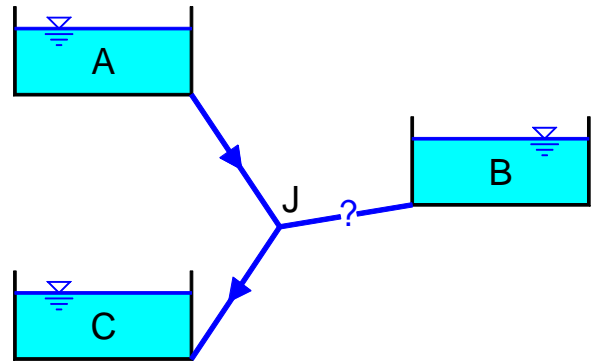
$\Delta H_1 = \Delta H_2$	same head change
$Q = Q_1 + Q_2$	add the flows

1.9.3 Branched Pipes – Single Junction

The simplest case is three pipes meeting at a single junction.

If the **flows** are known then the heads can be determined (relative to the head at one point) by calculating the head losses along each pipe.

If, however, the **heads** H_A , H_B and H_C are known (for example, from the water levels in reservoirs) then we have a classic problem known as the *three-reservoir problem*. (3 is just the smallest number that makes this non-trivial. Obviously, the n -reservoir problem can be solved in the same way, but with a proportionately larger amount of work.)



The head at J is adjusted (iteratively) to satisfy:

(a) the loss equation ($h = \alpha Q^2$) for each pipe; i.e:

$$|H_J - H_A| = \alpha_{JA} Q_{JA}^2 \quad \Leftrightarrow \quad Q_{JA} = \pm \sqrt{\frac{|H_J - H_A|}{\alpha_{JA}}}$$

$$|H_J - H_B| = \alpha_{JB} Q_{JB}^2 \quad \Leftrightarrow \quad Q_{JB} = \pm \sqrt{\frac{|H_J - H_B|}{\alpha_{JB}}}$$

$$|H_J - H_C| = \alpha_{JC} Q_{JC}^2 \quad \Leftrightarrow \quad Q_{JC} = \pm \sqrt{\frac{|H_J - H_C|}{\alpha_{JC}}}$$

(b) continuity at the junction J:

$$\text{net flow out of junction} = Q_{JA} + Q_{JB} + Q_{JC} = 0$$

Note the sign convention: Q_{JA} is the flow *from* J to A; it will be negative if the flow actually goes from A to J. **The direction of flow in any pipe is always from high head to low head.**

Although we consider only 3 reservoirs, the problem and its solution method clearly generalise to any number of reservoirs (and, in fact, to any number of junctions).

Solution Procedure

- (0) Establish the head-loss vs discharge (resistance) equations for each pipe;
- (1) guess an initial head at the junction, H_J ;
- (2) calculate flow rates in all pipes (from the head differences);
- (3) calculate net flow out of J;
- (4) as necessary, adjust H_J to reduce any flow imbalance and repeat from (2).

If the direction of flow in a pipe, say JB, is not obvious then a good initial guess is to set $H_J = H_B$ so that there is initially no flow in this pipe. The first flow-rate calculation will then establish

whether H_J should be lowered or raised and hence the direction of flow in this pipe.

Example.

Reservoirs A, B and C have constant water levels of 150, 120 and 90 m respectively above datum and are connected by pipes to a single junction J at elevation 125 m. The length (L), diameter (D), friction factor (λ) and minor-loss coefficient (K) of each pipe are given below.

Pipe	L (m)	D (m)	λ	K
JA	1600	0.3	0.015	40
JB	1600	0.2	0.015	25
JC	2400	0.25	0.025	50

(a) Calculate the flow in each pipe.

(b) Calculate the reading of a Bourdon pressure gauge attached to the junction J.

Solution.

First, prepare head-loss vs discharge relations for each pipe:

$$h = \left(\lambda \frac{L}{D} + K \right) \frac{V^2}{2g} \quad \text{where} \quad V = \frac{Q}{A} = \frac{Q}{\pi D^2/4}$$

$$\Rightarrow h = \left(\lambda \frac{L}{D} + K \right) \frac{8}{\pi^2 g D^4} \times Q^2$$

Substituting L , D , λ and K for each pipe we obtain the head-loss vs discharge relationships:

$$\text{Pipe AJ:} \quad |H_J - H_A| = 1224Q_{JA}^2 \quad \text{or} \quad Q_{JA} = \pm \sqrt{\frac{|H_J - 150|}{1224}}$$

$$\text{Pipe BJ:} \quad |H_J - H_B| = 7488Q_{JB}^2 \quad \text{or} \quad Q_{JB} = \pm \sqrt{\frac{|H_J - 120|}{7488}}$$

$$\text{Pipe CJ:} \quad |H_J - H_C| = 6134Q_{JC}^2 \quad \text{or} \quad Q_{JC} = \pm \sqrt{\frac{|H_J - 90|}{6134}}$$

The value of H_J is varied until the net flow out of J is 0.

- If there is net flow *into* the junction then H_J needs to be **raised**.
- If there is net flow *out* of the junction then H_J needs to be **lowered**.

After the first two guesses at H_J , subsequent iterations are guided by interpolation.

The working is conveniently set out in a table.

H_J (m)	Q_{JA} ($\text{m}^3 \text{s}^{-1}$) $= \pm \sqrt{\frac{ H_J - 150 }{1224}}$	Q_{JB} ($\text{m}^3 \text{s}^{-1}$) $= \pm \sqrt{\frac{ H_J - 120 }{7488}}$	Q_{JC} ($\text{m}^3 \text{s}^{-1}$) $= \pm \sqrt{\frac{ H_J - 90 }{6134}}$	Net flow out of J ($\text{m}^3 \text{s}^{-1}$) $= Q_{JA} + Q_{JB} + Q_{JC}$
120	-0.1566	0.0000	0.0699	-0.0867
140	-0.0904	0.0517	0.0903	0.0516
132.5	-0.1196	0.0409	0.0832	0.0045
131.8	-0.1219	0.0397	0.0825	0.0003

This is sufficient accuracy (0.0003/0.1219 or about 0.25%). The quantity of flow in each pipe is given in the bottom row of the table, with the direction implied by the sign.

(b) A Bourdon gauge measures **absolute** pressure. From the piezometric head at the junction:

$$H_J = \frac{p}{\rho g} + z$$

where p is the **gauge** pressure. Hence,

$$p = \rho g(H_J - z) = 1000 \times 9.81 \times (131.8 - 125) = 66708 \text{ Pa}$$

Taking atmospheric pressure as 101325 Pa, the **absolute** pressure is then

$$101325 + 66708 = 168033 \text{ Pa}$$

Answer: 1.68 bar.

1.10 Complex Pipe Networks (*Optional*)

1.10.1 Loop Method (Hardy-Cross, 1936)

Used for networks made up of a series of closed loops, where the external *flows* are known.

Basic Idea

Start with any flow satisfying continuity. Apply iterative *flow* corrections δQ until the net head change round each loop is 0.

Adopt a suitable sign convention (e.g. Q positive if clockwise) in each loop. The *signed* head loss for any particular pipe is then

$$h = s\alpha Q^2 \quad (19)$$

with the sign function s being $+1$ if Q is positive and -1 if Q is negative.

Initially, the net head loss round a closed loop probably won't be 0. To try to achieve this after perturbing the flow in all pipes of a loop by δQ require

$$\sum_{\text{loop}} s\alpha(Q + \delta Q)^2 = 0$$

where δQ is the same for every pipe in that loop. Expanding:

$$\sum s\alpha Q^2 + 2\left(\sum s\alpha Q\right)\delta Q + \left(\sum s\alpha\right)\delta Q^2 = 0$$

Neglecting the second-order δQ^2 term, and noting that $sQ = |Q|$, leads to a flow correction for this loop of

$$\delta Q = -\frac{\sum \alpha Q|Q|}{2\sum \alpha|Q|} \quad (20)$$

This update is applied successively to every pipe in the loop.

Algorithm

Divide the network into closed loops.

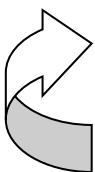
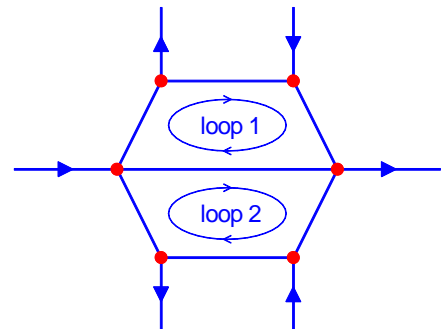
Start with any flow satisfying continuity.

For each loop in turn:

- calculate δQ using equation (20);
- update all pipes in this loop by δQ .

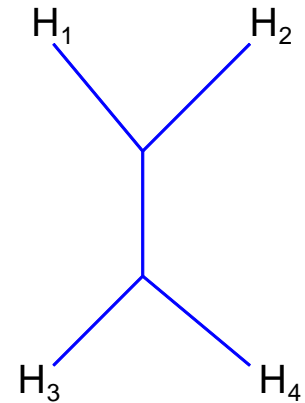
Repeat until the net head change around all loops is sufficiently small.

An example with two loops is given on the Example Sheet.



1.10.2 Nodal Method (Cornish, 1939)

Used for loops or branches where the external *heads* are known. In essence, this is a generalisation of the iterative technique for the *n*-reservoir problem with a more algorithmic head increment at each junction and several junctions whose head increments are connected by simultaneous equations.



Basic Idea

Start with guessed heads H_i at each internal junction and calculate the resulting flow in each pipe. Apply iterative *head* corrections δH_i so as to satisfy continuity at each junction.

As in the previous subsection, head changes at junctions i and j cause a change in the flow between them:

$$\delta H_i - \delta H_j = 2s_{ij}\alpha_{ij}Q_{ij}\delta Q_{ij} = 2\frac{H_i - H_j}{Q_{ij}}\delta Q_{ij} \quad (21)$$

where Q_{ij} is the flow rate from the i th node to the j th node, with appropriate sign.

Initially, the net outflow at the i^{th} junction won't be 0; to try to achieve this we aim to perturb the flow so that

$$\sum_{j \neq i} (Q_{ij} + \delta Q_{ij}) = 0, \quad i = 1, 2, 3, \dots$$

or

$$\sum_{j \neq i} Q_{ij} + \sum_{j \neq i} \frac{Q_{ij}}{2(H_i - H_j)} (\delta H_i - \delta H_j) = 0, \quad i = 1, 2, 3, \dots \quad (22)$$

Taken over all junctions i this gives a set of simultaneous equations for the δH_i .

A 2-junction example is given on the Example Sheet.

2. OPEN-CHANNEL FLOW

Flow in open channels (e.g. rivers, canals, guttering, ...) and partially-full closed conduits (e.g. sewers) is characterised by the presence of a free surface where the pressure is atmospheric.

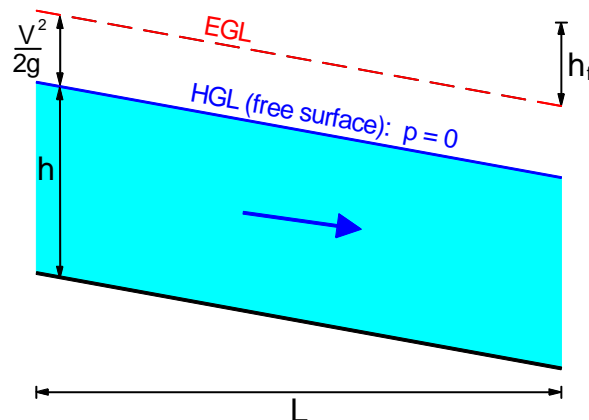
Unlike pipe flow, open-channel flow is always driven by gravity, not pressure.

	PIPE FLOW	OPEN-CHANNEL FLOW
Fluid:	LIQUIDS or GASES	LIQUIDS (free surface)
Driven by:	PRESSURE, GRAVITY or BOTH	GRAVITY (down slope)
Size:	DIAMETER	HYDRAULIC RADIUS
Volume:	FILLS pipe	Depends on DEPTH
Equations:	DARCY-WEISBACH (head loss) COLEBROOK-WHITE (friction factor)	MANNING'S FORMULA

2.1 Normal Flow

The flow is *uniform* if the velocity profile does not change along the channel. (This is at best an approximation for natural channels like rivers where the channel cross-section changes.)
The flow is *steady* if it does not change with time.

Steady uniform flow is called *normal flow* and the depth of water is called the *normal depth*. The normal depth, h , depends on the discharge, Q .

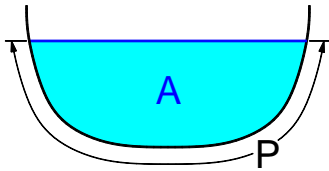


In normal flow, equal hydrostatic pressure forces at any cross-section mean no net pressure force. Hence, the downslope component of weight balances bed friction;

Note.

The following assumes the slope to be sufficiently small for there to be negligible difference between the depth h measured vertically (which determines the energy level) and that perpendicular to the bed of the channel (which determines the flow rate).

2.2 Hydraulic Radius and the Drag Law



In both open channels and partially-full pipes, wall friction occurs only along the *wetted perimeter*.

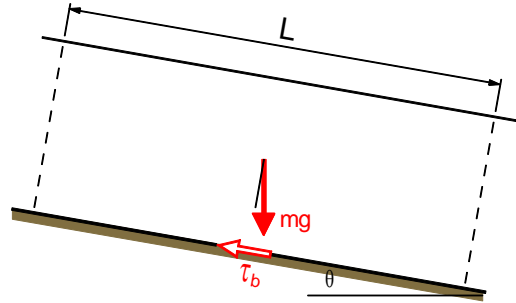
Let A be the cross-sectional area occupied by fluid and P the wetted perimeter.

For steady, uniform flow, the component of weight down the slope balances bed friction:

$$(\rho AL)g \sin \theta = \tau_b(PL)$$

where τ_b is the average bed stress. Hence,

$$\tau_b = \rho g \left(\frac{A}{P} \right) \sin \theta$$



Define:

Hydraulic radius

$$R_h \equiv \frac{A}{P} = \frac{\text{cross-sectional area}}{\text{wetted perimeter}} \quad (23)$$

Note that, in general, **the hydraulic radius depends on depth.**

Hence,

Normal flow relationship

$$\tau_b = \rho g R_h S \quad (24)$$

where S (= drop \div length) is the *slope*. (We have assumed $\tan \theta \approx \theta \approx \sin \theta$ for small angles.)

Examples.

(1) For a circular pipe running full,

$$R_h = \frac{A}{P} = \frac{\pi R^2}{2\pi R} = \frac{1}{2} R \quad (25)$$

i.e. for a full circular pipe, the hydraulic radius is half the geometric radius. (Sorry! Just one of those things!) As a result, it is common to define a *hydraulic diameter*, D_h , by

$$D_h = 4R_h$$

(2) For a rectangular channel of width b with water depth h ,

$$R_h = \frac{A}{P} = \frac{bh}{b + 2h} = \frac{h}{1 + 2h/b}$$

For a *wide* channel, $h/b \ll 1$, and hence

$$R_h = h$$

i.e. in a wide channel, R_h is equal to the depth of flow.

To progress we need an expression for the average bed stress τ_b .

2.3 Friction Laws – Chézy and Manning’s Formulae

From the balance of forces above:

$$\tau_b = \rho g R_h S$$

In principle, a (skin-)friction coefficient can be used to relate the (average) bed shear stress to the dynamic pressure. Hence,

$$c_f \left(\frac{1}{2} \rho V^2 \right) = \rho g R_h S \quad (26)$$

Friction factors $\lambda = 4c_f$ based on the Colebrook-White equation (using $4R_h$ as the hydraulic diameter) are unsatisfactory for open conduits because the shear stress is not constant on the wetted perimeter. Engineers use simpler empirical formulae due to Chézy⁹ and Manning¹⁰.

Rearranging equation (26) gives:

$$V^2 = \frac{2g}{c_f} R_h S$$

whence:

Chézy’s Formula:

$$V = C \sqrt{R_h S} \quad (27)$$

$C = \sqrt{2g/c_f}$ is Chézy’s coefficient. This gives the variation with slope for a particular channel, but it is not a helpful equation because C varies with channel roughness and hydraulic radius.

The most popular correlation for C is that of Manning who proposed, on the basis of a review of experimental data, that

$$C = R_h^{1/6} \times \text{function of roughness}$$

which he chose to write as

$$C = \frac{R_h^{1/6}}{n}$$

Combined with Chézy’s formula (27), this yields:

Manning’s Formula:

$$V = \frac{1}{n} R_h^{2/3} S^{1/2} \quad (28)$$

Very important.

Both Chézy’s C and Manning’s n are dimensional and depend on the units used. Typical values of n in metre-second units are given in the Appendix. Typical values for artificially-lined channels and natural water courses are $0.015 \text{ m}^{-1/3} \text{ s}$ and $0.035 \text{ m}^{-1/3} \text{ s}$ respectively.

⁹ Antoine Chézy (1718-1798); French engineer who carried out experiments on the canals in Paris.

¹⁰ Robert Manning (1816-1897); Irish engineer. Actually, if you live on the wrong side of the English Channel then what we call Manning’s equation is variously ascribed to Gauckler and/or Strickler.

2.4 Uniform-Flow Calculations

Assuming that the channel slope, shape and lining material are known, there are two main classes of problem:

(Type A - easy) Given the depth (h) determine the quantity of flow (Q)

Calculate:

- (1) cross-sectional area A and wetted perimeter P from geometry of channel;
- (2) hydraulic radius $R_h = A/P$.
- (3) average velocity from Manning's formula: $V = \frac{1}{n} R_h^{2/3} S^{1/2}$.
- (4) quantity of flow from velocity \times area: $Q = VA$.

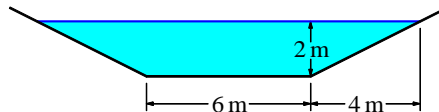
(Type B - harder) Given the quantity of flow (Q) determine the depth (h)

- (1) Follow the steps for Type A above to write algebraic expressions for, successively, Q in terms of depth h .
- (2) Invert the Q vs h relationship graphically or numerically; (e.g. by iteration or repeated trial).

Example. A smooth concrete-lined channel has trapezoidal cross-section with base width 6 m and sides of slope 1V:2H. If the bed slope is 1 in 500 and the normal depth is 2 m calculate the quantity of flow.

Solution.

We are given slope $S = 0.002$. From the Appendix, Manning's n is $0.012 \text{ m}^{-1/3} \text{ s}$.



Break the trapezoidal section into rectangular and triangular elements to obtain, successively:

$$\text{Area: } A = 6 \times 2 + 2 \times \frac{1}{2} \times 4 \times 2 = 20 \text{ m}^2$$

$$\text{Wetted perimeter: } P = 6 + 2 \times \sqrt{4^2 + 2^2} = 14.94 \text{ m}$$

$$\text{Hydraulic radius: } R_h = \frac{A}{P} = \frac{20}{14.94} = 1.339 \text{ m}$$

$$\text{Average velocity: } V = \frac{1}{n} R_h^{2/3} S^{1/2} = 3.727 \times R_h^{2/3} = 4.528 \text{ m s}^{-1}$$

$$\text{Quantity of flow: } Q = VA = 4.528 \times 20 = 90.56 \text{ m}^3 \text{ s}^{-1}$$

Answer: $90.6 \text{ m}^3 \text{ s}^{-1}$.

Example. For the channel above, if the quantity of flow is $40 \text{ m}^3 \text{ s}^{-1}$, what is the normal depth?

Solution. This time we need to leave all quantities as functions of height h . In metre-second units we have the following.

Area: $A = 6h + 2h^2$

Wetted perimeter: $P = 6 + 2\sqrt{5} h$

Hydraulic radius: $R_h \equiv \frac{A}{P}$

Average velocity: $V = \frac{1}{n} R_h^{2/3} S^{1/2} = \frac{\sqrt{S}}{n} \left(\frac{A}{P}\right)^{2/3}$

Quantity of flow: $Q = VA = \frac{\sqrt{S}}{n} \frac{A^{5/3}}{P^{2/3}} = 3.727 \frac{(6h + 2h^2)^{5/3}}{(6 + 2\sqrt{5} h)^{2/3}}$

Now simply try a few values of h to aim for $Q = 40 \text{ m}^3 \text{ s}^{-1}$:

h (m)	Q ($\text{m}^3 \text{ s}^{-1}$)
2	90.523
1	24.92
1.23	36.28
1.28	39.03
1.30	40.16

After the first two guesses, subsequent choices of h home in on the solution by interpolating/extrapolating from previous results.

Answer: $h = 1.30 \text{ m}$

Exercise. Use a spreadsheet or (better) write a computer program to solve this automatically.

2.5 Conveyance

Combining Manning's formula for average velocity ($V = \frac{1}{n} R_h^{2/3} S^{1/2}$) with expressions for hydraulic radius ($R_h = A/P$) and discharge ($Q = VA$) we obtain:

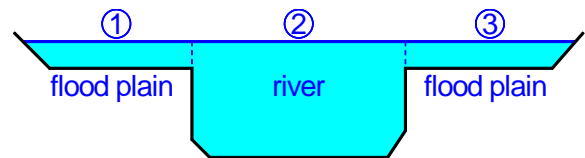
$$Q = KS^{1/2} \quad (29)$$

where

$$K = \frac{1}{n} \frac{A^{5/3}}{P^{2/3}} \quad (30)$$

K is a function of the channel geometry and the roughness of its lining. It is called the *conveyance* of the channel and is a measure of the channel's discharge-carrying capacity.

The primary use of K is in determining the discharge capacity of *compound* channels – for example river and flood plain. By adding the contribution to total discharge from individual components with different roughness:



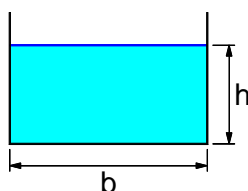
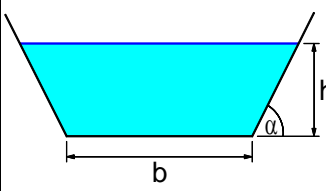
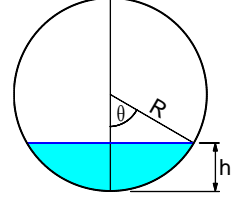
$$Q = \sum Q_i = \sum K_i S^{1/2} = K_{\text{eff}} S^{1/2}$$

the total conveyance is simply the sum of the separate conveyances:

$$K_{\text{eff}} = \sum K_i$$

2.6 Optimal Shape of Cross-Section

Expressions for A , P and R_h for important channel shapes are given below.

	rectangle	trapezoid	circle
			
cross-sectional area, A	bh	$bh + \frac{h^2}{\tan \alpha}$	$R^2(\theta - \frac{1}{2} \sin 2\theta)$
wetted perimeter, P	$b + 2h$	$b + \frac{2h}{\sin \alpha}$	$2R\theta$
hydraulic radius, R_h	$\frac{h}{1 + 2h/b}$	$h \frac{b + h/\tan \alpha}{b + 2h/\sin \alpha}$	$\frac{R}{2} \left(1 - \frac{\sin 2\theta}{2\theta}\right)$

The most *hydraulically-efficient* shape of channel is the one which can pass the greatest

quantity of flow for any given area. This occurs for the minimum hydraulic radius or, equivalently, for the minimum wetted perimeter corresponding to the given area.

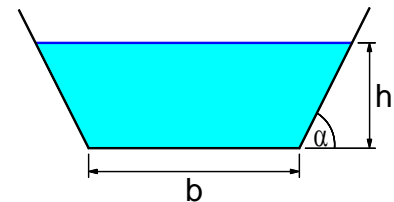
A semi-circle is the most hydraulically-efficient of all channel cross-sections. However, hydraulic efficiency is not the only consideration and one must also consider, for example, fabrication costs, excavation and, for loose granular linings, the maximum slope of the sides. Many applications favour trapezoidal channels.

Trapezoidal Channels

For a trapezoidal channel:

$$\text{cross-sectional area: } A = bh + \frac{h^2}{\tan \alpha}$$

$$\text{wetted perimeter: } P = b + \frac{2h}{\sin \alpha}$$



What depth of flow and what angle of side give maximum hydraulic efficiency?

To minimise the wetted perimeter for maximum hydraulic efficiency, we substitute for b in terms of the **fixed** area A :

$$P = b + \frac{2h}{\sin \alpha} = \left(\frac{A}{h} - \frac{h}{\tan \alpha} \right) + \frac{2h}{\sin \alpha} = \frac{A}{h} + h \left(\frac{2}{\sin \alpha} - \frac{1}{\tan \alpha} \right) \quad (31)$$

To minimise P with respect to water depth we set

$$\frac{\partial P}{\partial h} \equiv -\frac{A}{h^2} + \left(\frac{2}{\sin \alpha} - \frac{1}{\tan \alpha} \right) = 0$$

and, on substituting the bracketed term into the expression (31) for P , we obtain

$$P = \frac{2A}{h}$$

The hydraulic radius is then

$$R_h \equiv \frac{A}{P} = \frac{h}{2}$$

In other words, **for maximum hydraulic efficiency, a trapezoidal channel should be so proportioned that its hydraulic radius is half the depth of flow.**

Similarly, to minimise P with respect to the angle of slope of the sides, α , we set

$$\frac{\partial P}{\partial \alpha} \equiv h \left(\frac{-2}{\sin^2 \alpha} \cos \alpha + \frac{1}{\tan^2 \alpha} \sec^2 \alpha \right) = \frac{h}{\sin^2 \alpha} (1 - 2 \cos \alpha) = 0$$

This occurs when $\cos \alpha = \frac{1}{2}$. **The most efficient side angle for a trapezoidal channel is 60°.**

Substituting these results for h and α into the general expression for R_h one obtains $h/b = \sqrt{3}/2$; i.e. **the most hydraulically-efficient trapezoidal channel shape is half a regular hexagon.**

Circular Ducts

In similar fashion it can be shown that the maximum quantity of flow for a circular duct actually occurs when the duct is not full – in fact for a depth about 94% of the diameter (**Exercise**. Prove it; then try to explain in words why you might expect this).

Appendix

Material	k_s (mm)
Riveted steel	0.9 – 9.0
Concrete	0.3 – 3.0
Wood stave	0.18 – 0.9
Cast iron	0.26
Galvanised iron	0.15
Asphalted cast iron	0.12
Commercial steel or wrought iron	0.046
Drawn tubing	0.0015
Glass	0 (smooth)

Table 1. Typical roughness for commercial pipes (from White, 2021).

	n ($\text{m}^{-1/3}\text{s}$)
Artificial lined channels:	
Glass	0.01
Brass	0.011
Steel, smooth	0.012
painted	0.014
riveted	0.015
Cast iron	0.013
Concrete, finished	0.012
unfinished	0.014
Planed wood	0.012
Clay tile	0.014
Brickwork	0.015
Asphalt	0.016
Corrugated metal	0.022
Rubble masonry	0.025
Excavated earth channels:	
Clean	0.022
Gravelly	0.025
Weedy	0.03
Stony, cobbles	0.035
Natural channels:	
Clean and straight	0.03
Sluggish, deep pools	0.04
Major rivers	0.035
Floodplains:	
Pasture, farmland	0.035
Light brush	0.05
Heavy brush	0.075
Trees	0.15

Table 2. Typical values of Manning's n (from White, 2021).