Non-Embeddable Right LCM Semigroups

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Monoids

Let M be a monoid. We say that:

Definition

- M is *left cancellative* if $ca = cb \implies a = b$ for all $a, b, c \in M$.
- M is *right cancellative* if $ac = bc \implies a = b$ for all $a, b, c \in M$.
- M is *cancellative* if M is left cancellative and right cancellative.

For any $p \in M$, we may define a map $L_p : M \to M$ by $m \mapsto pm$. Then equivalently...

Definition / Lemma

A monoid M is *left cancellative* if and only if L_p is injective for all $p \in M$.

Definition

A monoid M is an *inverse monoid* if for all $m \in M$, there exists a unique $m' \in M$ such that

$$mm'm = m \text{ and } m'mm' = m'.$$

Any inverse monoid may be realised as partial isometries on a Hilbert space whose source and range projections commute.

Right Ideals and Right LCM

Definition

Let M be a monoid, $p \in M$. The *right ideal of M generated by p* is $pM := \{pm : m \in M\}$.

Definition

M is called *right LCM* if both:

1. M is a left cancellative monoid;
2. For any elements $p, q \in M$, either $pM \cap qM = \varnothing$, or $pM \cap qM = rM$ for some $r \in M$.

Examples

- Groups.
- Free [Commutative] Monoids.
- Nica's *Quasi-lattice ordered semigroups*.
- Dehornoy and Wehrung's M_B *Interval Monoid* of $B(4)$: $\mathrm{Mon}\langle a, b, c, d, e, f, g, h, i, j, k, l, O, P, Q, R, S, T, U, V, W, X, Y, Z \mid dO = eU, gQ = hU, aP = cS, dP = fW, jS = kW, bR = cT, gR = iY, jT = lY, eV = fX, hV = iZ, kX = lZ\rangle$.

A Potted History of Right LCM and C^*-algebras 1

X. Li (2012): Semigroup C^*-algebras and amenability of semigroups
Let M be a left cancellative monoid. Recall $L_p : M \to M$, $m \mapsto pm$.

Definition

The *left inverse hull* of M is the monoid $I_l(M) := \mathrm{Inv}\langle L_p : p \in M \rangle$.

Definition

The *full semigroup C^*-algebra of M* is $\mathrm{C}^*(M) := \mathrm{C}^*(I_l(M))$.

Theorem (Li, 2012)

Let M be left cancellative. Let $\mathcal{J}_M$ be the set of constructable right ideals of M. If M is right LCM, then

$$\mathrm{C}^*(M) \cong \mathcal{D} \rtimes_\alpha M$$

where

$$\mathcal{D} = \mathrm{C}^*(\{e_\mathcal{I} : \mathcal{I} \in \mathcal{J}_M\}) \text{ and } \alpha_p : \mathcal{D} \to \mathcal{D}, e_\mathcal{I} \mapsto e_{p\mathcal{I}}.$$

A Potted History of Right LCM and C^*-algebras 2

X. Li (2012): Semigroup C^*-algebras and amenability of semigroups
Let M be a left cancellative monoid. For each $m \in S$, define $V_m : \ell^2(S) \to \ell^2(S)$ via linearly extending $V_m(\delta_s) := \delta_{ms}$. The map $\lambda_M : m \mapsto V_m$ is called the *left regular representation* of M.

Lemma / Definition (Li, 2012)

Let M be a left cancellative monoid. Then V_m is an isometry for all $m \in M$, and the (*left reduced*) *semigroup* C^**-algebra of* M is

$$\mathrm{C}^*_\lambda(M) := \mathrm{C}^*(\{\lambda_M(m) : m \in M\}).$$

N. Brownlowe, N. Larsen, N. Stammeier (2017): On C^*-algebras associated to right LCM semigroups

Theorem (Brownlowe, Larsen, Stammeier, 2017)

Suppose M is right LCM. Then $\mathcal{D} \cong \overline{\mathrm{span}}\{V_m V_m^* : m \in M\}$.
If in addition M is group-embeddable and $\langle\langle$more nice conditions$\rangle\rangle$ then $\mathrm{C}^*(M) \cong \mathrm{C}^*_\lambda(M)$.

A Potted History of Right LCM and C^*-algebras 3

K.A. Brix, C. Bruce, A. Dor-On (2024): Normal Coactions extend to the C^*-envelope
Let $\mathfrak{C}$ be a left cancellative small category (e.g. monoid).

Definition (Informal)

The C^* envelope $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$ of an operator algebra $\mathcal{A}$ is the smallest C^*-algebra containing an isomorphic copy of $\mathcal{A}$.

Definition

Let Ω be the set of characters on $\mathcal{J}_{\mathfrak{C}}$. The left inverse hull $I_l(\mathfrak{C})$ acts on Ω: each $s \in I_l(\mathfrak{C})$ defines a partial homeomorphism

$$\Omega(\mathrm{dom}(s)) \to \Omega(\mathrm{im}(s)), \chi \mapsto \chi' \text{ where } \chi' : X \mapsto \chi(s'(X \cap \mathrm{im}(s))).$$

If $I_l(\mathfrak{C}) \ltimes \Omega$ is Hausdorff, then the *reduced boundary quotient* C^**-algebra* is

$$\partial \mathrm{C}^*_\lambda(\mathfrak{C}) := \mathrm{C}^*_\lambda\left(I_l(\mathfrak{C}) \ltimes \overline{\Omega_{\max}}\right).$$

A Potted History of Right LCM and C^*-algebras 4

Let $\mathfrak{C}$ be a left cancellative small category (e.g. monoid).

Definition

The *operator algebra* of $\mathfrak{C}$ is $\mathcal{A}_\lambda(\mathfrak{C}) := \overline{\mathrm{alg}}(\{\lambda_{\mathfrak{C}}(m) : m \in \mathfrak{C}\})$.

Theorem (Brix, Bruce, Dor-On, 2024)

If $\mathfrak{C}$ is groupoid-embeddable, then $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}_\lambda(\mathfrak{C})) \cong \partial \mathrm{C}^*_\lambda(\mathfrak{C})$.
If M is a cancellative, right LCM monoid, then $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}_\lambda(M)) \cong \partial \mathrm{C}^*_\lambda(M)$.

Note M does not have to be group-embeddable! (Not all monoids are group-embeddable)

Question

What are some cancellative, right LCM monoids which are not group-embeddable?

Theorem (Dehornoy, Wehrung, 2017)

The monoid M_B is cancellative, right LCM and not group-embeddable.

Any more...? **M. Edwardes, DH. (2024): A collection of cancellative, right LCM, not group-embeddable monoids**

Malcev Conditions

Define the formal $\mathcal{I}_n$ *Malcev implication as*:

$$\left\{\begin{aligned} da &= A_1C_1 \\ A_1D_1 &= A_2C_2 \\ &\vdots \\ A_{n-1}D_{n-1} &= A_nC_n \\ A_nD_n &= db \\ cb &= B_nD_n \\ B_nC_n &= B_{n-1}D_{n-1} \\ &\vdots \\ B_2C_2 &= B_1D_1 \end{aligned}\right\} \implies B_1C_1 = ca.$$

Theorem (Malcev, '40)

Let S be a semigroup. If the $\mathcal{I}_n$ Malcev implication is not satisfied for some assignment of S-elements to variables $\{a, b, c, d, A_i, B_i, C_i, D_i\}$, then S is not group-embeddable.

Constructing $\mathcal{M}_n$

Let X_n be the set $\{a, b, c, d, A_1, \ldots, A_n, B_1, \ldots, B_n, C_1, \ldots, C_n, D_1, \ldots, D_n\}$.
Let ρ_n be the set

$$\{da = A_1C_1,\ A_1D_1 = A_2C_2,\ \ldots,\ A_{n-1}D_{n-1} = A_nC_n,$$

$$A_nD_n = db,\ cb = B_nD_n,\ B_nC_n = B_{n-1}D_{n-1},\ \ldots,\ B_2C_2 = B_1D_1\}.$$

Definition

For any $n \geq 1$, define the monoid $\mathcal{M}_n := \mathrm{Mon}\langle X_n \mid \rho_n\rangle$.

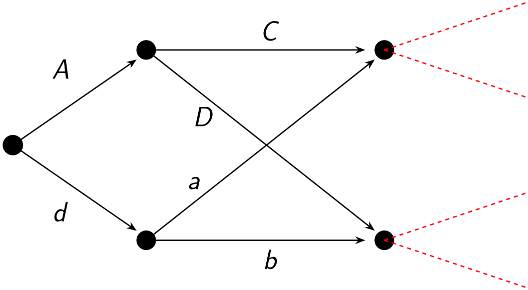
Theorem (Edwardes, H., 2024)

For any $n \geq 1$, $\mathcal{M}_n$ is cancellative and not group-embeddable.

Right LCM?

Theorem (Edwardes, H., 2024)

The monoid $\mathcal{M}_1 = \langle a, b, c, d, A, B, C, D \mid da = AC, db = AD, cb = BD \rangle$ is not right LCM.



Theorem (Edwardes, H., 2024)

The monoid $\mathcal{M}_1$ is 2-aligned. For $n \geq 2$, $\mathcal{M}_n$ is right LCM.

What's next?

Open Question 1

If M is a cancellative **finitely-aligned** monoid, is $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}_\lambda(M)) \cong \partial \mathrm{C}^*_\lambda(M)$?

Open Question 2

Can we create other cancellative, right LCM monoids which are not group-embeddable via other Malcev implications? Can we classify "good" Malcev implications?

Open Question 3

Can we obtain Dehornoy and Wehrung's M_B from such a construction?

Open Question 3.5

Is $\mathcal{M}_n$ constructable via an interval monoid construction?

Thank you and References

Thank you!

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