# Pretzel Monoids

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## Left Adequate Monoids

**Definition.** A *monoid* is a set equipped with an associative, binary operation, and containing a (two-sided) identity element.

**Definition.** An element $e$ of a monoid is called *idempotent* if $e^2 = e$.

**Definition.** On any monoid $M$, the equivalence relation $\mathcal{R}^*$ is defined by $a\mathcal{R}^*b$ if and only if

$$xa = ya \iff xb = yb \text{ for all } x, y \in M.$$

"*Elements are $\mathcal{R}^*$-related iff they 'share' right-cancellativity properties*".

**Definition.** A monoid $M$ is called *left adequate* if

1. Every $\mathcal{R}^*$-class contains an idempotent;
2. Idempotents commute, i.e. $ef = fe$ for idempotents $e, f$.

**Examples.** Groups, inverse monoids, left ample monoids, right cancellative monoids, free monoids ...

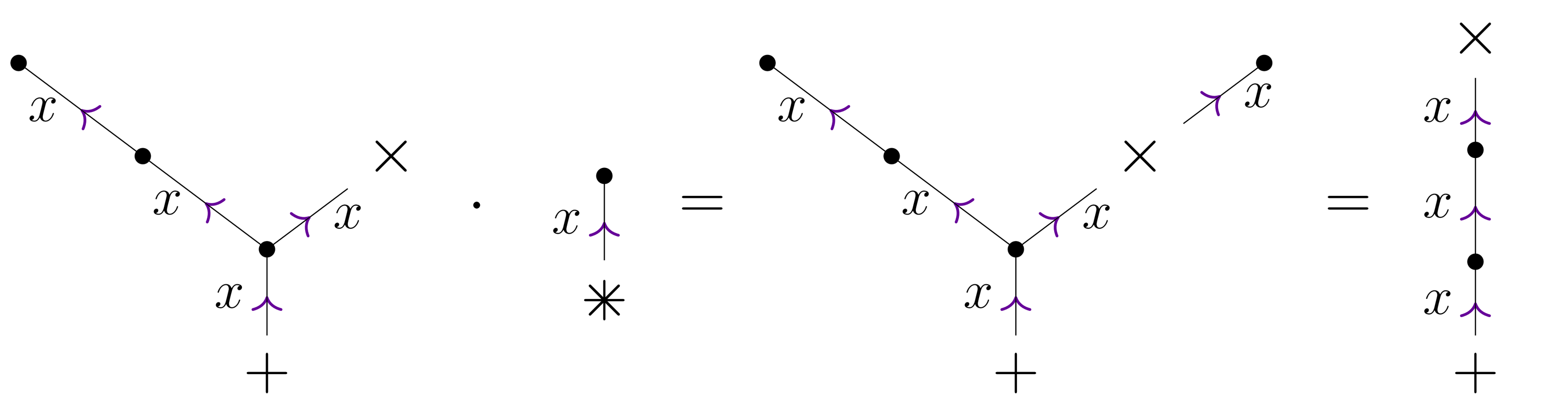
## Free Left Adequate Monoids and $X$-trees

**Theorem [Kambites, 2011].** Elements of the free left adequate monoid generated by a set $X$ are (isomorphism types of) directed, $X$-edge-labelled trees $T$, with two defined vertices called *start* ($+$) and *end* ($\times$), such that:

1. There is a path from the start vertex to every other vertex;
2. The graph admits no non-trivial *retractions*, that is an idempotent graph morphism $T \to T$ which fixes $+$ and $\times$.

The multiplication $ST$ of trees $S$ and $T$ is given by gluing $T$ to $S$ start-to-end, then retracting.

**Example.**



**Remarks.** The idempotents are trees with identified start and end. The idempotent $\mathcal{R}^*$-related to any tree is the tree obtained by moving the end to the start (then retracting).

## The Goal

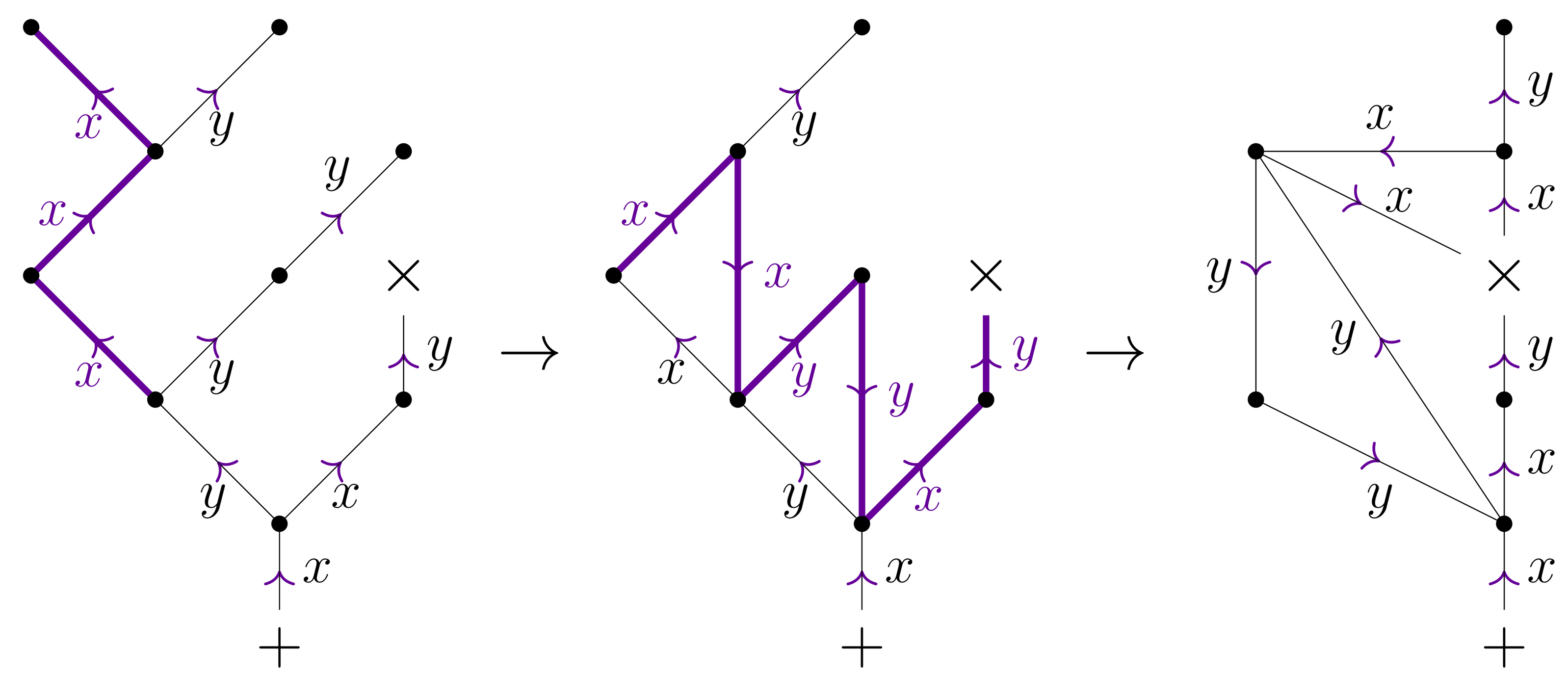
Can we generalise combinatorial descriptions from $E$-unitary inverse semigroup theory into left adequate land? Can we give geometric descriptions of certain *presentations* of left adequate monoids, or define what the generalisation of $E$-unitary should be?

## Idempath Identification

Fix a set $X$ and an $X$-generated right cancellative monoid $C$ (or even group).

**Definition.** An *idempath* in an $X$-labelled digraph $\Gamma$ is a path in $\Gamma$ labelled by a word $x_1x_2\cdots x_n$ which is equal to the identity in $C$. We take the empty path to be an idempath. An *idempath identification* on $\Gamma$ is the process of 'cycling up' an idempath, i.e. merging the initial and terminal vertex of the idempath.

**Example.** Take $X = \{x, y\}$ and $C = \mathbb{Z}_3 \times \mathbb{Z}_3 = \langle x, y \rangle$. Note that $xxx =_C yyy =_C xxyyxy =_C 1$.



**Lemma 1 [H., Kambites, Szakács, 2024].** Given a tree $T \in \mathrm{FLAd}(X)$, there exists a unique graph (up to isomorphism) obtainable by sequentially performing all non-trivial idempath identifications (in any order) to $T$.

## Pretzels

**Definition.** Given any tree $T \in \mathrm{FLAd}(X)$, perform the following:

1. Idempath identify as far as possible...
2. ...then retract anything in the result which can retract.

We call the (uniquely obtained) result the *pretzel* of $T$, denoted $\overline{\overline{T}}$.

## Margolis-Meakin Expansions

**Theorem [Margolis, Meakin, 1989].** Let $G$ be an $X$-generated group. Let $\mathcal{M}(G;X)$ be the set of pairs $(\Gamma, g)$ where $\Gamma$ is a finite connected subgraph of $\mathrm{Cay}(G;X)$ containing $1$ and $g$ as vertices. Define a multiplication on $\mathcal{M}(G;X)$ by $(\Gamma, g)(\Delta, h) = (\Gamma \cup g \cdot \Delta, gh)$ where $G$ acts on subgraphs of $\mathrm{Cay}(G;X)$ by translation. Then $\mathcal{M}(G;X)$ is the initial object in the category of $X$-generated $E$-unitary inverse monoids with maximal group image $G$. Moreover,

$$\mathcal{M}(G;X) \cong \mathrm{Inv}\langle X \mid w = w^2 \text{ for } w \in X^* \text{ with } w =_G 1\rangle.$$

## Results

Let $\mathcal{PT}(C;X) = \left\{\overline{\overline{T}} \mid T \in \mathrm{FLAd}(X)\right\}$. Define a multiplication $\Gamma \cdot \Delta$ on $\Gamma, \Delta \in \mathcal{PT}(C;X)$ by gluing $\Delta$ to $\Gamma$ start-to-end, performing all idempath identifications in the sense of Lemma 1, then retracting.

**Theorem 2 [H., Kambites, Szakács, 2024].** This multiplication is well-defined, associative, and $(\mathcal{PT}(C;X), \cdot)$ is a left adequate monoid. Moreover, $\mathcal{PT}(C;X)$ is the initial object in the category of $X$-generated left adequate monoids with maximal right cancellative image $C' = \mathrm{Canc}\langle X \mid w = 1 \text{ for } w \in X^* \text{ with } w =_C 1\rangle$, with morphisms the *idempotent-pure* $(2,1,0)$-morphisms.

**Theorem 3 [H., Kambites, Szakács, 2024].** For any $C = \mathrm{RC}\langle X\rangle$,

$$\mathcal{PT}(C;X) \cong \mathrm{LAd}\langle X \mid w = w^2 \text{ for } w \in X^* \text{ with } w =_C 1\rangle.$$

Pretzel monoids are one analogue of Margolis-Meakin expansions. It remains open if there are alternative ways to define left adequate expansions of $C$, perhaps with maximal right cancellative image $C$.

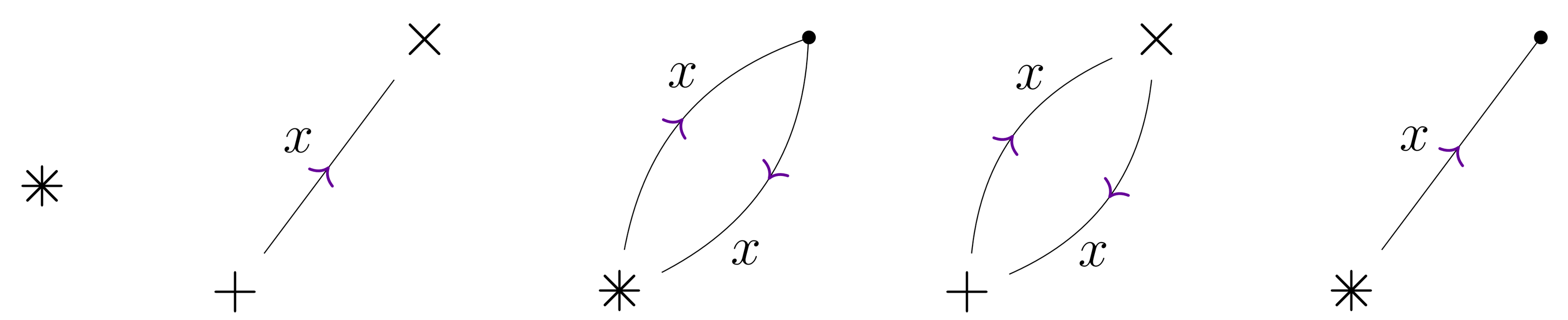


**Figure.** The 5 elements of $\mathcal{PT}(\mathbb{Z}_2; x)$.

## References

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[3] S. W. Margolis and J. C. Meakin. "*E-unitary inverse monoids and the Cayley graph of a group presentation*". In: J. Pure Appl. Algebra 58(1), pp. 45–76, 1989.

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