Pretzel Monoids:
A Dive into Geometric Semigroup Theory

Daniel Heath
(He/Him)
daniel.heath-2@manchester.ac.uk

University of Manchester
Joint work with Mark Kambites and Nóra Szakács

ITMAIA, University of Exeter
February 21–23, 2024

MANCHESTER 1824
The University of Manchester

Geometric Group Theory

A key goal of *Geometric Group Theory* is to understand properties and structure of *groups* using the geometry of their *Cayley Graphs*. E.g. *Hyperbolic Groups*[1].

Theorem (Gromov, 1987)

Hyperbolic groups have decidable word problem.

Geometric group theory emerged as a stand-alone field of study following Gromov's work.

Question

What techniques from Geometric Group Theory can generalise to *semigroups*?

Subquestion

What even is a *semigroup*?

[1] M. Gromov, "Hyperbolic groups", 1987.

Semigroups and Monoids

Definition

A *semigroup* $(S, \cdot)$ is a set S equipped with an associative binary operation $\cdot$.
A *monoid* is a semigroup which contains a (necessarily unique) identity element $\mathbf{1}$.

Examples of Monoids:

- Groups.
- Rings under their multiplication.
- $\mathbb{N}_0$ with $+$.
- All finite-length words over an alphabet X under concatenation.
  E.g. $X = \{x, y\}$. Then elements include $\epsilon, xxy, xyx, xxyxyx$ etc.
  This is the *free monoid over* X.

Examples of Semigroups:

- Monoids, Groups.
- $\varnothing$ (with any multiplication you like!)
- $\mathbb{N}$ with $+$.
- All finite-length, **non-empty** words over an alphabet X under concatentation.
  This is the *free semigroup over* X.

The $\mathcal{R}^*$ relation and left adequacy

Definition

Given a monoid M, define an equivalence relation $\mathcal{R}^*$ on M by $a\mathcal{R}^*b$ if and only if

$$\forall x, y \in M, \quad xa = ya \iff xb = yb.$$

Think: "*Elements are $\mathcal{R}^*$-related iff they 'share' right-cancellativity properties*".

Definition

An element e of a monoid is called *idempotent* if $e^2 = e$.

Definition

A monoid is called *left adequate* if:

1. Every $\mathcal{R}^*$-class contains a unique idempotent.
2. The idempotents of M commute with each other ($ef = fe$).

All groups are left adequate monoids (though not all left adequate monoids are groups)!

Why look at this class in particular?

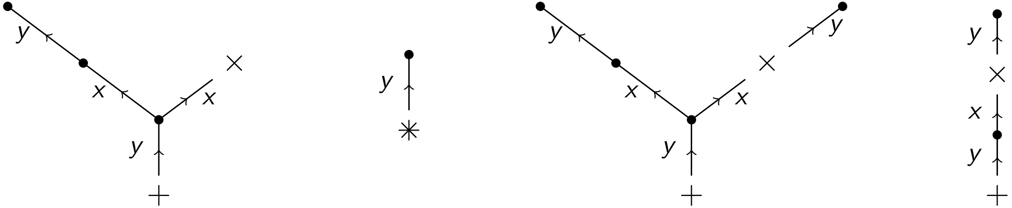
Free left adequate monoids (FLAds) have a geometric interpretation[2]!

Theorem (Kambites, 2009)

Elements of the free left adequate monoid generated by a set X may be treated as directed, X-edge-labelled trees, with two defined vertices called start $(+)$ and end $(\times)$, such that:

1. *There is a directed path from the start vertex to every other vertex.*
2. *No branches of the tree can be 'completely folded in', where we always fix the start/end vertices.*

The multiplication ST of trees S and T is given by gluing T to S start-to-end, then folding in any branches we can.



[2] M. Kambites, *Free left and right adequate semigroups*, 2009.

The Goal

Fact 1

$T \in \mathrm{FLAd}(X)$ is idempotent $\iff$ T has identified start and end vertex.

Fact 2

The unique idempotent $\mathcal{R}^*$-related to the tree T is the tree T with endpoint moved to the start (and possibly folded).

Fact 3

Any X-generated left adequate monoid is a quotient of $\mathrm{FLAd}(X)$.

Can we similarly describe non-trivial presentations of left adequate monoids? (**Hard**)

Pretzels!

Fix a set X and an X-generated group G.[3]

Definition

An *idempath* in an X-labelled digraph Γ is a directed path in Γ labelled by a word $x_1 x_2 \cdots x_n$ which is equal to the identity in G. We take the empty path with label ϵ to have $\epsilon =_G 1$.
An *idempath identification* in Γ is the process of 'looping up' an idempath.

Lemma (H., Kambites, Szakács, 2024)

Given a tree $T \in FLAd(X)$, there exists a unique graph obtainable by sequentially performing all non-trivial idempath identifications (in any order) to T.

Definition

Given any tree $T \in \mathrm{FLAd}(X)$, perform the following:

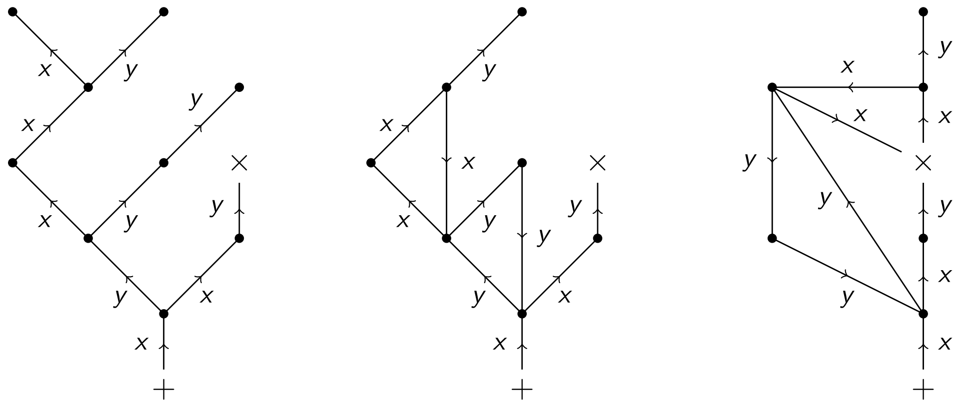
1. Idempath identify as far as possible...
2. ...then fold anything in the result which can fold.

We call the (uniquely obtained) result the *pretzel* of T, denoted $\overline{\widetilde{T}}$.

[3] D. Heath, M. Kambites, N. Szakács, *Pretzel Monoids*, 2024 (to appear).

An Example

Take $X = \{x, y\}$ and $G = \mathbb{Z}_3 \times \mathbb{Z}_3$. (so words equalling 1 include xxx, yyy, $xxyyxy$,...)



Gluing

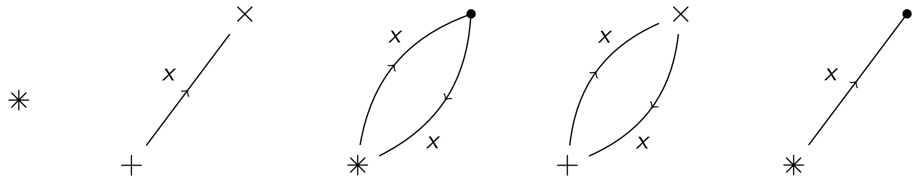
Take two trees S and T in $\mathrm{FLAd}(X)$ and pretzel-ify them w.r.t G.
Define a multiplication on pretzels as follows:

1. Glue $\overline{\widetilde{T}}$ to $\overline{\widetilde{S}}$, start-to-end.
2. Pretzel-ify the result (note that new idempaths could have been created!).

Theorem (H., Kambites, Szakács, 2024)

This multiplication on pretzels is well-defined and associative.
Under this multiplication, the set of all pretzels (w.r.t X and G) is a left adequate monoid.
The unique idempotent in the $\mathcal{R}^*$-class of a pretzel Γ is Γ with the endpoint moved to the start (and possibly folded).

Denote this left adequate monoid by $\mathcal{PT}(G;X)$. For example, the 5 pretzels of $\mathcal{PT}(\mathbb{Z}_2;x)$ are:



Properties of Pretzels

Properties

1. A pretzel Γ is idempotent in $\mathcal{PT}(G;X) \iff \Gamma$ has identified start and end vertex.
2. Any pretzel Γ is a tree of strongly connected subgraphs of $\mathrm{Cay}(G;X)$, connected via single edges (which has no non-trivial idempaths or folds).
3. $\mathcal{PT}(G;X)$ is X-generated (as a left adequate monoid).
4. G is finite $\iff \mathcal{PT}(G;X)$ is finite.
5. For $n \geq 1$, $|\mathcal{PT}(\mathbb{Z}_n;x)| = 2^n + n - 1$.
6. All of 1–5 works for G a right cancellative monoid, not necessarily a group!
7. The maximal group/right cancellative image of $\mathcal{PT}(G;X)$ is G.

And as promised...

Theorem (H., Kambites, Szakács, 2024)

$\mathcal{PT}(G;X) \cong LAd\langle X \mid w^2 = w \text{ for words } w =_G 1\rangle.$

Pretzel monoids give a family of left adequate monoids with a geometric interpretation.

Open Questions and What's Next

1. Can we describe other presentations using similar geometric methods?
2. ... maybe even all presentations of left adequate monoids?
3. Can we apply our methods to closely related semigroup classes (e.g. Ehresmann, ample, abundant, amiable, restriction etc.) and their left/right duals?
4. What 'traditional' semigroup theory can we apply to $\mathcal{PT}(G;X)$ (e.g. Green's $\mathcal{J}$-relation)?
5. What about the right adequate and two-sided adequate pretzel monoids?

References

Thank you!

1. M. Gromov. "Hyperbolic groups". In: *Essays in group theory*. Vol. 8. Math. Sci. Res. Inst. Publ. Springer, New York, 1987, pp. 75–263
2. M. Kambites. *Free left and right adequate semigroups*. 2009. arXiv: 0904.0916 [math.RA]
3. D. Heath, M. Kambites, N. Szakács. *Pretzel Monoids*. 2024 (to appear)