Pretzel Monoids: A Dive into Geometric Semigroup Theory

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HDP Student Day
Satellite to Heilbronn Annual Conference
University of Bristol, September 4, 2024

Semigroups and Monoids

Definition

A *semigroup* $(S, \cdot)$ is a set S equipped with an associative binary operation $\cdot$.
A *monoid* is a semigroup which contains a (necessarily unique) identity element $\mathbf{1}$.

Examples of Monoids:

- Groups.
- $\mathbb{N}_0$ with $+$.
- All finite-length words over an alphabet X under concatenation.
  E.g. $X = \{x, y\}$. Then elements include $\epsilon, xxy, xyx, xxyxyx$ etc.
  This is the *free monoid over* X, denoted X^*.

Examples of Semigroups:

- Monoids, Groups.
- $\emptyset$...
- $\mathbb{N} \setminus \{0\}$ with $+$.
- All finite-length, non-empty words over an alphabet X under concatentation.
  This is the *free semigroup over* X.

Generalising Results from Group Theory

In general, generalising properties of groups to semigroups or (even to monoids) is **very hard**. A lot of techniques cannot generalise immediately, e.g. *divisibility*, *normal subgroups*, *conjugacy*?

Thus semigroup theorists tend to consider special classes of semigroups and study those instead.

- Bands: For all $a \in S$, $a^2 = a$.
- Commutative semigroups: For all $a, b \in S$, $ab = ba$.
- Right cancellative semigroups: For all $a, b, x \in S$, $ax = bx \implies a = b$.
- Inverse semigroups: For all $x \in S$, $\exists! x'$ such that $xx'x = x$ and $x'xx' = x'$.
- Wikipedia has a list of 150 different classes of semigroups!
  `https://en.wikipedia.org/wiki/Special_classes_of_semigroups`

So which one to look at...?

The $\mathcal{R}^*$ relation and Left Adequacy

Definition

Given a monoid M, define an equivalence relation $\mathcal{R}^*$ on M by $a\mathcal{R}^*b$ if and only if

$$\forall x, y \in M, \quad xa = ya \iff xb = yb.$$

Think: "*Elements are $\mathcal{R}^*$-related iff they 'share' right-cancellativity properties*".

Definition

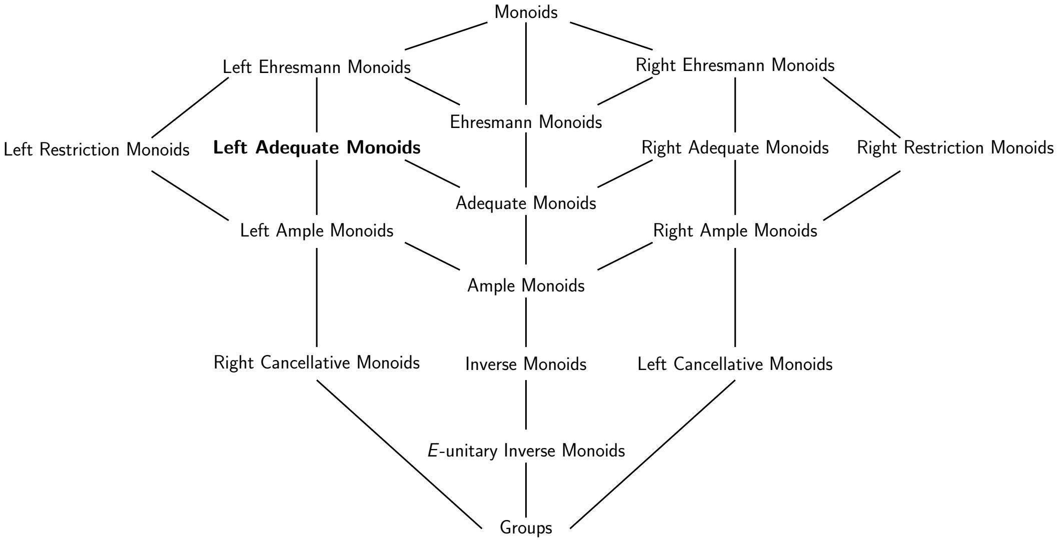
An element e of a monoid is called idempotent if $e^2 = e$.

Definition

A monoid is called left adequate if:

1. Every $\mathcal{R}^*$-class contains a unique idempotent.
2. The idempotents of M commute with each other ($ef = fe$).

A Big Diagram



FLAds

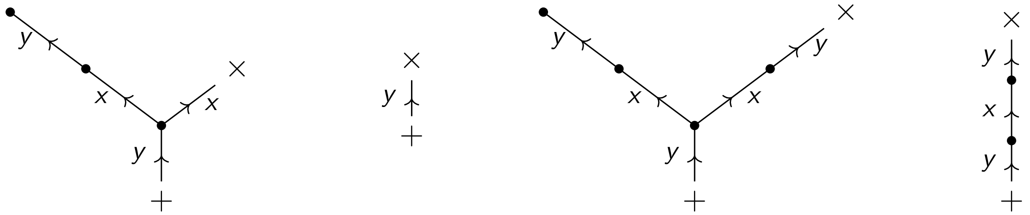
Free left adequate monoids (FLAds) exist and have a geometric interpretation.

Theorem (Kambites, 2009)

Elements of the free left adequate monoid generated by a set X may be treated as directed, edge-labelled trees, labelled by X, with two distinguished vertices called start and end, such that:

1. *There is a path from the start vertex to every other vertex.*
2. *No branches of the tree can be 'completely folded in', where we always fix the start/end vertices.*

The multiplication ST of trees S and T is given by gluing T to S start-to-end, then folding in any branches we can.



The Goal

Fact 1

The class of left adequate monoids is not a variety (not closed under quotients)...

Fact 1.5

...but 'nice' quotients and presentations can be defined.

Can we describe certain presentations of left adequate monoids similarly to FLAds? (**Hard**)

Pretzels!

Fix a set X and an X-generated group G.

Definition

An *idempath* in an X-labelled digraph Γ is a path labelled by a word $x_1 x_2 \cdots x_n$ which is equal to the identity in G. We take the empty path with label ϵ to have $\epsilon =_G 1$.
An *idempath identification* in Γ is the process of 'cycling up' an idempath.

Lemma (H., Kambites, Szakács, 2024)

Given a tree $T \in \mathrm{FLAd}(X)$, there exists a unique graph obtainable by sequentially performing all non-trivial idempath identifications (in any order) to T.

Definition

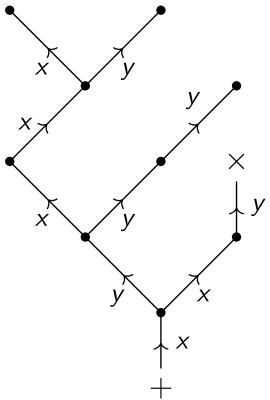
Given any tree $T \in \mathrm{FLAd}(X)$, perform the following:

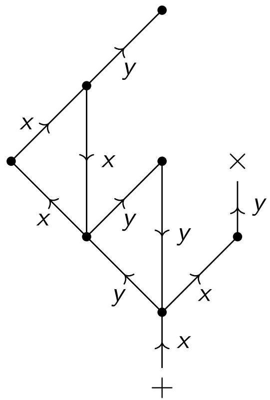
1. Idempath identify as far as possible...
2. ...then retract anything in the result which can retract.

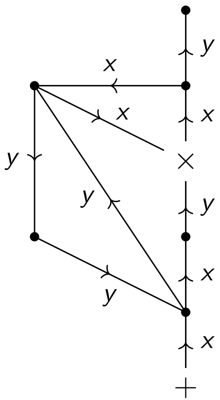
We call the (uniquely obtained) result the *pretzel* of T, denoted $\overline{\widetilde{T}}$.

Example

Take $X = \{x, y\}$ and $G = \mathbb{Z}_3 \times \mathbb{Z}_3 = \mathrm{Mon}\langle x, y \rangle$.







Gluing

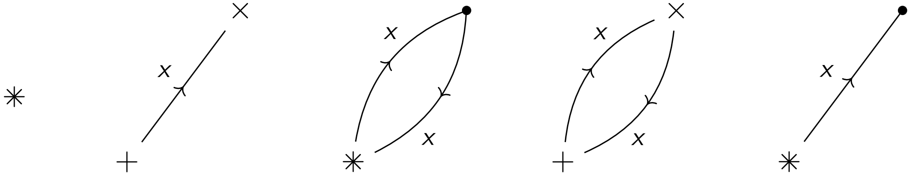
Take two trees S and T in $\mathrm{FLAd}(X)$ and pretzel-ify them w.r.t G.
Define a multiplication on pretzels as follows:

1. Glue $\overline{\widetilde{T}}$ to $\overline{\widetilde{S}}$, start-to-end.
2. Pretzel-ify the result (note that new idempaths could have been created!).

Theorem (H., Kambites, Szakács, 2024)

This multiplication on pretzels is well-defined and associative.
Under this multiplication, the set of all pretzels (w.r.t X and G) is a left adequate monoid.

Denote this left adequate monoid by $\mathcal{PT}(G; X)$. For example, the 5 pretzels of $\mathcal{PT}(\mathbb{Z}_2; x)$ are:



Properties of Pretzels

Properties

1. Pretzels are co-deterministic, but not necessarily deterministic.
2. $\mathcal{PT}(G;X)$ is X-generated (as a left adequate monoid).
3. $\mathcal{PT}(G;X)$ is finite $\iff$ G is finite.
4. For $n \geq 1$, $|\mathcal{PT}(\mathbb{Z}_n;x)| = 2^n + n - 1$.
5. **All** of this works for G a right cancellative monoid, not necessarily a group!
6. If G is a group, the maximal group image of $\mathcal{PT}(G;X)$ is G.
7. Even if G is not a group, any pretzel is a tree of strongly connected subgraphs of $\mathrm{Cay}(H)$ for some group H.

Theorem (H., Kambites, Szakács, 2024)

$\mathcal{PT}(G;X) \cong \mathrm{LAd}\langle X \mid w^2 = w \textit{ for } w \in X^* \textit{ s.t. } w =_G 1\rangle$.

Pretzel monoids are one analogue of *Margolis-Meakin expansions* from E-unitary inverse land.

$$\mathcal{M}(G;X) \cong \mathrm{Inv}\langle X \mid w^2 = w \text{ for } w \in X^* \text{ s.t. } w =_G 1\rangle.$$

Open Questions and What's Next

1. Can we describe other presentations using similar combinatorial methods?
2. Can we apply our methods to closely related semigroup classes (e.g. Ehresmann, ample, abundant, amiable, restriction, F-restriction etc.) and their left/right duals?
3. What about the right adequate and two-sided adequate pretzel monoids?
4. Can we find geometric interpretations of other analogues of Margolis-Meakin expansions in the left adequate setting?

References

Thank you!

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3. S. W. Margolis and J. C. Meakin. "*E*-unitary inverse monoids and the Cayley graph of a group presentation". In: *J. Pure Appl. Algebra* 58(1) (1989), pp. 45–76