Pretzel Monoids: Left Adequate Expansions of Right Cancellative Monoids

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The $\mathcal{R}^*$ relation and Left Adequacy

Definition

Given a monoid M, define an equivalence relation $\mathcal{R}^*$ on M by $a\mathcal{R}^*b$ if and only if

$$\forall x, y \in M, \quad xa = ya \iff xb = yb.$$

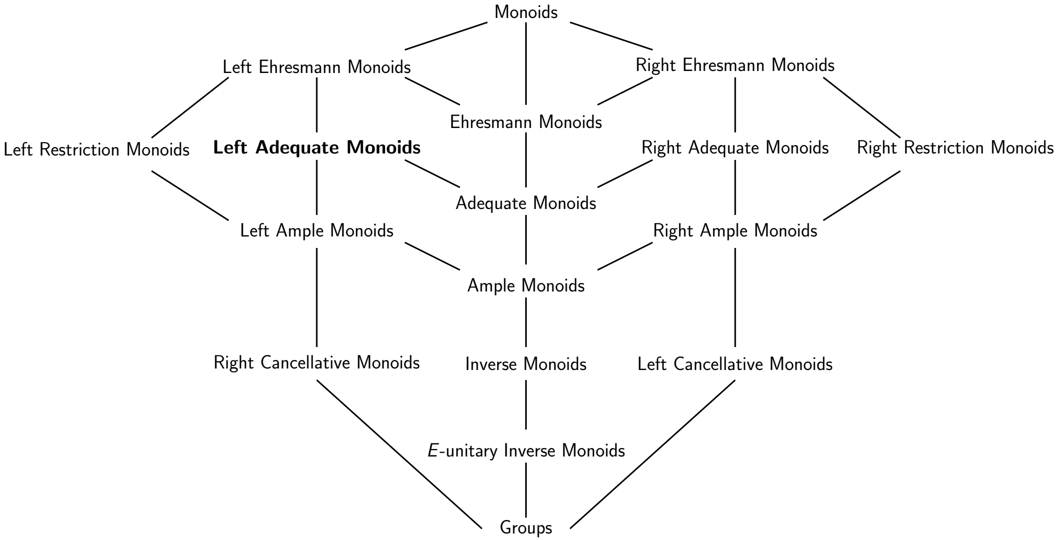
Think: "*Elements are $\mathcal{R}^*$-related iff they 'share' right-cancellativity properties*".

Definition

A monoid is called left adequate if:

1. Every $\mathcal{R}^*$-class contains a unique idempotent.
2. The idempotents of M commute with each other ($ef = fe$).

A Big Diagram



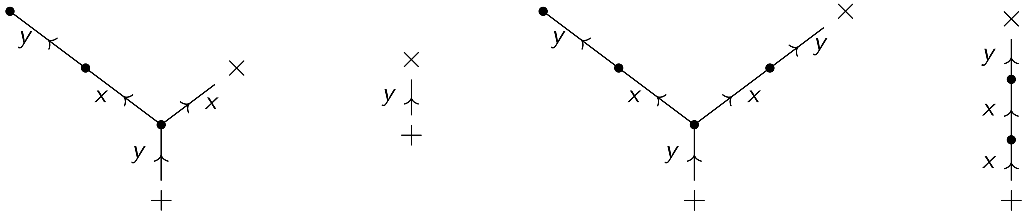
FLAds

Theorem (Kambites, 2009)

Elements of the free left adequate monoid generated by a set X may be treated as directed, edge-labelled trees, labelled by X, with two distinguished vertices called start and end, such that:

1. *There is a path from the start vertex to every other vertex.*
2. *No branches of the tree can be 'completely folded in', where we always fix the start/end vertices.*

The multiplication ST of trees S and T is given by gluing T to S start-to-end, then folding in any branches we can.



Pretzels!

Fix a set X and an X-generated right cancellative monoid C.

Definition

An *idempath* in an X-labelled digraph Γ is a path labelled by a word $x_1x_2 \cdots x_n$ which is equal to the identity in C. We take the empty path with label ϵ to have $\epsilon =_C 1$.
An *idempath identification* in Γ is the process of 'cycling up' an idempath.

Lemma (H., Kambites, Szakács, 2024)

Given a tree $T \in \mathrm{FLAd}(X)$, there exists a unique graph obtainable by sequentially performing all non-trivial idempath identifications (in any order) to T.

Definition

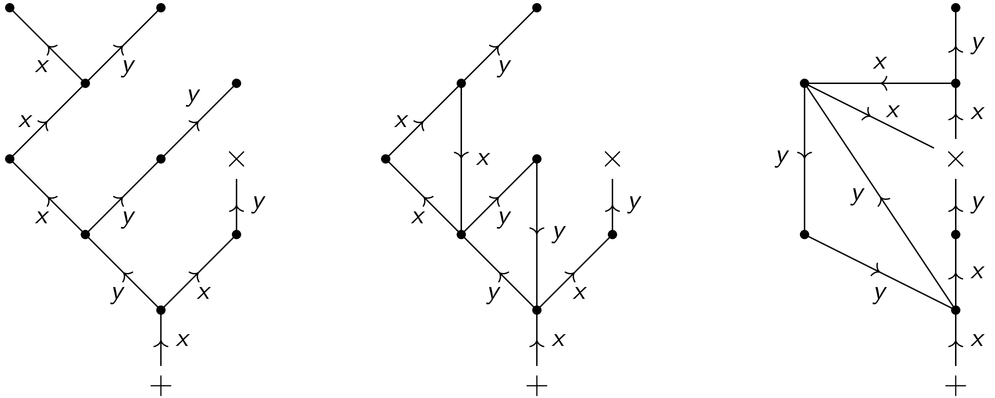
Given any tree $T \in \mathrm{FLAd}(X)$, perform the following:

1. Idempath identify as far as possible...
2. ...then retract anything in the result which can retract.

We call the (uniquely obtained) result the *pretzel* of T, denoted $\overline{\widetilde{T}}$.

Example

Take $X = \{x, y\}$ and $C = \mathbb{Z}_3 \times \mathbb{Z}_3 = \mathrm{Mon}\langle x, y\rangle$.



Gluing

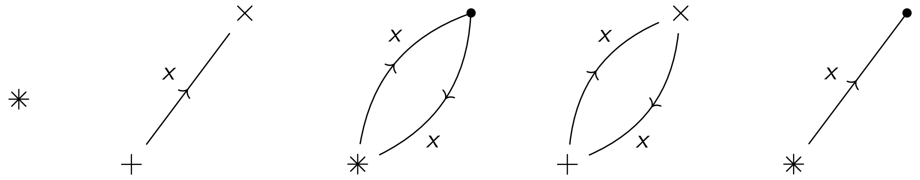
Take two trees S and T in $\mathrm{FLAd}(X)$ and pretzel-ify them w.r.t $C = \mathrm{RC}\langle X\rangle$.
Define a multiplication on pretzels as follows:

1. Glue $\overline{\widetilde{T}}$ to $\overline{\widetilde{S}}$, start-to-end.
2. Pretzel-ify the result (note that new idempaths could have been created!).

Theorem (H., Kambites, Szakács, 2024)

This multiplication on pretzels is well-defined and associative.
Under this multiplication, the set of all pretzels (w.r.t X and C) is a left adequate monoid.

Denote this left adequate monoid by $\mathcal{PT}(C; X)$. For example, the 5 pretzels of $\mathcal{PT}(\mathbb{Z}_2; x)$ are:



Properties of Pretzels

Properties

1. $\mathcal{PT}(C;X)$ is X-generated (as a left adequate monoid).
2. $\mathcal{PT}(X^*;X) \cong \mathrm{FLAd}(X)$.
3. For any $C = \mathrm{RCanc}\langle X \rangle$, there exists $C' = G * Y^*$ with $\mathcal{PT}(C;X) \cong \mathcal{PT}(C';X)$.
4. $\mathcal{PT}(C;X)$ is finite $\iff$ C is finite $\implies$ C is a group.
5. Any pretzel Γ is a tree of strongly connected subgraphs of $\mathrm{Cay}(G)$ for some group G.

Theorem (H., Kambites, Szakács, 2024)

$\mathcal{PT}(C;X) \cong \mathrm{LAd}\langle X \mid w^2 = w \textit{ for } w \in X^* \textit{ s.t. } w =_C 1 \rangle$.

Pretzel monoids are one analogue of Margolis-Meakin expansions from E-unitary inverse land.

$$\mathcal{M}(G;X) \cong \mathrm{Inv}\langle X \mid w^2 = w \text{ for } w \in X^* \text{ s.t. } w =_G 1 \rangle.$$

References

Thank you!

1. D. Heath, M. Kambites, and N. Szakács. *Pretzel monoids*. 2024. arXiv: 2405.00589
2. M. Kambites. "Retracts of trees and free left adequate semigroups". In: *Proc. Edinburgh Math. Soc.* 54(3) (2011), 731–747
3. S. W. Margolis and J. C. Meakin. "*E*-unitary inverse monoids and the Cayley graph of a group presentation". In: *J. Pure Appl. Algebra* 58(1) (1989), pp. 45–76