Growth of monogenic free adequate monoids

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They/Them

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Left Adequate Monoids

Definition

A monoid M is *left adequate* if:

1. Idempotents of M commute;
2. For all $a \in M$, there exists a unique idempotent $a^+ \in E(M)$ such that

$$\forall x, y \in M \quad xa = ya \iff xa^+ = ya^+.$$

Definition

Equivalently, a set M with type $(\cdot, ^+, 1)$ and signature $(2, 1, 0)$ is called *left adequate* if it satisfies the following quasi-identities:

$$a(bc) \approx (ab)c, \qquad a1 \approx a \approx 1a,$$

$$a^+a \approx a, \qquad (a^+b^+)^+ \approx a^+b^+, \qquad a^+b^+ \approx b^+a^+, \qquad (ab)^+ \approx (ab^+)^+,$$

$$a^2 \approx a \to a \approx a^+ \quad \text{and} \quad ac \approx bc \to ac^+ \approx bc^+.$$

Fact: If M is right cancellative, then $a^+ := 1$ makes M left adequate.

Fact: If M is inverse, then $a^+ := aa^{-1}$ makes M left adequate.

Adequate Monoids

Definition

A monoid M is *adequate* if M is left adequate and also:

3. For all $a \in M$, there exists a unique idempotent $a^* \in E(M)$ such that

$$\forall x, y \in M \quad ax = ay \iff a^* x = a^* y.$$

Definition

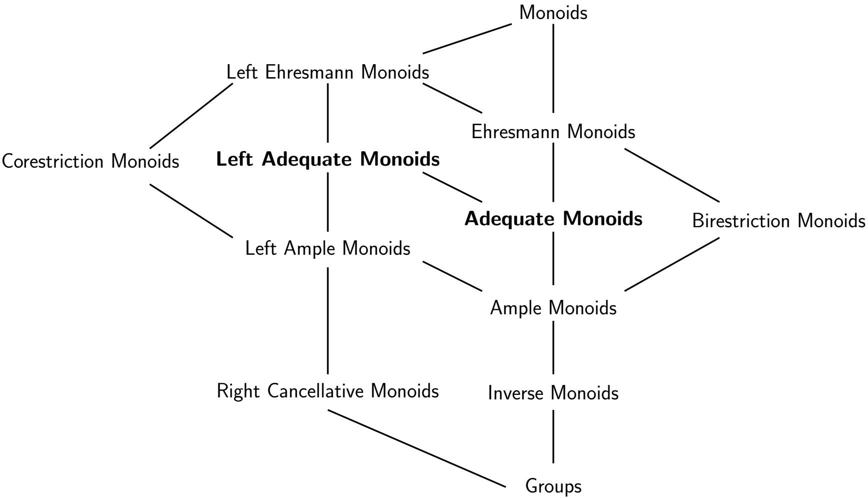
Equivalently, a set M with type $(\cdot, ^+, ^*, 1)$ and signature $(2, 1, 1, 0)$ is called *adequate* if it satisfies the quasi-identities for left adequate monoids plus:

$$aa^* \approx a, \qquad (a^* b^*)^* \approx a^* b^*, \qquad a^* b^* \approx b^* a^*, \qquad (ab)^* \approx (a^* b)^*,$$

$$a^2 \approx a \to a \approx a^* \quad \text{and} \quad ca \approx cb \to c^* a \approx c^* b.$$

Fact: If M is cancellative, then $a^+ := 1$ and $a^* := 1$ makes M adequate.
Fact: If M is inverse, then $a^+ := aa^{-1}$ and $a^* := a^{-1}a$ makes M adequate.

(Bi)unary Classes Floating Around

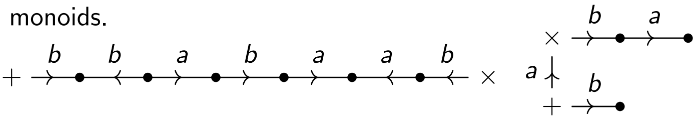


Free Objects

Many of these (quasi)varieties have free objects described by operations on directed graphs.

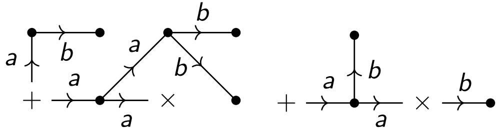
Munn 1974: Free inverse monoids.

$bb^{-1}abaa^{-1}b^{-1}$



Fountain, Gomes, Gould 2009: Free ample / birestriction monoids.

$(ab)^{+}a(ab^{+}b)^{+}a$



Kambites 2009, 2011: Free adequate / Ehresmann monoids.

Trees

Definition

An *a-tree* Γ is a directed graph, with a *start vertex* and an *end vertex* such that:

1. The underlying undirected graph of Γ is a tree,
2. There is a directed path from the start vertex to the end vertex.

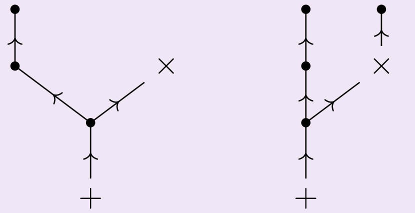
Definition

A *retraction* is an idempotent endomorphism $\Gamma \to \Gamma$.

Definition

Γ is called *retract-free* if Γ admits no non-trivial retractions.

Examples



$a(aa)^+a$ $\qquad$ $a(aa)^+aa^+$

Fact: Any Γ admits a unique retract-free core up to isomorphism, which we denote $\overline{\Gamma}$.

Free Objects

Theorem (Kambites 2009, 2011)

FAd_1 *is the set of all retract-free a-trees with:*

1. $\Gamma\Delta := \overline{\Gamma \times \Delta}$,
2. Γ^+ *given by moving the end vertex to the start and retracting,*
3. Γ^* *given by moving the start vertex to the end and retracting.*

Definition

An *a*-tree is a *left a-tree* if every vertex is reachable (directed) from the start vertex.

Theorem (Kambites 2009, 2011)

FLAd_1 *is the set of all retract-free* **left** *a-trees with multiplication and* $+$ *as above.*

Growth

Definition

- The *ball of size n* $B(n)$ in FAd_1 consists of elements which may be created by a $+$-word with at most n characters.
- The *sphere of size n* is $S(n) := B(n) \setminus B(n-1)$.

We similarly define $B_L(n)$ and $S_L(n)$ for balls and spheres in FLAd_1.

Proposition (Aird, H. 2025+)

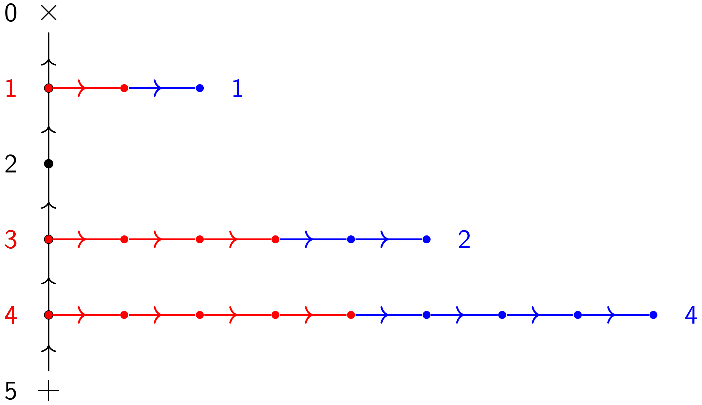
1. $T \in B(n)$ if and only if T has at most n edges.
2. $T \in B_L(n)$ if and only if T has at most n edges.

Corollary (Aird, H. 2025+)

1. $T \in S(n)$ if and only if T has **exactly** n edges.
2. $T \in S_L(n)$ if and only if T has **exactly** n edges.

The Goal: Akin to free inverse monoids, examine $|S(n)|$ and $|S_L(n)|$, i.e. trees with n edges.

What do trees in FLAd_1 look like?



Observation 1: Retract-free left a-trees consist of some length trunk, with single, non-splitting branches off the trunk, of length strictly greater than the remaining trunk.
Observation 2: If there are branches on trunk vertices Y, the number of mandatory red edges is $\sum_{i \in Y} i$.
Observation 3: The blue edges form a strictly decreasing sequence with sum $n - k - \sum_{i \in Y} i$. The blue edges correspond to a partition of $n - k - \sum_{i \in Y} i$ into $|Y|$ distinct parts.

Partitions

Theorem (Aird, H. 2025+)

The number of trees in FLAd_1 *with* n *edges and* k *trunk edges is*

$$\sum_{Y \subseteq \{0,\dots,k\}} Q\left(n - k - \sum_{i \in Y} i, |Y|\right)$$

where $Q(m, l)$ *is the number of partitions of* m *into* l *distinct parts.*

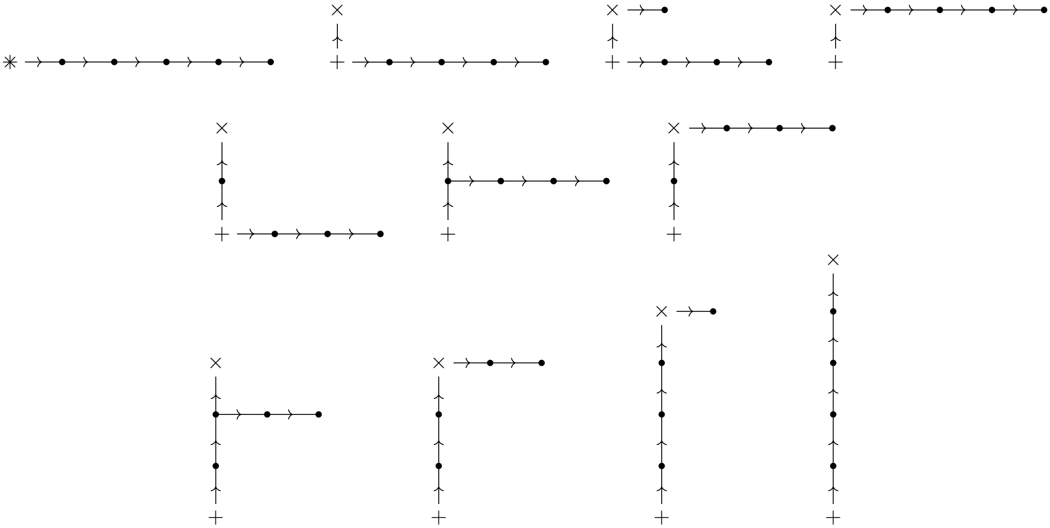
Corollary (Aird, H. 2025+)

The number of trees in FLAd_1 with n edges and k trunk edges is $P(n+1, k+1)$ where $P(m, l)$ is the number of partitions of m into l parts.

Corollary (Aird, H. 2025+)

In FLAd_1, the number of trees with n edges is $|S_L(n)| = P(n+1)$.

FLAd_1 trees with 5 edges



$(6), (5,1), (4,2), (3,3), (4,1,1), (3,2,1), (2,2,2), (3,1,1,1), (2,2,1,1), (2,1,1,1,1), (1,1,1,1,1,1).$

Growth

Theorem (Aird, H. 2025+)

The monogenic free left adequate monoid has intermediate growth.

Proof.

By a famous result of Hardy and Ramanujan,

$$P(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \sim \exp\left(\sqrt{n}\right). \qquad \square$$

Theorem (Aird, H. 2025+)

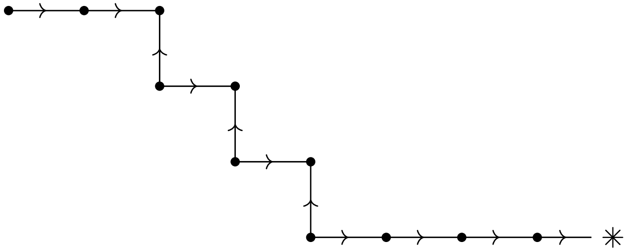
Free left adequate monoids have exponential growth for rank ≥ 2.

Proof.

Note FAd_n contains the free monoid of rank n. $\square$

What do trees in FAd_1 look like?

In *adequate* land, retract-free trees can look much stranger...



Theorem (Aird, H. 2025+)

The semilattice of idempotents $E(\mathrm{FAd}_1)$ *grows exponentially of degree at least* 2.

Enumeration and Open Problems

n	0	1	2	3	4	5	6
$\lvert S_E(n)\rvert$	1	2	3	6	11	28	63
$\lvert S(n)\rvert$	1	3	6	14	29	74	?

Table: Size of spheres in FAd_1 for $0 \leq n \leq 5$.

Questions

1. What is the idempotent growth rate of FAd_1?
2. Is the growth rate of FAd_1 governed by its idempotents?
3. Higher rank? In both FLAd_n and FAd_n?

Thank you!