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# 5 Operating on Spatial Relations

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## Introduction

It is a fact of everyday experience that we can efficiently perceive the spatial relations between objects and between the parts of objects. Indeed without this capacity, we would have difficulty performing many basic visual tasks, from identifying faces to making sense of the environment and moving safely through it. Judgments about whether one object is near to or far from another, whether it is longer or shorter, or above or below are all performed apparently instantly and effortlessly. Visual performance, however, goes beyond this level of competence, for many spatial relations are changed as soon as the object is moved or the viewpoint of the observer is altered. For example, the description 'the hinge is on the left of the door' becomes 'the hinge is on the right of the door' when the door is viewed from the opposite side. Despite this seeming transience of spatial relations we continue to recognize objects defined in terms of their spatial relations and make appropriate assertions about them.

How is this achieved? Given that the visual system makes use of spatial relations, it must be possible to operate internally on them to compensate for the changes induced as object and viewer change position with respect to each other. In effect, these internal operations convert the spatial relation 'right of' to the spatial relation 'left of' in the description 'the hinge is on the right of the door', thereby identifying the second view of the object with the first. In principle, the capacity to extract spatial relations from an image and the ability to operate internally on spatial relations should provide a solution to the general problem posed by visual pattern recognition. What then is the nature of these spatial relations and internal operations?

## Spatial Relations

Spatial relations specify that certain distinct items, taken two or more at a time, have a certain spatial property; for example, that one item is 'joined to' another, that one item is 'between' two others, that three or more items are 'collinear', that one item is 'inside' another, and so on. Other examples and a formal definition are given later. As might be anticipated, the notion of spatial relations is fundamental to the understanding of geometry, now regarded not as the study of 'geometrical objects' but of the relations between those objects. The point was made forcibly by the German mathematician David Hilbert, who allegedly said that the words 'point', 'line' and 'plane' could be replaced by 'table', 'chair' and 'beer-mug' without changing geometry in the least (Bourbaki, 1968, p 317).

Although examples of visually plausible spatial relations may be produced easily, there are a number of problems that arise when one considers more carefully the possible kinds of spatial properties. These properties may be numerical, for example, scalars (the distance between two objects), or vectors (the distance between two objects and direction the one defines with respect to the other), or they may be predicates asserting something geometrical or topological (the closure of a curve). It would be desirable to classify the spatial relations of vision within the framework of one or more formal mathematical structures (see e.g. Bourbaki, 1968, Chapter 4), or possibly some fuzzy version of these structures (Kaufmann, 1975). But which structures are the most appropriate for vision, how they should be combined with each other, and how they should be modified to reflect the limitations of the visual apparatus are largely unknown.

In addition to these foundational matters, there are uncertainties concerning the status of some commonly cited examples of spatial relations. Consider the set given by Barlow *et al.* (1972), here slightly modified for later reference:

| | |
|---|---|
| *left of,* | *right of,* |
| *above,* | *below,* |
| *inside,* | *outside,* |
| *at the centre of,* | *surrounds,* |
| *near to,* | *far from,* |
| *attached to,* | *separate from,* |
| *in front of,* | *behind,* |
| *larger than,* | *smaller than,* |
| *longer than,* | *shorter than,* |
| *more than,* | *less than.* |

It is not known which, if any, of these spatial relations are true visual primitives, that is, not derived from combinations of other spatial relations. Because observers can make judgments of say whether one object is above and to the left of another does not necessarily mean that the spatial relation 'above and to the left of' is actually part of an internal visual representation. (One method of assessing whether certain spatial relations are primitive is to decide the issue operationally, for example, by testing combinations of spatial relations defining an object of interest in an appropriate experimental paradigm; see e.g. Treisman and Paterson, 1984; Pomerantz and Pristach, 1989.)

## Transformations and Operations

Spatial relations make possible the generation of compact and immediate descriptions of visual scenes. Some spatial relations are also robust against naturally occurring transformations in the retinal image as the position of the observer changes in relation to the scene and the objects in it (Sutherland, 1968; Barlow *et al.*, 1972). These transformations include translations (as the point of gaze is shifted over the frontoparallel plane, dilatations (as the distance between viewer and object is varied), and rotations (as the head is tilted, but within limits). Some spatial relations may also be robust against line-preserving, that is affine, transformations (as the object is tilted towards or away from the viewer). They may also be robust against non-affine image transformations, such as jitter. For example, the relation 'inside' between two objects is still true even when the 'inside' object is given a modest displacement.

For spatial relations that are not robust against image transformations the internal compensatory operations must be invoked if visual recognition is to be achieved (Shepard, 1975). Clearly the choice of which internal operations to apply in a particular situation is not arbitrary. They must obey certain rules otherwise the identifications they achieve will be meaningless or misleading. Consider, for example, the impact of changing 'on the left of' to 'above' or to 'inside' in the description 'the hinge is on the left of the door'.

Some aspects of these rules may be decided theoretically: for a given type of spatial relation, there is often a class of operation that is complementary to it. Thus, for spatial relations that specify an ordering of items in space (e.g. 'left of'), there are operations that act on that order, replacing one value ('left of') by another ('right of'); for spatial relations that specify continuous distances between items (e.g. '1° away from'), there are operations that act on those distances, smoothly changing one numerical value ('1° away from') to another ('0.5° away from'). Nevertheless, not all operations that are theoretically admissible for a given type of spatial relation are actually appropriate. An example is given later concerning topological relations.

## Scope of this Chapter

By their nature, spatial relations provide a mechanism for the analysis of how entities are localized within the framework of visual space, a problem that was classically approached through the notion of local sign. This chapter therefore gives a brief introduction to the ideas of local sign and localization and to the special role of the vertical and horizontal in defining a visual reference frame. Some basic definitions are then established. These include the notions of general spatial relations and spatial-order and global-position relations, along with the operations applied to them, and other possible types of spatial relations based on geometrical-topological structures.

A sequence of experiments is then reviewed, largely based on forced-choice 'same-different' visual judgments of spatially transformed and randomly paired patterns, each presented for periods too short for shifts in the point of fixation of the eye. 'Same'-detection of rotated patterns is considered first. In one experiment, variations in performance with rotation angle were compared with those expected on the basis of an internal matching operation that monitors the number of unaltered spatial-order relations. To account for observed non-uniformities in performance, it was found necessary to introduce an internal operation that inverts the direction or *sense* of spatial-order relations in the transformed patterns. In two other experiments the effects of pattern position on 'same'-detection performance were measured for patterns that were identical, rotated through 180°, or reflected about a vertical axis. Observed performance was explained in terms of two internal operations: the one reversing the sense of spatial-order relations and the other modifying, progressively, global-position relations specifying the approximate positions of the patterns relative to the point of fixation.

The type of visual framework for spatial-order relations and their operations is considered next. Three experiments are described requiring 'same-different' judgments of identical, 180°-rotated, and reflected patterns in a variety of spatial configurations. The assumption of a horizontal-vertical framework (as opposed to any other orthogonal or isotropic framework) was found to provide an accurate predictor of 'same-different' discrimination performance.

The question of whether internal operations can be applied selectively to patterns is examined in an experiment that required 'same-different' judgments of patterns made up of sets of variously transformed 'subpatterns'. The results of this experiment suggested that a sense-reversal operation can be applied selectively either to spatial-order relations or to global-position relations, provided that the spacing of the subpatterns within the pattern is sufficiently large. A different kind of experiment is then considered in which judgments of perceived symmetry are made in patterns with 'bilateral' and other symmetries. An explanation of performance is offered in terms of the spatial relations in patterns and the effects of possible operations upon them. It is shown that judgments of perceived symmetry in symmetric patterns may be predicted by the same rules that govern 'same-different' judgments of transformed patterns without symmetry.

Finally, two issues concerning the expression and generation of spatial relations are considered. In one experiment, performance in 'same-different' judgments of shape was compared with 'same-different' judgments of dot number to determine whether spatial relations in patterns can be effectively suppressed when they are not required. In another experiment, performance in visually tracking various numbers of arbitrarily designated target elements in a field of identical elements was measured to test the notion of a neutral 'indexing' operation that might be a precursor to the formation of spatial relations.

There are a number of topics not covered in this review. These include spatial relations and internal operations associated with the recognition of words or faces or other special and familiar classes of images; limits on spatial-relation judgments of the hyperacuity kind; cognitive issues concerning, for example, lateralization or mental imagery; and developmental or clinical aspects of spatial relations and their operations.

The treatment in this chapter is not formal, although some technical material is introduced in the section on the classification of spatial relations. With regard to terminology, the term 'local feature' is used occasionally as a referent for spatial relations, and it is intended to have its conventional meaning, notwithstanding Hilbert's maxim concerning geometrical objects.

## Localization and Reference Frames

How spatial relations are perceived and operated on is intimately related to the notion of how we localize objects within the external world. There are two problems: how a structure is imposed on the visual field and how spatial non-uniformities in the visual apparatus might influence that structure.

### Local Sign

Philosophical consideration of the problem of localization has a long history. An account of some of the early issues concerning localization has been given by von Kries in a number of appendices to Helmholtz's *Treatise on Physiological Optics* (Helmholtz, 1925). One enduring concept has been that of *local sign*, due primarily to the German metaphysician Hermann Lotze. He proposed a theory of localization (Lotze, 1887) in which graduated signs or tokens were attached (as 'extra-impressions') to the sensations originating from different points on the retina. But he recognized that simply differentiating distinct points of excitation, that is introducing a system of labelling, was not sufficient to solve the problem of localization:

> *If the local signs π κ ρ merely differ generally in quality, it is true that they would suffice to prevent three perfectly similar stimuli from coalescing, and to make them appear as three instances of the same felt content. But the only result would be an impulse to hold the sensations apart in a general way; there would be nothing to lead us on to give to the sensations thus produced a definite localisation in space. It is this that is left unnoticed by those who regard the isolated conduction of three impressions by three fibres as a sufficient reason, taken by itself, for their being perceived as spatially separate. Even if (in the absence of the extra local signs) this isolation were a sufficient condition of the three impressions being distinguished as three, yet the question whether they were to be represented at the corners of a triangle or in a straight line, could only be decided by a soul which already possessed that capacity of localisation which we are trying to understand.* (pp 259–260)

Labelling separate points of retinal excitation or fibres does succeed when it is part of a more general description of activity that extracts some of the structure of the visual field (Lotze, 1887, p 260; Koenderink, 1984). Thus if activity in one fibre is always correlated with activity in another, the assumption might be made that their inputs were spatially close to each other on the retina (or, in binocular viewing, perhaps close to corresponding points on the two retinae). The collection of such correlations, or relations, allows the generation of a geometrical structure, for example, based on the relation 'being a neighbour of' (Koenderink, 1988). This structure may, in turn, offer a basis for judgments about retinal location. Some consideration of the topological implications of this approach is given in Toet *et al.* (1987) and Koenderink (1988).

### Anisotropy of the Visual Field

In the present context, variations in visual performance with eccentricity of the stimulus may be neglected, but departures from isotropy are relevant to any general theory of spatial relations and their operations. The *oblique effect* (Appelle, 1972) is well known and is traditionally associated with a reduction in visual discrimination performance for stimuli oriented along the oblique axes (Rochlin, 1955; Onley and Volkmann, 1958; Campbell *et al.*, 1966; Mitchell *et al.*, 1967; Berkeley *et al.*, 1975; Orban *et al.*, 1984; Vandenbussche *et al.*, 1986). These effects, due primarily to neurophysiological or neuroanatomical factors, have been called *Class 1* oblique effects (Essock, 1980).

A special role for the horizontal and vertical axes has also been noted in other, more perceptual or cognitive, aspects of visual function that do not require high acuity (Mach, 1897, Chapt. 6; Koffka, 1935; Attneave, 1955, 1968; Attneave and Olson, 1967; Olson and Attneave, 1970; Rock, 1973; Kahn and Foster, 1986). For example, a pattern comprising horizontal and vertical lines has been found to give better grouping or segmentation effects than one comprising lines oriented at $-45°$ and $+45°$ to the vertical, despite the fact that the difference between the slopes of the lines in each of the patterns was identically 90° (Olson and Attneave, 1970). Analogous effects have been observed in peripheral form discrimination under conditions of stimulus uncertainty (Beck, 1972), and in the classification and discrimination of lines of different orientations (Lasaga and Garner, 1983) and of dot positions within different surrounds (Cecala and Garner, 1986). In a task requiring the reproduction of dot patterns from immediate memory, the order of the dots on the horizontal and vertical axes has been found to be more accurately produced than their order on the diagonal axes (Attneave and Curlee, 1977). These effects have been called *Class 2* oblique effects (Essock, 1980).

Retinal, cortical, gravitational, and visual (or environmental) frames of reference have been variously suggested as determining the actual directions of these orthogonal axes (Attneave and Olson, 1967; Annis and Frost, 1973; Rock, 1973; Corballis and Roldan, 1975; Timney and Muir, 1976; Switkes *et al.*, 1978; Vandenbussche *et al.*, 1986; Heeley and Timney, 1988). In general, Class 1 oblique effects should be fixed to the retinal frame of reference and Class 2 oblique effects should be labile, since they are associated with the *perceived* vertical–horizontal (Essock, 1980). For spatial relations and their operations, it is Class 2 anisotropies that are the more germane.

# Classification of Spatial Relations

This section now makes explicit some basic concepts concerning the kinds of structure captured by spatial relations and their operations. First, a formal definition of a spatial relation is given, then the role of a visual framework in classifying spatial relations is developed, and finally two important types of spatial relations and their complementary operations are defined.

## General Spatial Relations

It is convenient to be able to express the idea of spatial relations in a more formal language, briefly as follows. A *relation R* (sometimes called an *n-ary relation*) in a set of items $\{f_i\}$, where the index $i$ ranges in some index set, is a function that assigns to any sequence $(f_1, f_2, ..., f_n)$ of items one of two values, that is

$$R(f_1, f_2, ..., f_n) = \begin{cases} 1, & \text{if } f_1, f_2, ..., f_n \text{ are related} \\ 0, & \text{otherwise} \end{cases} \quad (5.1)$$

When $n=2$, $R$ is a *binary relation*, and $R(f_1, f_2)=1$ may be written more directly as $f_1 R f_2$, pronounced '$f_1$ is related to $f_2$'. If $R$ is the relation 'near to', then $R(f_1, f_2)=1$ is equivalent to '$f_1$ is near to $f_2$'. When $n=3$, $R$ is a *ternary relation*, the minimum required to define the collinearity of (isotropic) items, and when $n=4$, a *quaternary relation*, a form of which may be used in symmetry detection (and which is discussed later). By an abuse of notation, a spatial relation is sometimes referred to in the plural when it is considered in conjunction with the items $(f_1, f_2, ..., f_n)$ for which it is true. For example, in a horizontal array of $n$ items, one may refer to the $n(n-1)/2$ relations 'left of' rather than to the $n(n-1)/2$ pairs of items for which the relation 'left of' is true.

The definition (Equation 5.1) may be extended so that $R$ also takes on values between 0 and 1, thus becoming a *fuzzy relation* (Kaufmann, 1975). It is this possibility of $R$ taking on values other than 0 and 1 that makes Equation 5.1 so useful. For example, the spatial relation 'near to' can be quantified as a function that associates with each pair of items $(f_1, f_2)$ a number between 0 and 1 such that when $f_1 = f_2$ the value of the function is 1 and as the distance between $f_1$ and $f_2$ increases the value of the function gets closer to zero.

An *operation* $\phi$ on a spatial relation $R$ is simply a procedure for taking that relation into some other spatial relation $R'$, written $\phi(R)=R'$. For example, an operation $\sigma$ will be introduced that replaces the spatial relation 'left of' by the spatial relation 'right of'.

## Intrinsic and Extrinsic Spatial Relations

Spatial relations may be classified as *intrinsic* if they are invariant under the natural transformations of the retinal image as the position of the observer changes in relation to the object. As noted earlier, these transformations include translations, dilatations, rotations (partly), and some other affine transformations. Spatial relations that are not intrinsic are *extrinsic*. Extrinsic relations depend on the space in which the object is embedded (three-dimensional Euclidean space), or on the viewpoint of the observer, or on both. The spatial relation 'inside' is intrinsic: it does not depend directly either on Euclidean space or on the observer's viewpoint. The spatial relation 'near to' is extrinsic: although not dependent on the observer's viewpoint, it does depend on the coordinates of Euclidean space. The spatial relation 'left of' is also extrinsic: it depends on the observer's viewpoint. In principle, some extrinsic relations can be made intrinsic if a suitable coordinate system is attached to the object (or group of objects), possibly by making use of a preferred axis such as an axis of elongation (Yakimoff and Mitrani, 1979; Lánský *et al.*, 1988).

Whether a spatial relation is intrinsic or extrinsic is intimately related to the choice of general coordinate system. Marr (1982) distinguished (at least) two types of coordinate systems: a *viewer-centred* system, in which spatial positions were specified relative to the viewer, and an *object-centred* system, in which spatial positions were defined with respect to the object and did not depend on the position or orientation from which the object was viewed. Object-centred coordinate systems are attractive: by definition they ensure that descriptions formed by the visual system are independent of the relative position of object and observer.

Unfortunately, as has been argued elsewhere (Shepard, 1981, p 292; Foster, 1984, pp 85–86), there is a body of experimental data, some reviewed here, that suggests that the visual system constructs *hybrid* coordinate systems: part viewer-centred, part object-centred. The interaction or 'mesh' (Shepard, 1981) of internal and external coordinate systems is nicely illustrated in a description taken from Marr (1982, p 42) 'To say that the tip of a certain cat's tail is above and to the left of its body is a remark in a coordinate frame that is centered on the cat.' In fact, this description is not properly object-centred: the spatial relation 'above' is gravitational, and 'left of' would normally refer to the observer's left, not the cat's left. Thus the natural interpretation of this spatial-relation description is an extrinsic one, and this is obvious if the cat is viewed from the opposite direction or is turned (rigidly) upside down.

## Preorder Relations and their Inverses

Many spatial relations of visual interest are binary and constitute *preorders*. A relation $o$ in a set of items is called a preorder if (and only if) for every item $f_j, f_k, f_l$

1. $o(f_j, f_j) = 1$ (reflexivity)
2. $o(f_j, f_k) = 1$ and $o(f_k, f_l) = 1$ implies $o(f_j, f_l) = 1$(transitivity)

Any preorder $o$ has an *inverse* (or *opposite*) $o^{-1}$, where $o^{-1}(f_j, f_k) = 1$ if and only if $o(f_k, f_j) = 1$. The operation $\sigma$ that takes a preorder $o$ into its inverse $o^{-1}$ is called an *inversion.*

The extrinsic spatial relation 'left of' is a preorder, and its inverse is simply 'right of'. The intrinsic spatial relation 'inside' is also an example of a preorder, with inverse 'outside'. In this terminology, 'left of' and 'inside' do *not* exclude 'equal to'. An example of a spatial relation that is not a preorder is 'near to': if $f_j$ is 'near to' $f_k$ and $f_k$ is 'near to' $f_l$, it does not necessarily follow that $f_j$ is 'near to' $f_l$.

If a third condition is added to the above definition, namely

3. $o(f_j, f_k) = 1$ and $o(f_k, f_j) = 1$ implies $f_j = f_k$ (antisymmetry)

then the preorder becomes an *order*. Notice that 'inside' is an order, but 'left of' is not an order: if $f_j$ is 'left of' $f_k$ and $f_k$ is 'left of' $f_j$, it does not necessarily follow that $f_j = f_k$, for $f_j$ may be directly above (or below) $f_k$.

## Two-Dimensional Spatial-Order Relations

Because the spatial relations 'left of' and 'right of' are inverses of each other and 'above' and 'below' are also inverses of each other, it is possible to modify the notation introduced earlier to provide an economical formulation of these relations suitable for computation. Let $f_j, f_k$, be local features and let

$$r_x(f_j, f_k) = \begin{cases} -1, & \text{for } f_j \text{ 'left of' } f_k \\ 1, & \text{for } f_j \text{ 'right of' } f_k \\ 0, & \text{otherwise} \end{cases} \qquad (5.2)$$

$$r_y(f_j, f_k) = \begin{cases} -1, & \text{for } f_j \text{ 'below' } f_k \\ 1, & \text{for } f_j \text{ 'above' } f_k \\ 0, & \text{otherwise} \end{cases}$$

By another abuse of notation, $r_x$ and $r_y$ are referred to as horizontal and vertical *spatial-order* relations, respectively. (There are two kinds of 'orders' being used here; the technical terms 'preorder' and 'order' should not be confused with 'spatial order', which applies to a particular kind of preorder, namely that based on spatial position.) For a spatial-order relation such as $r_x$, the effect of applying the inversion operation $\sigma = \sigma_x$ takes on a particularly simple form. Let $\sigma_x(r_x) = r'_x$. Then, $r'_x(f_j, f_k) = r_x(f_k, f_j) = -r_x(f_j, f_k)$. That is,

$$\sigma_x(r_x) = -r_x$$

Likewise for the vertical spatial-order relation $r_y$ and inversion $\sigma_y$. Operations such as $\sigma_x$ and $\sigma_y$ are thus referred to as *sense-reversal* operations.

## Global-Position Relations

It is useful to be able to treat the point of fixation as special and to define a notion of approximate position of a group of local features in the visual field in terms of the distance and direction of the group from this point. Let 0 denote the point of fixation, $f$ the group of local features, and let

$$d_x(f) = r_x(f, 0) \cdot \text{horizontal distance of } f \text{ from } 0$$

$$d_y(f) = r_y(f, 0) \cdot \text{vertical distance of } f \text{ from } 0$$

The $d_x$ and $d_y$ are referred to as horizontal and vertical global-position relations, respectively. They could more realistically be represented as fuzzy quantities of 'type 2' (Zadeh, 1974; Mizumoto and Tanaka, 1976), as follows. For ordinary fuzzy entities, such as the fuzzy relation $R$ defined earlier, values are drawn from the unit interval [0, 1]. For $d_x$ and $d_y$ it is the value itself that is ill-defined, and this fact is captured by making each value of $d_x$ and $d_y$ into a function that maps possible distances into [0, 1]. Thus a value of $d_x$ such as '1°' would be represented by a function that associates a value of unity with the particular distance 1° and values closer to zero with distances correspondingly smaller or larger than 1°. For convenience, phrases like '1° to the left of the point of fixation' will continue to be used.

Global-position relations are continuously varying quantities, and the operations (or family of operations) applied to them can be similarly continuous. For example, let $\alpha(t)_x$, $0 \leq t < \infty$, be such a family of operations acting on the horizontal global-position relation $d_x$. Let $\alpha(t)_x$ take $d_x$ into $d'_x$, that is, $\alpha(t)_x(d_x) = d'_x$. Then the new value of $d'_x$ at $f$ may be written as $d'_x(f) = d_x(f) + a_x(t)$, where $a_x$ is a continuous function of the parameter $t$, with $a_x(0) = 0$. That is, with yet another small abuse in notation,

$$\alpha(t)_x(d_x) = d_x + a_x(t)$$

Likewise for the vertical global-position relation $d_y$ and operation $\alpha(t)_y$, $0 \leq t < \infty$. The operations such as $\alpha(t)_x$ and $\alpha(t)_y$ are referred to as *continuous-modification* operations, or *continuous-shift* operations when $a_x(t) = a_x \cdot t$, that is, a linear displacement from the original position.

## Topological, Affine, and Metric Relations

For each of the major mathematical structures related to geometry, it is possible to find intrinsic or extrinsic spatial relations of visual significance. For topology, there are intrinsic spatial relations 'connected to', 'inside', 'in the neighbourhood of'; for affine geometry, there are intrinsic spatial relations 'collinear', and 'parallel'; for a metric structure, there are extrinsic spatial relations of the form 'separated by 1° (of visual angle)'; and so on.

It is, however, important to distinguish between objective and visual definitions of some of these relations. The difference is illustrated in Fig. 5.1 (adapted from Minsky and Papert, 1969), for the intrinsic topological spatial relation 'inside'. Without scrutiny, it is impossible to decide if the dot is inside or outside the contour (or indeed whether there is one closed contour or two, another topological property). The question is well defined, and the difficulty in performing the task is not a matter of visual acuity. Although the problem may be resolved by restricting the domain of definition of the spatial relation 'inside' to some subset of less-convoluted figures, that approach is *ad hoc* and suggests that for vision the spatial relation 'inside' is expressed within an inappropriate topological-geometrical structure.

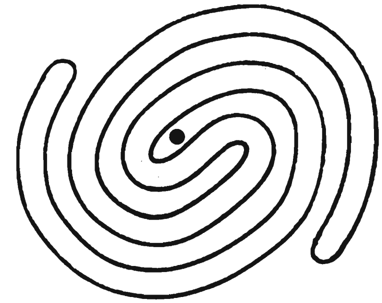

Fig. 5.1 *A test of the ability to perceive an intrinsic spatial relation: The task is to determine whether the dot is inside a closed contour. (Adapted from Minsky and Papert, 1969, Fig. 5.1, with permission.)*

## Image Rotations and Invariances

By definition, intrinsic spatial relations provide properties that are independent of the space in which the object is embedded and of the observer's viewpoint. In practice, however, visual performance in recognizing objects is not invariant under all the expected transformations: translations, dilatations, rotations and affine transformations. The most notable failure in invariance is under rotations.

### Pattern Rotation

It is well established that the perception, identification, and discrimination of planar figures depends on the orientation of the figures in the plane (Mach, 1897, Chapter 6; Dearborn, 1899; Aulhorn, 1948; Arnoult, 1954; Kolers and Perkins, 1969, 1975; Rock, 1973, Chapter 3; Foster, 1978; Kahn and Foster, 1981). For displays of limited duration, too short for deliberate saccades and scrutiny of the image, performance in discriminating rotated 'same' patterns (identical patterns related by a rotation) from 'different' patterns (patterns paired at random) declines with rotation angle for angles up to about 90°, and then *increases* again with rotation angle for angles up to 180° (see e.g. Fig. 5.4, open symbols, discussed in detail later). This upturn in performance for discriminating 'same-different' patterns at 180° angle of rotation (Foster, 1978) is not specific to particular types of patterns: it occurs with randomly contoured shapes (Dearborn, 1899; Rock, 1973), with random-dot patterns (Foster, 1978; Kahn and Foster, 1981, 1986), and with alphabetic shapes (Aulhorn, 1948). It also occurs with drawings of natural objects (Jolicoeur, 1985). Because this performance is obtainable with randomly formed patterns, it is not a simple consequence of the meaning, conventional orientation, or handedness of the stimuli.

As an aside, it may be noticed that this form of the angular dependence is very different from that obtained by Shepard and his colleagues in 'mental rotation' experiments (see e.g. Shepard and Metzler, 1971; Cooper and Shepard, 1973; Shepard, 1975; Shepard and Cooper, 1982). There a monotonic dependence of reaction time for a correct response on angle of rotation was obtained: the larger the angle of rotation, the longer the reaction time. The experimental task typically involved the accurate discrimination of rotated 'same' patterns from rotated 'same-but-reflected' patterns. Reaction times for these sense discriminations were of the order of seconds. A critical factor in inducing this monotonic behaviour may have been the requirement to make a 'left-right' or sense discrimination (Cooper and Shepard, 1973; Corballis and McLaren, 1984; Jolicoeur, 1985), as opposed to a 'same-different' judgment.

Nonmonotonic performance obtained in 'same-different' discriminations has implications for the kinds of spatial relations used by the visual system; in particular, it suggests that some spatial relations must be extrinsic. The rotation through 180° has a particular status: it is equivalent to a reflection through the origin and, for convenience, it will occasionally be referred to as a *point-inversion*, denoted by PI (which has the additional mnemonic value of representing the angle turned through in radians).

## Matching Relations, Sense-Reversal and Pattern Rotation

Given two sets of spatial relations from two patterns, how should they be compared? The simplest internal matching process is one that counts or estimates the number of identical (or non-identical) spatial relations in the two patterns. Such a process has been proposed in the analysis of dot-displacement detection in random-dot patterns and in the analysis of 'same'-detection performance with rotated random-dot patterns. A brief account of these analyses follows.

### Counting Spatial Relations

In an experiment (French, 1953) to investigate the discrimination of differences between dot patterns, subjects were presented with pairs of dot patterns, each comprising two to seven dots distributed almost randomly in a region of average extent approximately 1.5°. The patterns were either identical or differed in that a single dot was displaced by approximately 0.3° (Fig. 5.2). Each pattern was

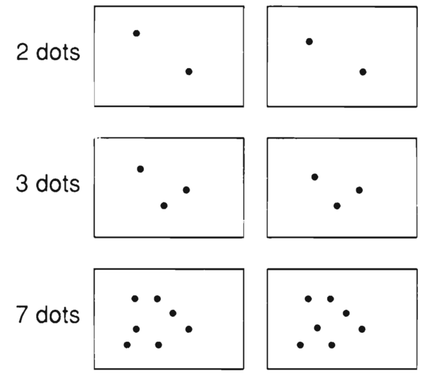


Fig. 5.2 *Illustrations of the types of 'different' pairs of dot patterns used in a displacement discrimination task. One dot in one member of each pair of patterns is displaced relative to the other dots. (Adapted from French, 1953.)*

presented for 3 s. The mean percentage of errors in subjects' 'same-different' judgments declined as the number of dots increased from two to three, and then increased monotonically as the number increased from three to seven. In more exhaustive measurements of the effects of dot-number on the detection of dot displacement (Pollack, 1972), a monotonic increase in proportion of errors with dot-number from two to 64 was obtained.

It was suggested (French, 1953) that one of the factors contributing to the monotonic increase in errors with dot number was the decreasing proportion of altered relations in the patterns as the number of dots increased. Thus, in a pattern with $n$ dots, the total number of relations was $(n(n-1))/2$, and the number of relations modified by the displacement of a single dot was $n-1$. The ratio of the number of modified relations to the total number of relations was therefore $(n-1)/((n(n-1))/2) = 2/n$. Although this function decreases with $n$, no quantitative comparisons with the increasing experimental error rate were made. The decline in error rate as the number of dots increased from two to three was thought to be the result of a change in the kind of relations available (French, 1953).

It should be emphasized that no assumption was made about the types of spatial relations underlying this performance, other than that they were binary, with one relation to each pair of dots (any more being accounted for by a scale factor), and that the relations depended on the relative positions of the dots (French, 1953).

In a different analysis, of 'same'-detection performance with rotated random-dot patterns (Foster, 1978), an explicit assumption was made about the nature of the spatial relations used for comparing the patterns. Subjects were presented with pairs of patterns that were either rotated versions of each other, forming 'same' pairs (Fig. 5.3(a)), or paired at random, forming 'different' pairs (Fig.5.3(b)). Each pattern contained 10 dots distributed randomly

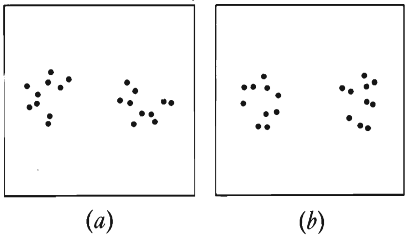


Fig. 5.3 *Illustrations of the types of random-dot patterns used to test the effects of pattern rotation on 'same-different' discriminations (Foster and Mason, 1979). In (a) the patterns have the same shape and differ only in orientation; in (b) the patterns are paired at random. The rectangular frame was not part of the stimulus display.*

within an imaginary circle of 0.75° angular subtense, and the centre-to-centre spacing of the patterns was 1.25°. Display duration was 200 ms. In the analysis, the assumption was made that matching was based on the extrinsic spatial-order relations 'left of', 'above', and their opposites (Foster and Mason, 1979). Other intrinsic spatial relations such as 'near to' that might have been included were ignored since they would have been invariant under pattern rotation. Equation 5.2 shows each of these spatial-order relations combined with its opposite to yield spatial-order relations $r_x, r_y$ between local features (here dots) $f_j, f_k$:

$$r_x(f_j,f_k) = \begin{cases} -1, & \text{for } f_j \text{ 'left of' } f_k \\ 1, & \text{for } f_j \text{ 'right of' } f_k \\ 0, & \text{otherwise} \end{cases}$$

$$r_y(f_j,f_k) = \begin{cases} -1, & \text{for } f_j \text{ 'below' } f_k \\ 1, & \text{for } f_j \text{ 'above' } f_k \\ 0, & \text{otherwise} \end{cases}$$

Thus each pattern of $n$ dots was represented as a set of $n(n-1)$ horizontal and vertical relations (see Foster and Kahn, 1985, for a more complete formulation), that is,

$$\{r_x(f_j,f_k),\ r_y(f_j,f_k) : 1 \leq j < k \leq n\} \qquad (5.3)$$

This description and the description of the same pattern rotated through a very small angle would be the same, but, it was argued, as the angle of rotation increased, the proportion of altered relations increased, so that some relations $r_x(f_j,f_k) = -1$ and $r_y(f_j,f_k) = -1$ ('left of' and 'below'), for example, changed respectively to $r_x(f_j,f_k) = 1$ and $r_y(f_j,f_k) = 1$ ('right of' and 'above'). This increasing mismatch between sets of relations accounted for the decline in observed performance with increasing angle of rotation. Beyond 90°, however, observed performance improved, whereas the proportion of altered relations continued to increase, until, at 180° rotation, the whole set of relations was inverted, resulting in all occurrences of $r_x(f_j,f_k) = -1$ and $r_y(f_j,f_k) = -1$ being replaced respectively by $r_x(f_j,f_k) = 1$ and $r_y(f_j,f_k) = 1$, and vice versa. To explain the improvement in performance, a simple internal compensatory operation was postulated, called *global sense-reversal.*

## Global Sense-Reversal

It was proposed (Foster and Mason, 1979) that before two sets of spatial-order relations were matched, an internal operation $\sigma = (\sigma_x, \sigma_y)$ could be applied, globally, to one of the sets, the effect of which would be to transform each spatial-order relation into its inverse:

$$\sigma_x(r_x) = -r_x$$
$$\sigma_y(r_y) = -r_y \qquad (5.4)$$

All occurrences of $r_x(f_j,f_k) = 1$ ('right of') were therefore replaced by $r_x(f_j,f_k) = -1$ ('left of') and vice versa,

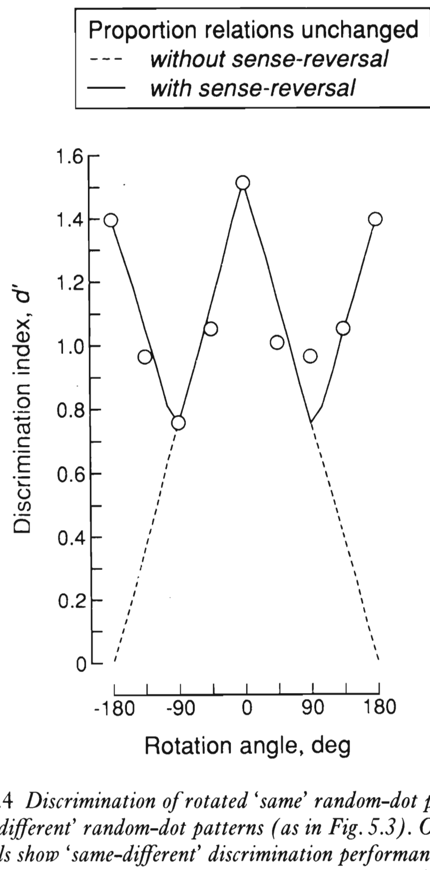


Fig. 5.4 *Discrimination of rotated 'same' random-dot patterns from 'different' random-dot patterns (as in Fig. 5.3). Open symbols show 'same-different' discrimination performance as a function of rotation angle. The continuous and broken lines are the respective predicted performances based on counts of the number of unchanged spatial relations in the patterns with and without a sense-reversal operation. (Adapted from Foster and Mason, 1979.)*

and all occurrences of $r_y(f_j, f_k) = 1$ ('above') were replaced by $r_y(f_j, f_k) = -1$ ('below') and vice versa. This global sense-reversal thereby compensated for the effects of point-inversion precisely. It should be noted that global sense-reversal is equivalent to a simple relabelling operation. The detailed variation of 'same'-detection performance with rotation angle was predicted by the counting measure, that is the number of unchanged spatial-order relations, expressed as a proportion of the total, with and without global sense-reversal.

Fig. 5.4 shows expected 'same'-detection performance without the sense-reversal operation (broken line) and with the sense-reversal operation (continuous line) as a function of pattern rotation, based on an exhaustive evaluation of the counting measure over 15° intervals for 21 different random-dot patterns, each comprising 10 dots (Foster and Mason, 1979). The open symbols in Fig. 5.4 show observed 'same-different' discrimination performance with the same set of patterns taken from another study (Foster, 1978). Performance is expressed in terms of the criterion-free discrimination index $d'$ from signal-detection theory (Tanner and Swets, 1954). The index is zero when performance is at chance level and increases monotonically (without limit) as performance improves. Given certain assumptions, it is bias free and additive (Durlach and Braida, 1969).

The predicted performance allowing sense-reversal was clearly superior, and with that operation the description based on the spatial relations 'left of' and 'below' accounted well for the detailed variation of observed discrimination performance with rotation angle.

# Global-Position Relations and Continuous-Shift Operations

Experiments on the effects of pattern rotation showing an upturn in performance at 180° rotation have typically employed symmetric arrangements of the stimuli, presented either as side-by-side pairs (Foster, 1978) or one at a time, centrally in the visual field (Dearborn, 1899; Rock, 1973). But judgments of the perceptual similarity of figures (Attneave, 1950), discrimination of mirror images (Sekuler and Rosenblith, 1964; Sekuler and Pierce, 1973), and identification of parafoveal figure pairs (Banks *et al.*, 1977, 1979; Chastain and Lawson, 1979) have all been shown to depend on the relative positions of the stimuli in the field. The time taken to report the sameness of mirror pairs has been found to be shorter when the patterns are presented symmetrically about the point of fixation than when they are both presented to one side (Corballis and Roldan, 1974; Bradshaw *et al.*, 1976). Similarly, it has been demonstrated that symmetry in a complex random-dot pattern is best perceived when the observer fixates a point on the axis of symmetry (Julesz, 1971; Barlow and Reeves, 1979; see also Bruce and Morgan, 1975).

## Positional Symmetry and Separation

How do changes in positional symmetry and separation influence the detection of rotated patterns? Two experiments designed to test these factors (Kahn and Foster, 1981) used simultaneous and sequential presentations of pairs of random-dot patterns that were identical, rotated through 180°, reflected about a vertical axis, or paired at random. Each pattern comprised 10 dots distributed randomly within an imaginary circle of 0.5° radius, and was positioned along a horizontal meridian at $-0.5°$, 0°, or 0.5° with respect to the point of fixation (and, in another experiment, at $-1.0°$, 0°, 1.0°). Each pattern was displayed for 100 ms. The symmetry and separations of the positions were quantified by the sums and differences of the pattern positions with respect to the point of fixation. Symmetry was thus treated as a continuous variable (see Barlow and Reeves, 1979). Results may be summarized as follows.

1. 'Same'-detection of pairs of identical patterns was strongly affected by the distance between the patterns and not by the symmetry of their positions with respect to the point of fixation. The greater the separation of the patterns, the worse the performance.
2. 'Same'-detection of pairs of patterns that were point-inverted (rotated through 180°) or reflected versions of each other was strongly affected by the symmetry of the positions of the patterns with respect to the point of fixation and not by the distance between the patterns. Performance was best when the patterns were positioned symmetrically about the point of fixation.

Since spatial-order relations and the associated internal sense-reversal operations were, *a priori*, insensitive to the position of the stimulus in the field (given that spatial resolution performance was not the limiting factor), it was concluded that some information about the approximate position of the pattern with respect to the point of fixation must be included in the image representation, along with a corresponding internal operation.

## Continuous-Shift Operations

In addition to the spatial-order relations $r_x(f_j, f_k)$, $r_y(f_j, f_k)$ signifying whether local feature $f_j$ was 'left of' or 'below' local feature $f_k$, it was proposed that there are additional global-position relations $d_x$, $d_y$, specifying the approximate position of the pattern in a horizontal-vertical coordinate system centred on the point of fixation (Kahn and Foster, 1981; Foster and Kahn, 1985). Attneave (1968) suggested a similar framework of separate local and global Cartesian axes for the representation of stimuli. Note that spatial-order relations are discrete-valued and global-position relations are continuous-valued. (There is some evidence that discrete and continuous spatial relations are processed differently by the visual system; Kosslyn *et al.*, 1989.)

By extension, the sense-reversal operations $\sigma_x, \sigma_y$ (Equation 5.4) apply not only to the spatial-order relations $r_x$, $r_y$ but also to the global-position relations $d_x, d_y$, thus

$$\sigma_x(d_x) = -d_x$$
$$\sigma_y(d_y) = -d_y$$

An additional assumption was made, namely that the global-position relations, since they specified continuously varying quantities, could be modified individually in a progressive, continuous fashion. Thus if $(\alpha(t)_x, \alpha(t)_y)$, $0 \leq t < \infty$, was this continuous sequence of operations, parameterized by time $t$, then:

$$\alpha(t)_x(d_x) = d_x + a_x \cdot t$$
$$\alpha(t)_y(d_y) = d_y + a_y \cdot t$$

where $a_x$, $a_y$ are constants, governing the speed at which the transformation of the global-position relations $d_x$, $d_y$ to their new values was effected. This continuous sequence of operations is similar to that proposed by Shepard and his colleagues (see earlier) in the analysis of mental rotation experiments. Both of these operations, $\sigma$ and $\alpha$, it was assumed, could be used in the internal comparison of two pattern representations, but with an efficiency depending on the size of the operation needed to bring the representations into coincidence (Kahn and Foster, 1981; Foster and Kahn, 1985).

The dependence of detection performance on positional symmetry and separation of reflected and rotated patterns, summarized earlier, was then explained as follows.

Pairs of identical patterns differing only in position were detected as 'same' by an application of the continuous-shift operations $\alpha(t)_x$, $\alpha(t)_y$, $0 \leq t < \infty$, to the global-position relations $d_x, d_y$ in the two patterns, until the representations coincided. The spatial-order relations $r_x(f_j, f_k)$, $r_y(f_j, f_k)$ were identical in the two patterns and did not require adjustment. Increased pattern separation required more modification of $d_x, d_y$ before the match could be achieved, and so performance was reduced.

Pairs of symmetrically positioned patterns that were related by point-inversion were detected as 'same' by a global application of the sense-reversal operations $\sigma_x, \sigma_y$. All the spatial-order relations $r_x(f_j, f_k)$, $r_y(f_j, f_k)$ and global-position relations $d_x, d_y$ were inverted. Thus the spatial-order relation 'above' relating one local feature to another in the pattern became 'below', and the positional relation '1° to the left of the fixation point' became '1° to the right of the fixation point'. The result was that the two representations were brought into coincidence. If the two point-inverted patterns were not symmetrically positioned with respect to the point of fixation, this operation was not sufficient. In that case, the 'sameness' of the stimuli was less detectable, since further modification of the positional relations was necessary to achieve a match.

Pairs of symmetrically positioned patterns that were related by reflection in a vertical line were detected as 'same' by a global application of the sense-reversal operation $\sigma_x$. All the spatial-order relations $r_x(f_j, f_k)$ and the global-position relation $d_x$ were inverted. The spatial-order relations $r_y(f_j, f_k)$ and the global-position relation $d_y$ were unchanged. Thus the spatial-order relation 'left of' relating one local feature to another became 'right of', and the positional relation '1° to the left of the fixation point' became '1° to the right of the fixation point'. This again brought the two representations into coincidence. If the two reflected patterns were not positioned symmetrically with respect to the point of fixation, this operation was not sufficient and performance was again reduced.

This description of spatial-order and global-position relations and their internal operations was based on data obtained for just three possible horizontal positions for each pattern (Kahn and Foster, 1981). This number was increased to five (−1.0°, −0.5°, 0°, 0.5°, 1.0°) in a more detailed measurement of the effects of symmetry and separation on the recognition of point-inverted and identical patterns (Foster and Kahn, 1985). The results were similar: 'same'-detection of identical patterns was affected only by positional separation, that of point-inverted patterns only by positional symmetry.

## Implications of a Horizontal-Vertical Reference System

The assumptions that spatial relations are defined with respect to a horizontal-vertical reference system, centred on the point of fixation, explains the effects of positional symmetry and separation. If the results obtained with reflected patterns are ignored, it is possible to make a weaker but otherwise equally effective assumption, namely that the axes of the reference system are merely orthogonal, not necessarily oriented along the horizontal and vertical. Thus, in principle, 'oblique' spatial relations could be defined, the global sense-reversal of which would explain the detectability of symmetrically positioned point-inverted patterns.

The assumption of a horizontal-vertical reference system for spatial-order relations does imply certain constraints on performance that would not be expected from a purely object-centred pattern description, or from viewer-centred descriptions that are isotropic, such as those using

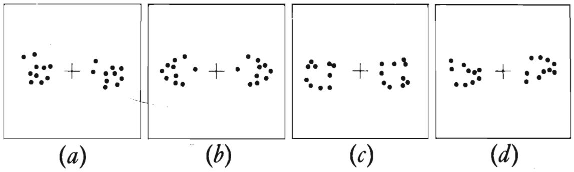

Fig. 5.5 *Illustrations of the types of random-dot patterns used to test for a reflection-axis effect in 'same-different' discriminations. In each of (a)–(d), one pattern is obtained from the other by reflection in an axis oriented at −45°, 0°, 45° and 90° clockwise from the vertical. The cross shows the point of fixation, but neither it nor the rectangular frame was visible during the simultaneous presentation of the patterns. (Adapted with permission from Kahn and Foster, 1986.)*

polar coordinate systems (Leibovic *et al.*, 1971; Schwartz, 1980). Some of the implications of a horizontal-vertical reference system have been tested experimentally (Kahn and Foster, 1986), as follows.

## Reflection-Axis Effect

Suppose that 'same' pattern pairs were related by reflection in an axis of variable orientation (Fig. 5.5). If, as illustrated, the patterns were positioned horizontally and symmetrically about the fixation point, the highest 'same'-detection performance should occur when the reflection axis is perpendicular to an imaginary line joining the centres of the patterns (Fig. 5.5(b)). All that is needed to bring the representations into coincidence is a global application of the sense-reversal operation $\sigma_x$; the spatial-order relations $r_x(f_j,f_k)$ and the global-position relation $d_x$ would be inverted. Additional internal operations would be required in all the other conditions (Fig. 5.5(a),(c),(d)).

'Same-different' pattern discrimination performance was obtained (Kahn and Foster, 1986) as a function of the orientation of the reflection axis, −45°, 0°, 45°, 90° to the vertical. The patterns consisted of 10 dots distributed randomly within an imaginary circle of diameter 0.5°, one pattern centred 0.5° to the left of the fixation target, the other 0.5° to the right (as in Fig. 5.5). Display duration was 100 ms. (The number of subjects in this and subsequent experiments in the series varied from four to nine.) The display was presented for 100 ms.

As expected 'same'-detection performance was found to be high for patterns reflected about a vertical axis, 0°, and low at all other angles (see also Sekuler and Rosenblith, 1964, and Foster and Mason, 1979).

## Selective Oblique Effect

Suppose that 'same' patterns were related by a reflection in an axis perpendicular to an imaginary line joining the centres of the patterns (Fig. 5.6(a)–(d)). If the patterns were positioned symmetrically about the fixation point,

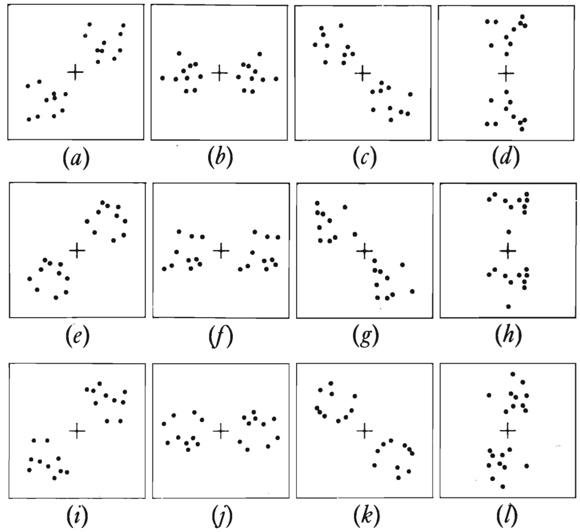

Fig. 5.6 *Illustrations of the types of random-dot patterns and their transformations used to test for a selective oblique effect in 'same-different' discriminations. In each of (a)–(d) one pattern is obtained from the other by reflection in an axis perpendicular to an imaginary line joining the centres of the patterns; in each of (e)–(h) the patterns are identical; in each of (i)–(l) one pattern is obtained from the other by point-inversion. Other details as in Fig. 5.5. (Adapted with permission from Kahn and Foster, 1986.)*

then 'same'-detection performance should be lower when the imaginary line joining the centres of the patterns was oblique (Fig. 5.6(a),(c)) than when it was horizontal or vertical (Fig. 5.6(b),(d)). When the imaginary line was oblique, the representations of the patterns could not, in principle, be brought into coincidence by a global application of the sense-reversal operations $\sigma_x,\sigma_y$. When the imaginary line was horizontal or vertical, $\sigma_x$ or respectively $\sigma_y$ was sufficient.

Suppose that the patterns were identical (Fig. 5.6(e)–(h)). Then there should be no oblique effect of the kind discussed for reflected patterns, although an effect due to differences in visual acuity might have been anticipated (see earlier references). Independent of the orientation of the imaginary line joining the patterns, they differed by a constant separation, and their representations could, in principle, be brought into coincidence by the application of the continuous-shift operations $\alpha(t)_x$, $\alpha(t)_y$, $0 \leq t < \infty$, to the global-position relations $d_x$, $d_y$.

Finally suppose that the patterns were related by point-inversion (Fig. 5.6(i)–(l)). There should also be no oblique effect of the reflected-patterns kind. Independent of the orientation of the imaginary line joining the patterns, their representations could, in principle, be brought into coincidence by global application of the sense-reversal operations $\sigma_x, \sigma_y$ inverting all spatial-order relations $r_x(f_j,f_k)$, $r_y(f_j,f_k)$, and all global-position relations $d_x$, $d_y$.

'Same-different' pattern discrimination performance was obtained (Kahn and Foster, 1986) as a function of display orientation, $-45°, 0°, 45°, 90°$ to the vertical, for each of the pattern transformations: identity, reflection, and point-inversion. Experimental details were similar to those of the previous experiment, but as a further precaution the patterns were 'normalized' by individual linear horizontal and vertical scaling so that the maximum horizontal dot-separation and the maximum vertical dot-separation were always 0.5°. It was thus impossible for subjects to use inappropriate strategies (such as testing for equal pattern widths) to discriminate 'same' from 'different' patterns.

It was found that, consistent with the foregoing analysis, 'same'-detection performance for patterns related by a reflection showed a strong oblique effect: discrimination was best for vertical and horizontal axes, and worst for oblique axes. For identical or point-inverted patterns, there was no little or no oblique effect.

## Selective Midline Effect

Suppose that the patterns were related by reflection in a vertical axis (Fig. 5.7(a),(b)). If the patterns were positioned symmetrically about the vertical midline, then 'same'-detection performance should be independent

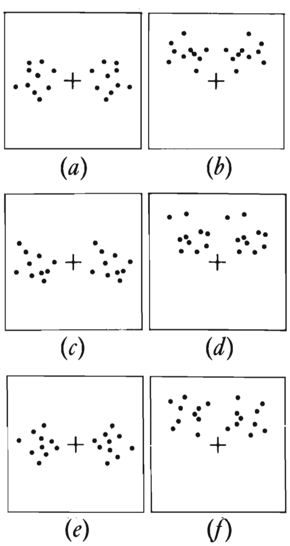


Fig. 5.7 *Illustrations of the types of random-dot patterns and their transformations used to test for a selective midline effect in 'same-different' discriminations. In each of (a) and (b) one pattern is obtained from the other by reflection in the vertical midline; in each of (c) and (d) the patterns are identical; in each of (e) and (f) one pattern is obtained from the other by point-inversion. In conditions (b), (d) and (f), the vertical offset occurred upwards and downwards equally often. Other details as in Fig. 5.5. (Adapted with permission from Kahn and Foster, 1986.)*

(within some limits) of the vertical positions of the patterns. In principle, the representations of the patterns could be brought into coincidence by a global application of the sense-reversal operation $\sigma_x$; the spatial-order relations $r_x(f_j, f_k)$ and the global-position relation $d_x$ would both be inverted.

Suppose that the patterns were identical and positioned symmetrically about the vertical midline (Fig. 5.7(c),(d)). Then as for reflected patterns, 'same'-detection performance should also be independent (within limits) of vertical position.

Suppose, finally, that the patterns were related by point-inversion and positioned symmetrically about the vertical midline (Fig. 5.7(e),(f)). Then 'same'-detection performance should be lower for patterns above or below the horizontal midline than for patterns on the horizontal midline. In principle, only when the patterns were in line with the fixation point could the representations be brought into coincidence, by a global application of the sense-reversal operations $\sigma_x$, $\sigma_y$, inverting all spatial-order relations $r_x(f_j, f_k)$, $r_y(f_j, f_k)$ and all global-position relations $d_x, d_y$. When the patterns were above or below the fixation point, additional operations would be required.

'Same-different' pattern discrimination performance was obtained (Kahn and Foster, 1986) as a function of the two pattern-position combinations, in-line and offset, for each of the three pattern transformations: identity, reflection, and point-inversion. Experimental details were similar to those of the previous experiment, and normalized patterns were used. Again, as anticipated, 'same'-detection performance showed a marked worsening in the offset condition for patterns related by point-inversion, but no worsening in the offset condition for patterns that were identical or related by a reflection.

It was concluded (Kahn and Foster, 1986) from these three sets of experiments (Figs. 5.5–5.7) that if the visual system did use spatial-order and global-position relations in image descriptions, a horizontal-vertical reference system was essential to their implementation.

# Spatially Selective Internal Operations

In the analysis of the data reviewed in the last section, it was assumed implicitly that in the detection of point-inverted patterns it was not possible to apply, at least efficiently, the sense-reversal operations $\sigma_x, \sigma_y$ to the spatial-order relations $r_x(f_j, f_k)$, $r_y(f_j, f_k)$ alone; that is, any sense-reversal of the $r_x(f_j, f_k)$, $r_y(f_j, f_k)$ was accompanied by a sense-reversal of the global-position relations $d_x, d_y$.

In an investigation (Bischof *et al.*, 1985) designed to test whether sense-reversal could be applied selectively, subjects were presented with stimulus patterns (examples illustrated in Fig. 5.8(a)–(d)) consisting of a number of small subpatterns (shown in the upper section of Fig. 5.8). The subpatterns were chosen for their asymmetry under point-inversion. Each pattern was generated by random selection, with replacement, of five subpatterns. The orientations of the subpatterns were chosen randomly, as were their locations, subject to the constraint that their centres were within a limiting circle of diameter 0.5°, in one experiment, and 1.0° in another. In addition to the identity transformation and global point-inversion PI, two *non*-uniform point-inversions were applied to each pattern:

1. $PI_S$, inverting all the subpatterns about each of their centres, but leaving their positions unaltered;
2. $PI_P$, inverting the positions of all the subpatterns about the centre of the pattern, but leaving the orientations of the subpatterns unaltered.

First consider subpattern point-inversion $PI_S$. Fig. 5.8(a) and (c) shows two patterns related by this transformation. If there existed an efficient way to apply the sense-reversal operations $\sigma_x,\sigma_y$ solely to the spatial-order relations $r_x(f_j,f_k),r_y(f_j,f_k)$ associated with each of the subpatterns, thus leaving any global-position relations $d_x,d_y$ intact, then 'same'-detection performance for patterns related by transformation $PI_S$ should be high.

Second, consider position point-inversion $PI_P$. Fig. 5.8(a) and (b) shows two patterns related by this transformation. If there existed an efficient way to apply the sense-reversal operations $\sigma_x,\sigma_y$ solely to the global-position relations $d_x,d_y$, leaving spatial-order relations $r_x(f_j,f_k)$, $r_y(f_j,f_k)$ intact, then 'same'-detection performance for patterns related by transformation $PI_P$ should also be high.

If, however, none of these selective applications of the sense-reversal operations $\sigma_x,\sigma_y$ were possible, then the only way in which these patterns could be detected as 'same' would be by the use of other inefficient procedures, such as the continuous-shift operation considered earlier.

Note that the composition of subpattern point-inversion $PI_S$ and position point-inversion $PI_P$ (that is, the application of the transformations in turn) results in global point-inversion PI. The transformations also commute. Symbolically,

$$PI = PI_S \circ PI_P = PI_P \circ PI_S \tag{5.5}$$

which may be compensated for precisely and efficiently by global application of the sense-reversal operations $\sigma_x, \sigma_y$. (As in another study dealing with the detailed effects of pattern rotation on 'same'-detection performance, there is an equivalent version of this analysis in which the subpatterns are treated as distinct local features rather than as clusters of dots associated with spatial relations $r_x(f_j,f_k), r_y(f_j,f_k)$; see Foster and Mason, 1979; Bischof *et al.*, 1985.)

In the experiment, pairs of 'same' and 'different' random-dot patterns were presented in sequence, at the point of fixation (Bischof *et al.*, 1985). Each pattern was presented for 100 ms, with a 1 s interval between each member of the pair. 'Same-different' discrimination performance was obtained from four subjects, as a function of pattern transformation: identity transformation, position point-inversion $PI_P$, subpattern point-inversion $PI_S$, and global point-inversion PI. Large (1° diameter) and small (0.5° diameter) patterns were tested. Details of the classes of 'different' patterns used to compute the discrimination index values $d'$ are given in Bischof *et al.* (1985).

For both large and small patterns, discrimination was found to be high under global point-inversion PI. For

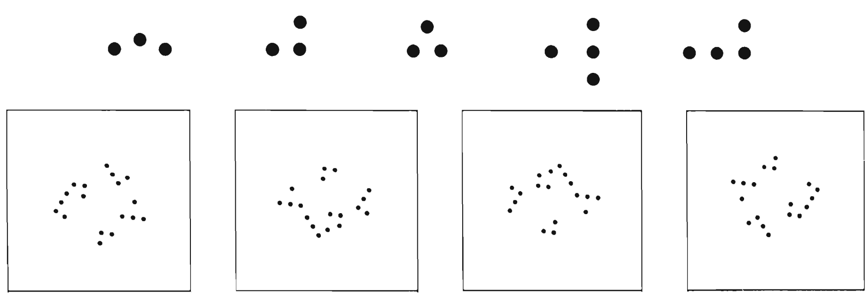


Fig. 5.8 *Illustrations of the types of random-dot patterns used to test effects of non-uniform point-inversion transformations on 'same-different' discriminations. Each pattern in (a)–(d) was composed of five subpatterns chosen randomly, with replacement, from the set shown in the upper section of the figure, with the orientations of the subpatterns and their positions also chosen randomly (within constraints). 'Same' pattern pairs were related by one of the following transformations: identity (pattern (a) and its duplicate), position point-inversion* $PI_P$ *(patterns (a) and (b)), subpattern point-inversion* $PI_S$ *(patterns (a) and (c)), and global point-inversion* PI *(patterns (a) and (d)). The rectangular frame was not part of the stimulus display. (Adapted, with permission, from Bischof* et al., *1985.)*

large patterns, discrimination was also well above chance level under the non-uniform point-inversion transformations $PI_P$ and $PI_S$, but, for small patterns, performance was almost indistinguishable from chance level.

The only difference between large and small patterns was the mean spacing of the subpatterns. The inference was made that the sense-reversal operations $\sigma_x, \sigma_y$ could in effect be applied selectively either to spatial-order relations $r_x(f_j,f_k)$, $r_y(f_j,f_k)$ or to global-position relations $d_x, d_y$ provided that the separations of the components of a pattern, in this case the subpatterns, were sufficiently large.

The efficiency of these selective operations, as assessed by the corresponding levels of discrimination performance with large patterns, was a little less than when the sense-reversal operations were applied globally. It was hypothesized (Bischof *et al.*, 1985) that one of the selective operations could have been effected indirectly, by virtue of the fact that position point-inversion $PI_P$ is formally equivalent to the composition of global point-inversion PI and subpattern point-inversion $PI_S$, that is, $PI_P = PI \circ PI_S$ (Equation 5.5). Thus, it was argued that the reason for a poorer discrimination performance under position point-inversion $PI_P$ was that the $\sigma_x, \sigma_y$ could *not* be applied selectively to global-position relations $d_x, d_y$ independently of the spatial-order relations $r_x(f_j,f_k), r_y(f_j,f_k)$, and that the observed better-than-chance performance was achieved by a relatively inefficient composition of the $\sigma_x, \sigma_y$ applied globally and then to the $r_x(f_j,f_k), r_y(f_j,f_k)$ alone.

# Symmetry

## A Symmetry-Detection Rule

The representation of patterns in terms of spatial-order relations and internal operations offers a simple rule for the perception of symmetry: any pattern that has a description in terms of spatial-order relations which is invariant under global sense-reversal operations should be perceived as symmetric. Although the task of discriminating two 'same' patterns related by a reflection from two 'different' patterns paired at random can be treated as an implicit detection of symmetry, a number of studies have examined explicitly the question of symmetry detection. As will be seen, the symmetry-detection rule gives a good account of the results.

It has long been known that symmetry in a pattern is most obvious perceptually when the axis of symmetry is vertical (Mach, 1897; Rock and Leaman, 1963; Goldmeier, 1972; see also Rock, 1973, Chapter 2). More comprehensive measurements have provided further confirmation. The effects of symmetry-axis orientation on response time for detecting bilateral symmetry have been determined (Palmer and Hemenway, 1978) for closed polygons with single, double, quadruple, rotational, and near (incomplete) symmetry. Detection was found to be fastest for vertical, next fastest for horizontal, and slowest for oblique axes. At each orientation, correct responses to quadruple symmetries were fastest, then double symme-

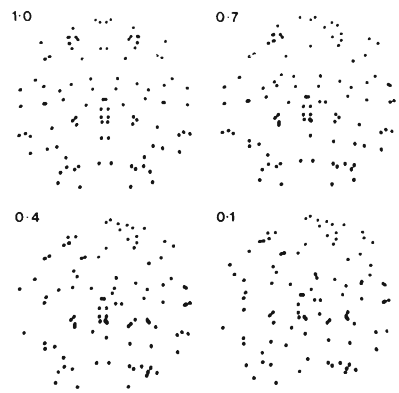


Fig. 5.9 *Illustrations of the types of random-dot patterns with varying grades of symmetry used to test the effect of orientation of axis of symmetry. Each pattern contains 100 dots with the proportion indicated belonging to pairs and the remainder placed at random. (From Barlow and Reeves, 1979, reprinted with permission.)*

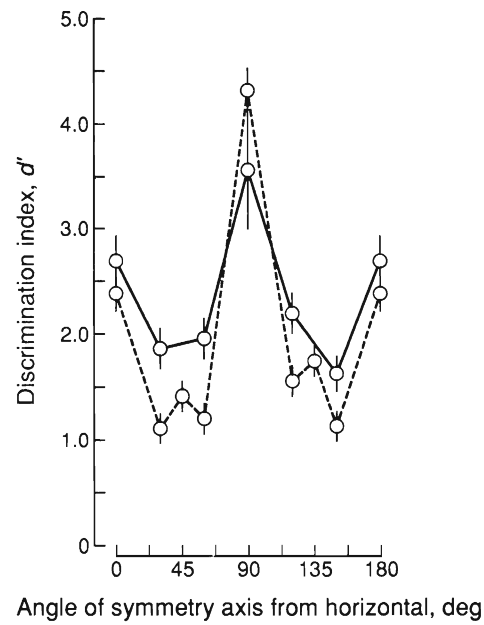


Fig. 5.10 *Effect of orientation of axis of symmetry on detection of symmetry. Discrimination of 0.8 symmetric dot-patterns (Fig. 5.9) from random dot-patterns is shown as a function of symmetry-axis orientation. Data are shown for two subjects (continuous and broken lines). (Adapted from Barlow and Reeves, 1979, with permission.)*

tries, then single. Rotational symmetries showed no orientation effect.

The effects of symmetry-axis orientation on the discriminability of bilaterally symmetric patterns have been investigated (Barlow and Reeves, 1979) with random-dot displays (Fig. 5.9). It was found that discrimination of almost bilaterally symmetric dot patterns (100 dots with a proportion 0.8 forming symmetric pairs; Fig. 5.9) from random-dot patterns (100 randomly placed dots) was best when the orientation of the symmetry axis was vertical, next best when it was horizontal, and worst when it was oblique or near oblique (Fig. 5.10).

This dependence of discrimination index $d'$ on angle was similar to that obtained in 'same-different' judgments of reflected patterns (Kahn and Foster, 1986; Fig. 5.6(a)–(d) here).

Displacing the symmetry axis away from the point of fixation has been found to worsen symmetry-detection performance (see also Julesz, 1971, Bruce and Morgan, 1975; Barlow and Reeves, 1979), a result which parallels the 'same-different' judgments of Fig. 5.7.

The type of judgments required of subjects has been examined (Corballis and Roldan, 1974) in measurements of the discriminability of identical and mirror-image patterns. Two instructional conditions were tested: the one requiring judgments 'symmetrical' and 'asymmetrical', and the other requiring judgments 'mirror' and 'same'. For random-dot patterns, instructions had no effect on reaction time. Separation of the patterns, however, was important, and for adjacent patterns, which were assumed to favour a holistic percept, it was found that symmetry was perceived more rapidly than repetition, whereas for separated patterns, which were assumed to favour the perception of distinct figures, there was no significant difference in reaction times.

## Correlation Quadrangles

An approach to the problem of detecting symmetry as a case of a more general perceptual grouping operation has been proposed (Wagemans *et al.*, 1989, 1990) in terms of a certain class of quaternary spatial relations called *correlation quadrangles*. The notion was that elements $\{f_i\}$ in a display are grouped by presenting all possible virtual lines between them and then choosing subsets of elements $(f_j,f_k,f_l,f_m)$ that form virtual quadrangles $q(f_j,f_k,f_l,f_m)$ which are *regular*, that is, the opposite sides of the virtual quadrangle are either parallel or symmetric. These regular quadrangles were argued to facilitate a 'bootstrapping' effect that reduced the potential computational effort in analysing all possible groupings. The method was shown to explain the detection of different kinds of symmetries in dot patterns including bilateral symmetry (when the virtual quadrangles were symmetrical trapeziums), and double symmetry (around the vertical and horizontal axes, when the virtual quadrangles were rectangles; Palmer and Hemenway, 1978). It also predicted the effects of pattern position and symmetry (see earlier) on relative detectabilities for identical and reflected dot patterns (Kahn and Foster, 1981) by calculating the smallest possible extent of the correlation quadrangles and their positions with respect to the point of fixation (Wagemans and Van Gool, 1989, private communication).

## Operations Independent of Spatial Relations

All the foregoing has supposed that the internal comparisons of pattern representations involve some action on spatial relations. A basic question is whether it is possible to make internal comparisons of patterns *independent* of spatial relations.

In judgments of dot-number, it is certainly possible to identify the number of dots in any configuration in a pattern quickly and accurately, provided that the number of dots is six or less (Jevons, 1871; Taves, 1941). The process is called *subitization* (Kaufman *et al.*, 1949). With more than six dots, not arranged in some regular pattern or grouped according to colour, spacing, or local collinearity (Atkinson *et al.*, 1976), a less accurate estimation process takes place (Mandler and Shebo, 1982).

The effect on number discriminations of gross changes

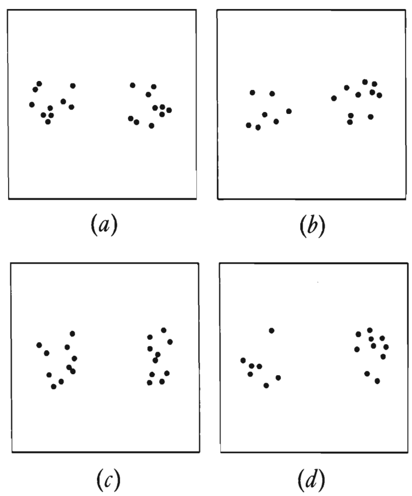


Fig. 5.11 *Illustrations of the types of random-dot patterns used to test for effects of pattern orientation and dot-number on 'same-different' discriminations. In (a) the patterns have same shape but different orientation; in (b) they have part-same shape and different dot-number; in (c) they have different shape and same dot-number; and in (d) they have different shape and different dot-number. (Adapted from Foster, 1978.)*

in stimulus configuration of patterns with more than six dots was examined by Frith and Frith (1972) who showed that one large cluster appears to contain more elements than several small clusters, where clustering was defined by Gestalt principles of contiguity and spatial separation. Taves (1941) also found a configurational effect in that placing dots in a simple arrangement (a circle) made their number appear less than when they were randomly distributed in the plane. Given, then, two patterns of dots differing only by a rigid transformation (a translation, rotation, reflection, or some combination of all three), they will necessarily possess identical clusterings, and they should therefore appear to have the same numbers of elements, provided that spatial-order relations are irrelevant.

Fig. 5.11 shows patterns used in a study (Foster, 1978) of the effects of dot-number and pattern orientation on judgments of equality of dot-number and shape. In Fig. 5.11(a) the patterns have the same shape, and differ only in orientation; in (b) they have part-same shape, but different dot-number; in (c) they have different shape but the same dot-number; and in (d) they have different shape and different dot-number. The number of dots in the patterns was either seven or 10. The dots were distributed randomly within an imaginary circle of diameter 0.75°. Pairs of patterns were presented simultaneously each side of the point of fixation. Their centre-to-centre separation was 1.25°. Display duration was 200 ms, the same as that used by Taves (1941) and by Kaufman *et al.* (1949). Twenty-four subjects made judgments about the equality of dot-number and another group of 24 subjects made judgments about the equality of shape.

'Same-different' discrimination performance $d'$ was determined for the dot-number criterion and for the shape criterion, as a function of pattern rotation angle. It was found that in judgments of dot-number-equality, with patterns of the same shape, discrimination index depended on the relative orientation of the patterns, in a manner closely similar to that for judgments of shape-equality (Foster, 1978; see Fig. 5.4).

Thus even when the distribution of dot-clusters was identical, information about extrinsic spatial relations appeared not to be suppressed when visual comparisons of numerosity were made. It was inferred (Foster, 1978) that spatial relations were bound to local features at some very early stage in visual processing. The next section deals with a possible precursor to this binding operation.

## Operations that Precede Spatial Relations: Indexing

In a number of models of the early stages of image processing (Foster, 1980; Ullman, 1984; Pylyshyn, 1989), it has been hypothesized that there is some procedure by which elements of patterns can be picked or *indexed*, before being assigned properties or attributes and their values. The hypothesis has been expressed most clearly by Pylyshyn (1989) who proposed that, as a prerequisite for detecting various relational properties between local features, there is a primitive visual process capable of indexing and tracking local features or clusters of local features. The process was assumed to be preattentive and it assigned an index or internal reference, called a FINST, to the local features of interest, *without* requiring an explicit encoding of the locations within some coordinate system or an encoding of the feature type (Pylyshyn, 1989, p 67). (The name FINST was derived from an observation that the indexing acts like sticky 'instantiation fingers', effectively pointing to the local features.)

A simple test of the plausibility of this hypothesis was made in an experiment (Pylyshyn and Storm, 1988) in which subjects had to track multiple targets in an animated display containing a number of randomly moving objects, under conditions where targets and non-targets were, apart from their histories, identical. A display consisting of 10 stationary crosses was presented to subjects who had to note the subset of one to five crosses that were flashing. After 10 s, the flashing stopped and all 10 crosses started moving randomly (within some constraints). Subjects had to track the subset that had been flashing and indicate whenever one of the target elements briefly changed shape. On average, subjects performed extremely well, even when there were five targets. Performance ranged from about 98% correct to about 86% correct as the number of targets increased from one to five.

The fact that tracking performance declined somewhat with target number led to the suggestion that there was either some serial component to the processing or there was an interaction between global attentional load and processing rate associated with individual targets, the latter equivalent to the assumption of a resource-limited parallel process (Pylyshyn and Storm, 1988). It may be relevant that some measurements of subitization performance with up to four targets have also indicated the possibility of a serial component (Mandler and Shebo, 1982; Folk *et al.*, 1988; but see Sagi and Julesz, 1985). Nevertheless, in both tracking and subitization the dominant process appears parallel. Some discussion of how spatial relations might be associated with FINSTs is given in Pylyshyn (1989). Different approaches to the notion of indexing have been developed by Ullman (1984) and Strong and Whitehead (1989). In fact, Strong and Whitehead proposed a mechanism of indexing, called *tagging*, that was essentially spatial and linked to eye-movements, an idea that may be traced back to a solution of the localization problem suggested by Lotze (1887, pp 266–267).

## Conclusion

It is significant that some of the earliest contributions to the analysis of spatial relations and their operations, most notably the works of Lotze (1887) and of Mach (1897), continue to be highly relevant. In an earlier review, Dodwell (1978) commented that the question of 'how the features are related to one another ... has received too little attention from most workers in this line of investigation' (p 533). Some progress in understanding spatial relations and their operations has been made since then, but as noted in the introduction and elsewhere there remain substantial theoretical and empirical problems, particularly in the analysis of spatial relations that are independent of the spatial framework and observer, that is, the intrinsic spatial relations.

Most experimental work has been concerned with extrinsic spatial relations, and this chapter has concentrated on those spatial relations that include the specification of a sense of direction and the possible operations that might be applied to such relations. Data were reviewed from a range of experiments requiring 'same-different' judgments of identical, rotated, point-inverted, and reflected patterns with varying positional symmetry, separation, and alignment in the visual field. Data were also reviewed from experiments requiring judgments of symmetry in patterns with varying symmetries and orientations of axes of symmetry. It was shown that the assumption of horizontal and vertical spatial-order and global-position relations with complementary sense-reversal and continuous-shift operations provided a parsimonious basis for predicting and explaining visual performance. Evidence was examined for the spatially selective action of sense-reversal operations and for the early binding of spatial relations to local features. The notion of an indexing operation as a prerequisite for forming spatial relations was also briefly considered.

The assumption of a system of spatial relations and internal operations, although economical, does not necessarily offer a unique interpretation of data derived in 'same-different' and symmetry-detection tasks. Shepard proposed an alternative explanation of the asymmetries observed in discriminating reflected and rotated patterns as a function of retinal position using the notion of mental rotation of an image about preferred (horizontal and vertical) axes (see Kahn and Foster, 1986, p 431). In a limited fashion, there is a formal duality of such a scheme and a scheme based on spatial relations and internal operations; for example, a horizontal sense-reversal operation can be represented as a 180° rotation (in depth) about a vertical axis. The duality, however, is not complete and different predictions of the two schemes were obtained for patterns that were centred on the point of fixation and rotated in the plane about that point; although not conclusive, observed performance has been argued (Kahn and Foster, 1986) to favour an interpretation based on sense-reversal operations.

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