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The Cue for Contour-Curvature Discrimination

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Seven potential geometric cues for contour-curvature discrimination were tested: *curvature, turning-angle, arc-length, arc-length-divided-by-chord-length, maximum-deviation (sag), mean-deviation* and *area*. Three experiments were performed, each requiring the discrimination of two simultaneously presented, 1-sec-duration, curved-line stimuli, whose chord-lengths ranged from 12 to 48 arcmin visual angle and whose curvatures ranged from 0 to 0.13 arcmin^{-1}. Experiments 1 and 2 determined for each cue the smallest detectable increment (the increment threshold) as a function of cue value, for a set of spatial transformations of the stimulus (one- and two-dimensional scalings) equivalent to changes in viewing distance and direction. In accordance with statistical estimation theory, the "best" cue was defined as the most efficient one, that is, the one which best accounted for the variance in the data. As a control, Expt 3 compared increment-threshold functions for circular and elliptical arcs of constant chord-length and circular arcs of constant arc-length. Over all three experiments, only sag and its linear approximation, mean-deviation, accounted well for the variance in the data; sag provided the best predictor, and its increment-threshold function satisfied Weber's law over almost all of the stimulus range. Additionally, sag has a special theoretical property (shared only with mean-deviation and area): the relationship it defines (in proportional terms) between curved contours in an image of an object or a scene is constant and independent of viewing distance and direction.

Curvature Weber's law Efficiency Variance Cue Contour Invariance Constancy Scaling Perspective

INTRODUCTION

The importance of curvature in the perception of spatial structure is well established (Attneave, 1954; Koenderink & van Doorn, 1982; Richards, Dawson & Whittington, 1986; Link & Zucker, 1988; Lehky & Sejnowski, 1988). Measurements of the ability of the eye to discriminate stimuli of different curvatures reveal an appropriately high level of performance (Ogilvie & Daicar, 1967; Watt & Andrews, 1982; Wilson, 1985; Fahle, 1991): in terms of their two-dimensional retinal projections, curved contours may be discriminated when their differences in shape are less than the spacing of individual cones, a level of performance that has been classed as "hyperacute" (Westheimer, 1975). What, then, is the geometric property of a curved contour that allows it to be characterized perceptually and discriminated from other curved contours?

Previous analyses, although establishing critical limits on performance, have not been able to identify unambiguously the cue for contour-curvature discrimination, nor indeed to determine whether there is only one such cue operating over the stimulus range (see e.g. Ogilvie & Daicar, 1967; Watt & Andrews, 1982; Watt, 1984; Wilson, 1985; Koenderink & Richards, 1988; Wilson & Richards, 1989). Candidate cues have included curvature itself, the angle turned through by a tangent moving along the curve, the deviation from linearity of the curve (or "sag"), and the area enclosed by the curve.‡ This question of the identity of the cue for contour-curvature discrimination may be considered from the standpoint of statistical estimation theory (Fisher, 1922; Stuart & Ord, 1991). For each candidate cue, let c represent the variable value of the cue. It is possible to summarize discrimination behaviour, at some criterion level of performance, by the size of the smallest detectable increment, the *increment threshold*, Δc in c at each c.§ The relationship between Δc and the reference (or standard) value c may be described by a function w, the *Weber function* (Falmagne, 1985, p. 199), thus

$$\Delta c = w(c). \tag{1}$$

Weber's law asserts that w should be linear. Its exceptions are well documented (Laming, 1986), and the

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‡Sag has been given several other names: "sagitta" (Della Valle, Andrews & Ross, 1956); "height", for a horizontal curved line (Watt & Andrews, 1982); and "arc-chord distance" (Phillips & Rosenfeld, 1987).

§This nomenclature is the same as that in Falmagne (1985, p. 197). The terms *just noticeable difference* and *difference limen* are also used, sometimes with different interpretations. Laming (1986, pp. 67–69) distinguished between the use of *difference* threshold and *increment* threshold, and in his nomenclature Δc is a difference threshold. The value of the increment threshold Δc is of course dependent on the actual criterion level of performance used to define threshold, but this level, like other non-geometrical factors, is for the present taken to be constant across conditions.

assumption of linearity is *not* necessary here, although it may be considered desirable.* The increment threshold Δc and c are in the same units in equation (1) (Laming, 1986), and so the function w is dimensionless.

In the sense of statistical estimation theory, the best cue out of a set of cues is the most *efficient* one, i.e. the one which is associated with the smallest variance.† Variance, in this case, refers to the variance of the observed increment-threshold values $\Delta c_i'$, $i = 1,2, \ldots, n$ ($n > 1$), about the sample mean $\overline{\Delta c'} = (\Sigma_i \Delta c_i')/n$, at each reference value c (the sample mean $\overline{\Delta c'}$ providing the best estimate of Δc at c). Thus, in this interpretation, the best cue is the one that accounts for the most variance in the observed data. Notice that the variance of the observed values is assumed to be due to variations in stimuli with identical values c (e.g. different-sized circular arcs with identical turning angles can have different curvatures); the effect of variations in sensory performance is considered shortly. The problem may then be approached by standard analysis-of-variance (ANOVA) techniques to estimate the significance of the "lack of fit" or scatter of the observed values about the function w in equation (1).

The value of the increment threshold Δc at each c [i.e. the function w of equation (1)] does not completely describe discrimination behaviour for a given criterion proportion correct. Such a description would require knowledge at each c of the probability density function for Δc. Additional information is, however, provided by the second moment about the mean, that is, the variance of Δc or its square root, the standard deviation σ.

For each cue it is possible to summarize this second aspect of discrimination behaviour by the relationship between σ and c, thus

$$\sigma = u(c), \tag{2}$$

for some function u. It is not assumed that u is necessarily linear. Notice that there are two sources of variability: that which underlies the non-zero value of σ and which results from the inherent variability of the observer, and that which is associated with variations between stimuli with identical values of c and which is used to assess the quality of the candidate cues.

In practice it may be necessary to execute many more trials to estimate σ than to estimate Δc. Traditional asymptotic methods for estimating the standard deviation of a threshold (e.g. Finney, 1971) may be unreliable under some conditions. A robust resampling method called the bootstrap (Efron, 1982) was therefore used (Foster & Bischof, 1991).

As was suggested earlier, the best cue in the sense of statistical estimation theory is the most efficient one, that is, the one which is associated with the smallest variance, in this case in the observed values of the standard deviation σ_i', $i = 1,2, \ldots, n$, about the sample mean $\overline{\sigma'} = (\Sigma_i \sigma_i')/n$, at each reference value c (the sample mean $\overline{\sigma'}$ providing the best estimate of σ at c). The variance of the observed values σ_i' is similarly assumed to be due to variations in stimuli with identical cue values c.

To summarize the present approach: each candidate cue for contour-curvature discrimination was assessed in relation to its capacity to estimate performance defined by the increment threshold Δc and by its standard deviation σ as functions of the cue value c. Seven attributes of curved lines were considered as candidate cues: *curvature*, *turning-angle*, *arc-length*, *arc-length-divided-by-chord-length*, *maximum-deviation* (*sag*), *mean-deviation* and *area*. The following section gives definitions of each of these attributes and the corresponding computational formulae. Three experiments were performed, each requiring the discrimination of two simultaneously presented, 1-sec-duration, curved-line stimuli. The chord-lengths of the stimuli ranged from 12 to 48 arcmin visual angle and curvatures ranged from 0 to 0.13 arcmin^{-1}. All curved lines were either arcs of circles or very close to arcs of circles (see Definitions). It was not the intention of this study to determine the cue or cues for curved-line stimuli in which curvature varied rapidly over the curved lines (cf. Wilson & Richards, 1989).

Experiments 1 and 2 determined increment threshold as a function of reference value for curved contours subjected to a range of spatial transformations (one- and two-dimensional scalings). In terms of a projected retinal image, such transformations corresponded to the natural changes in shape associated with changes in viewpoint. Performance was expressed in terms of each of the seven curved-line attributes. Each such set of threshold data was tested against equation (1). In Expt 2, the number of stimulus trials was such that reliable estimates of individual standard deviations could also be made, and for each attribute these estimates were tested against equation (2). As a control on these measurements and on the choice of stimulus parameters not specified by the spatial transformations, Expt 3 compared increment-threshold functions (for one of the curved-line attributes) for circular and elliptical arcs of constant chord-length and circular arcs of constant arc-length.

It was found that under all these conditions only two of the seven attributes accounted well for the variance in the data: sag and mean-deviation. Sag provided the best fit overall, and the increment-threshold function for sag satisfied Weber's law over almost all of the stimulus range. Sag also has a special theoretical property: the difference between the sag values of two curved contours, expressed as a proportion of the sag value of one of them, is constant and independent of one- and two-dimensional scalings, a result which has implications for the perceptual invariance of relationships between contours in an image of an object or a scene under changes in viewing distance and direction.

*Indeed Falmagne (1985) asserted that "A discrimination model is judged acceptable only if, to a good approximation, it is consistent with Weber's law at medium intensities" (p. 7).

†Additional criteria of consistency and unbiasedness (Stuart & Ord, 1991) are less relevant here.

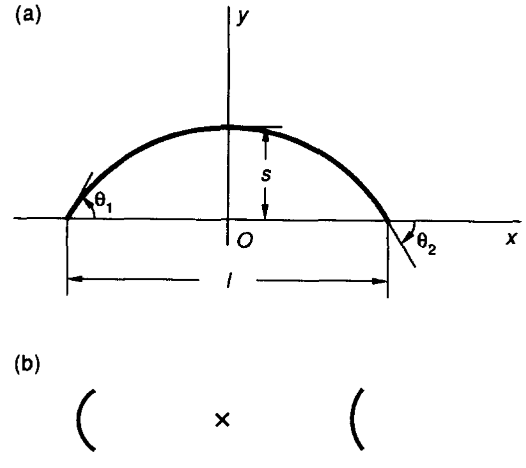

FIGURE 1. (a) Geometry of curved-line stimuli. For computational formulae used to describe curvature attributes, see Definitions section in text. (b) Display configuration. Stimulus eccentricities were 1.4 deg visual angle (measured to the midpoint of the chord); chord-lengths did not exceed 0.8 deg.

DEFINITIONS

Figure 1(a) shows a horizontally oriented curve with height s and width l located at the origin O of a local Cartesian coordinate system, with the endpoints of the curve on the x-axis and the midpoint of the curve on the y-axis. Suppose that the curve has the equation $y = f(x)$. If it is an arc of a circle, let ρ be the corresponding radius. The seven attributes considered as candidate cues were as follows.

Sag. The sag of the curve coincides with the height s as shown in Fig. 1(a), i.e.

$$\max\{f(x) \mid -l/2 \leqslant x \leqslant l/2\}.$$

Curvature. At each point $(x, f(x))$ on the curve, curvature is given by

$$|f''(x)|/(1 + (f'(x))^2)^{3/2}, \tag{3}$$

where $f'(x)$ and $f''(x)$ are the first and second derivatives of f at x. If the curve is a circular arc, then curvature is constant and equal to the reciprocal of the radius ρ of the circle; in terms of the variables s and l, curvature is then given by

$$8s/(l^2 + 4s^2). \tag{4}$$

The units of curvature are thus "length^{-1}". Expressing curvature as a rate of change of line slope in "radians·length^{-1}", as preferred by some authors, is dimensionally equivalent.

Turning-angle. The angle θ turned through by a tangent to the curve is given by

$$\arctan(f'(l/2)) - \arctan(f'(-l/2)),$$

that is, $\theta_2 - \theta_1$ [Fig. 1(a)], which, if the curve is circular, is $2 \arcsin(l/(2\rho))$. Notice that the tangent to the curve in Fig. 1 is decreasing from $x = -l/2$ to $x = l/2$, and that the value of θ_2 is negative. For the particular case in which the curve is circular and the turning angle is 120 deg, $s = \rho/2 = l/\sqrt{12}$.

Arc-length. The arc-length of the curve is

$$\int_{-l/2}^{l/2} (1 + (f'(x))^2)^{1/2}\, dx, \tag{5}$$

which reduces to $\rho\theta$ or $2\rho \arcsin(l/(2\rho))$ if the curve is circular.

Arc-length-divided-by-chord-length. The quotient of arc-length by chord-length is

$$(1/l)\int_{-l/2}^{l/2} (1 + (f'(x))^2)^{1/2}\, dx,$$

which reduces to $\rho\theta/l$ if the curve is circular.

Mean-deviation. The mean deviation of the curve is

$$(1/l)\int_{-l/2}^{l/2} f(x)\, dx,$$

which reduces to $(\rho^2/(2l))(\theta - \sin\theta)$ if the curve is circular.

Area. The area enclosed by the curve and its chord is

$$\int_{-l/2}^{l/2} f(x)\, dx,$$

which reduces to $(\rho^2/2)(\theta - \sin\theta)$ if the curve is circular.

Transformations. As detailed in the Methods section of Expt 1, all stimuli were generated from circular arcs by linear transformations of the plane; that is, with respect to the local coordinate system of Fig. 1(a): (1) by a two-dimensional scaling (*enlargement*) in the x- and y-directions $(x,y) \rightarrow (ax,ay)$, where the scale factor $a > 0$; or (2) by a one-dimensional scaling (*compression*) in the y-direction alone $(x,y) \rightarrow (x,ay)$, where $0 \leqslant a \leqslant 1$; or (3) by a one-dimensional scaling (*elongation*) in the x-direction alone $(x,y) \rightarrow (ax,y)$, where $a \geqslant 1$.* For a circular arc embedded in a plane perpendicular to the line of sight, the first operation is equivalent to a change in viewing distance, the second and third operations to a change in viewing direction possibly combined with a change in viewing distance.†

Under enlargements a circular arc remains circular and each such arc may therefore be assigned a unique value of curvature [given by formula (4)]. Under elongations and compressions, a circular arc becomes an elliptical arc and therefore no longer has a unique value of curvature (except when $a = 0$ or $a = 1$). For the curved-line stimuli used here, however, the differences between elliptical arcs and best-fitting circular arcs were very small: in the worst case, the maximum deviation in the y-direction was not more than 3% of the sag of the best-fitting circular arc. As is shown in Expt 3, such differences were not significant in determining visual

*There is no standard nomenclature for these transformations; enlargements are also called "dilatations with fixed centre", "stretchings", "homothetic transformations", and "transformations of similitude"; elongations are also called "expansions", "lengthenings", and "stretchings".

†Here and elsewhere it is assumed that this imaginary plane is sufficiently distant from the eye that perspective effects may be ignored.

performance. For elliptical arcs, therefore, curvature was defined as the curvature of the circular arc with the same chord-length and sag as the elliptical arc.

EXPERIMENT 1: EFFECT OF CURVE ENLARGEMENT ON INCREMENT-THRESHOLD FUNCTIONS

Curved-line increment thresholds were determined as a function of reference value for each of the curved-line attributes listed in the previous section. Stimuli were subjected to four levels of enlargement. Turning angle and arc-length-divided-by-chord-length are invariant under enlargements, the other attributes are not. It should be noted that in this and subsequent experiments, values of the curved-line attributes were defined for the stimuli actually presented to the eye, that is, *after* the spatial transformations had been applied.

Methods

Stimuli and apparatus. All stimuli were generated from circular arcs by linear transformations of the plane: in the local coordinate system of Fig. 1(a), by an enlargement $(x,y)\rightarrow(ax,ay)$, with $a > 0$, and by a compression in the y-direction $(x,y)\rightarrow(x,ay)$, with $0 \leqslant a \leqslant 1$. Thus if the curved lines were embedded in a plane perpendicular to the line of sight, then these transformations would correspond to moving the plane closer to or further from the eye, and rotating the plane about the chord of a curved line, that is, about the local x-axis. For each fixed chord-length, a continuum of curved lines was thus generated from a circular arc of turning angle 120 deg [Fig. 1(a)]. Intermediate members of the continuum (the elliptical arcs) were therefore not themselves circular arcs, although their departures from circularity were small (see Definitions).

The configuration of the stimuli was as illustrated in Fig. 1(b): two curved lines, oriented vertically and curved in the same direction, were presented one on each side of a small fixation target at an eccentricity of 1.4 deg visual angle.* Chord-lengths were identical in each display and values ranged from 12 to 48 arcmin visual angle. Therefore stimuli were always well separated, by a distance at least twice their individual chord-lengths. The thickness of each curved line was 1.3 arcmin visual angle. The duration of each presentation was 1 sec.

In each stimulus display, one of the curved lines had attribute value c and the other had attribute value $c + \delta c$ [Fig. 1(a,b)]. The value of c, the common direction of the curved lines (left- or right-facing), and the position of the more-curved line (left or right of the point of fixation) varied randomly from trial to trial. For each c, the value of the increment δc was varied according to a sequential adaptive testing algorithm (PEST, Taylor & Creelman, 1967; modified by Hall, 1981). Effectively, seven PEST routines ("n-fold PEST"), corresponding to seven different reference values c, were run concurrently and independently for each chord-length.

The stimuli were white and appeared superimposed on a spatially uniform, white, 30 × 35 deg background field, luminance 40 cd m^{-2}. The intensity of the stimuli was adjusted by each subject at the beginning of each experimental session (see later) to be 10-times luminance increment threshold (typically 350 μcd).

The stimuli were produced on the screen of a 21-in. X–Y display cathode-ray tube (Hewlett–Packard, U.S.A., Type 1321A) with white (P4 sulphide) phosphor (90–10% decay time approx. 100 μsec) controlled by a 16-bit laboratory computer through 12-bit digital-to-analogue converters and a 10-bit vector-graphics true line generator (Sigma Electronic Systems, England, QVEC 2150). Each curved line was composed of 12, 24, 36, or 48 concatenated straight-line segments (depending on chord size), and over the range of curvatures used here appeared smooth to the eye. The half-height full width of each line-segment drawn on the screen was 0.65 mm. The two curved lines (and the fixation cross) were each drawn (by the vector-graphics generator) with linear endpoint precisions of 1 part in 1024 in the x- and y-directions over separate "patches" of extent 12 × 12 mm or 32 × 32 mm, depending on chord size. Each patch containing each curved line could be positioned (by the DACs) with linear precision of 1 part in 4096 in the x- and y-directions over the screen. The two curved lines and the fixation cross could all be displayed on the screen during the 20-msec refresh interval, and displays of long duration were produced by identically refreshing the display at 50 Hz. The nominal 1-sec presentation thus comprised 50 refresh cycles. (This fine temporal structure was not apparent to the subject.)

The display screen was viewed binocularly at a distance of 1.7 m through a view-tunnel and optical system that produced the uniform background field. Head position was stabilized with a chin-rest and head-rest. The geometry of the screen was calibrated at the beginning of each experimental session after at least 20-min warm-up of the CRT. The intensity of the stimuli was set with the aid of a 1-log-unit neutral density filter placed directly over the screen.

Procedure and experimental design. The task of the subject was to report the location (left or right) of the more curved line in each display. The order of operations in each trial was as follows. The subject fixated the central fixation target and, when ready, initiated a trial by pressing the appropriate switch on a push-button box connected to the computer. After a 40-msec delay, a display containing two curved lines appeared for 1 sec (the fixation target still present). When the subject had signalled his or her response on the push-button box, the fixation target reappeared after about 2-sec delay. The reappearance of the target indicated that the next trial could be started. The subject maintained central fixation during the presentation period.

Trials were performed in sequences of 70. In each such sequence, the chord-lengths of the curved lines were constant. In each trial the reference values c were chosen

*The local coordinate system of Fig. 1(a) should not be confused with the Cartesian coordinate system associated naturally with the display configuration.

at random from seven different values, spread uniformly over the curved-line continuum under test, and the increments δc were determined by their corresponding PEST algorithms. Subjects were given no feedback on their performances, other than that implied by the alogorithm's manipulation of stimulus level. Each sequence of 70 trials was preceded by a practice run of 7 trials, the results of which were automatically discarded. Sequences of trials were grouped into blocks with four different values of chord-length, chosen randomly (without replacement) from the given range. Values of chord-length and reference value c were balanced within and across sequences. Each subject performed one block of measurements in each experimental session of < 1 hr, with not more than two sessions in one day.

For each subject, each estimate of increment threshold Δc at each reference value c and chord-length was thus based on approx. 80 trials spread uniformly over 7–8 sessions.

Data analysis. Each set of response scores, expressed as percent correct as a function of testing level δc, was transformed by the inverse of the standardized normal integral and fitted with a quadratic psychometric function by a maximum-likelihood procedure (GLIM, NAG, 1987). Apart from the assumption that the function was quadratic and had to pass through the value 50% (corresponding to chance performance) at zero testing level ($\delta c = 0$), there were no other constraints on the fitting procedure. This function gave satisfactory fits to the data under all conditions. [A χ^2 statistic for each set of increment-threshold data was calculated, based on the empirical logistic transform (Cox, 1970), and the sum of these values was compared with a χ^2 distribution; the result was not significant: $\chi^2(1053) = 800$.] There were no significant trends in fit with either chord-length [$F(1,108) = 0.00$, $P > 0.5$] or reference value of the parameter sag [$F(1,108) = 0.61$, $P > 0.4$]. The estimated increment threshold was defined as the testing level δc at which performance was 75%.

For each of the seven curved-line attributes, the corresponding set of increment-threshold data was subjected to an analysis of variance (ANOVA) with repeated measures (Winer, 1971, section 4.2). The adequacy of fit of equation (1) was investigated by an analysis of trend (Winer, 1971, section 4.6) with respect to c. The standard method (Winer, 1971, section 4.6) of computing orthogonal polynomials in c was modified (Robson, 1959) to allow for the non-uniform spacing of the values of c for several of the curved-line attributes. The form of the function w in equation (1) was, in principle, unknown, and several polynomial functions were explored. Tests were made of the extent to which the variance in Δc estimates was accounted for by linear and higher-order (up to quartic) polynomials in the values of c. The lack of fit was specified by the value of the corresponding F-statistic.

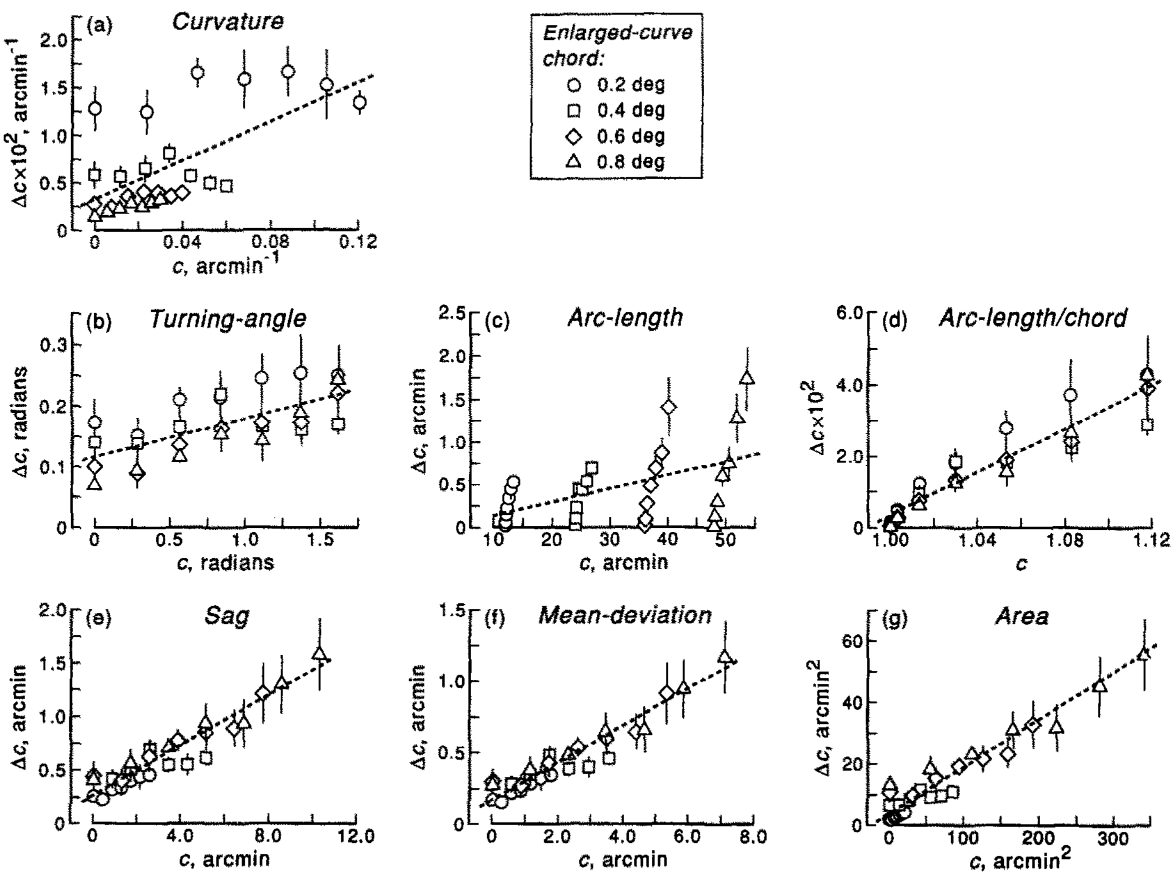

FIGURE 2. Increment-threshold functions under curved-line enlargements. For each of the specified curved-line attributes, estimates of increment threshold Δc are plotted against reference value c for four chord-lengths, indicated by different symbols. Each data point is the weighted mean over five subjects, and the vertical bars show ± 1 SEM. Broken lines are least-squares linear regressions.

Subjects. Five subjects, aged 19–36 yr, participated in the experiment. Two were male and three were female. Each had normal or corrected-to-normal visual acuity (Snellen acuities were each not worse than 6/5). No subject was aware of the purpose of the experiment.

Results and comment

Figure 2 shows estimates of increment threshold value Δc as a function of reference value c for each of the seven curved-line attributes. Each data point is the weighted mean over the five subjects, and vertical bars show ± 1 SEM. Different symbols correspond to different enlargements of the curved lines, the amount of enlargement specified by the chord-length, as indicated in the key to the figure. The broken lines are least-squares linear regressions, each with two degrees of freedom.

It is evident that curvature and arc-length provided the worst fits to the data, and that mean-deviation, sag and area provided the best. This assessment was confirmed by the formal ANOVA, the results of which are summarized in Table 1. Mean-deviation, sag, area and arc-length-divided-by-chord-length accounted well for the variance in the data; curvature and arc-length both failed highly significantly.

Theoretically, sag and mean-deviation are closely related. It is possible to write mean-deviation as a Taylor's series in sag s, thus

$$s(2/3 + O((s/l)^2)). \tag{6}$$

Over the range of curved-line stimuli used here, the maximum error in writing mean-deviation as $2s/3$ was approx. $0.04s$.

For mean-deviation, sag and area, the dependence of Δc on c was linear (Table 1). This result is consistent with a modified version of Weber's law which has an additive constant Δc_0 to represent the absolute threshold. For mean-deviation, the value of Δc_0 was 0.17 ± 0.03 arcmin (mean $\pm$ SEM), and for sag 0.27 ± 0.04 arcmin. The latter value is larger than previously reported absolute-threshold values (e.g. Watt & Andrews, 1982) obtained with simultaneously presented circular arcs. The sag gradient $\Delta c/c$ (the Weber fraction) was 0.12 ± 0.01, which is also larger than previously reported values (Wilson & Richards, 1989) obtained with sequentially presented, centrally fixated, parabolic arcs. Two factors probably contributed to the larger values found here. First, the stimuli were presented with separations of 2.8 deg visual angle, to avoid possible interaction between stimuli with large chord-lengths (see e.g. Watt & Andrews, 1982). Second, although subjects received practice in the task, they were not given prolonged training, which is known to reduce hyperacuity thresholds (McKee & Westheimer, 1978).

The next experiment obtained a more finely sampled subset of increment-threshold data and also tested estimates of the standard deviations of the increment thresholds against equation (2). Subjects were given more practice and performed approximately four times as many trials for each threshold determination.

TABLE 1. Consistency of increment-threshold data (Fig. 2) with equation (1) under four levels of curved-line enlargement for each of seven curved-line attributes

Geometric attribute	Lack of fit: Linear $F(26,108)$	Lack of fit: Quartic $F(23,108)$
Mean-deviation	0.90	0.46
Sag	0.96	0.63
Area	1.04	0.92
Arc-length/chord-length	1.23	1.23§
Turning-angle	1.61*	1.61*‡
Curvature	8.72†	7.06†
Arc-length	9.80†	8.32†

*$P < 0.05$, †$P \ll 0.001$.

‡Negligible improvement for quadratic and higher.

§Negligible improvement for cubic and higher.

The F-ratios measure the lack of fit of linear and higher-order (quartic, degree 4) polynomials in reference values c fitted to observed estimates of increment threshold Δc for each attribute.

EXPERIMENT 2: EFFECT OF CURVE ELONGATION ON INCREMENT-THRESHOLD AND VARIANCE FUNCTIONS

Curved-line stimuli were subjected to four levels of elongation along the direction of their chords. Since sag and mean deviation are invariant under this transformation, the corresponding increment-threshold functions should have been identical if either of these attributes was the cue for discrimination. As in Expt 1, curved-line increment thresholds were determined as a function of reference value for each of the seven curved-line attributes. The standard deviations σ of the increment thresholds Δc were estimated by the bootstrap method (Foster & Bischof, 1991).

Methods

Stimuli and apparatus. Stimuli were again generated from circular arcs by linear transformations of the plane: in the local coordinate system of Fig. 1(a) (see footnote on p. 332), transformations were an elongation in the x-direction $(x,y)\rightarrow(ax,y)$, with $a \geqslant 1$, and a compression in the y-direction $(x,y)\rightarrow(x,ay)$, with $0 \leqslant a \leqslant 1$. Thus, as in the preceding experiment, if the curved lines were embedded in a plane perpendicular to the line of sight, then these transformations would correspond to moving the plane closer to or further from the eye, and rotating the plane about the chord of a curved line, that is, about the local x-axis. Other details were as in Expt 1.

Procedure and experimental design. The procedure was the same as in Expt 1; moreover, since the range of threshold values was known, it was more practicable to use a method of constant stimuli to set the stimulus levels δc. Trials were performed in sequences of 28. Values of the chord-length, reference value c, and increments δc in c were balanced within and across sequences of trials. Although there is evidence (Taylor, Forbes & Creelman, 1983) that fixed-levels testing may yield lower performance levels and steeper slopes of the psychometric function than with adaptive methods such as PEST, the lengths of the trial sequences were relatively short and less likely to lead to lapses in memory for "cues to the presence of the signal" (Taylor *et al.*, 1983, p. 1373), particularly given the variety of stimuli presented. Even so, a formal test was made for a change in the slopes of the psychometric functions. For each subject, each estimate of the increment threshold Δc and estimate of the standard deviation σ of Δc at each reference value c and chord-length was thus based on approx. 300 trials spread uniformly over sessions.

Data analysis. The methods of data analysis were the same as in Expt 1, except that the analysis was extended to the variation of the estimates of the standard deviations σ with c.

Subjects. Four subjects, aged 26–29 yr, participated in the experiment. All were male and each had normal or corrected-to-normal visual acuity (Snellen acuities were each not worse than 6/6). All subjects except one were unaware of the purpose of the experiment.

Results and comment

Figure 3 shows estimates of increment threshold Δc as a function of reference value c for each of the seven curved-line attributes. Each data point is the weighted mean over the four subjects, and vertical bars show ± 1 SEM. Different symbols correspond to different elongations of the curved lines along the direction of their chords, the amount of elongation specified by the chord-length (0.2 deg corresponding to unit elongation), as indicated in the key to the figure. The broken lines are least-squares linear regressions, each with two degrees of freedom.

The results of the formal ANOVA are summarized in Table 2. Sag, mean-deviation and arc-length-divided-by-chord-length (but only for a quadratic fit) accounted well for the variance in the data, whereas turning-angle, curvature and arc-length all failed highly significantly. Sag provided the best fit, and the dependence of Δc on c was again linear, consistent with a modified Weber's law with an absolute threshold Δc_0 of value 0.19 ± 0.02 arcmin, which was smaller than the value obtained in Expt 1.

The lower threshold was assumed not to be due to the change in the method of setting stimulus levels (Taylor *et al.*, 1983). The mean of the slopes at the threshold value of the underlying quadratic psychometric function (before transformation by the inverse of the standardized normal integral; see Data Analysis, Expt 1) was 2.96 ± 0.11 arcmin^{-1}, which was not significantly different from the corresponding value (for the same range of chord-lengths) in Expt 1, namely 2.49 ± 0.26 arcmin^{-1} [$t(145) = 1.90$, $P > 0.05$, two-tailed test].

Figure 4 shows bootstrap estimates of the standard deviation σ of Δc as a function of reference value c for each of the seven curved-line attributes. Each data point

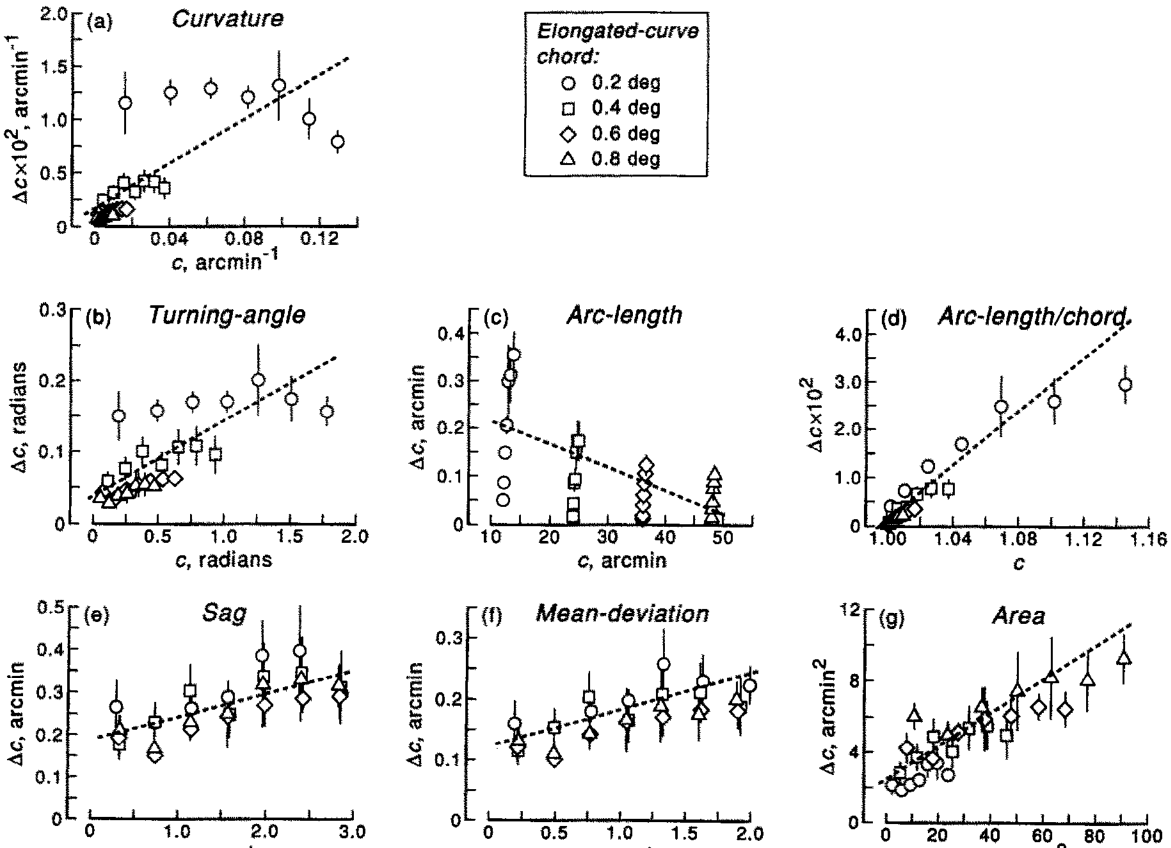

FIGURE 3. Increment-threshold functions under curved-line elongations. For each of the specified curved-line attributes, estimates of increment threshold Δc are plotted against reference value c for four chord-lengths, indicated by different symbols. Each data point is the weighted mean over four subjects, and the vertical bars show ± 1 SEM. Broken lines are least-squares regressions.

TABLE 2. Consistency of increment-threshold data (Fig. 3) with equation (1) under four levels of curved-line elongation for each of seven curved-line attributes

Geometric attribute	Lack of fit: Linear $F(26,81)$	Lack of fit: Quartic $F(23,81)$
Sag	1.38	0.58
Mean-deviation	1.58	0.91
Area	2.08*	2.08*‡
Arc-length/chord-length	2.95†	1.38§
Turning-angle	5.69†	3.68†
Curvature	9.17†	6.14†
Arc-length	10.71†	10.33†§

$^*P < 0.01$, $^\dagger P \ll 0.001$.
‡Negligible improvement for quadratic and higher.
§Negligible improvement for cubic and higher.
The F-ratios measure the lack of fit of linear and higher-order (quartic) polynomials in reference values c fitted to observed estimates of increment threshold Δc for each attribute.

is the mean over the four subjects, and vertical bars show ± 1 SEM. Different symbols correspond to different levels of elongation of the curved lines along the direction of their chords, the amount of elongation again specified by the chord-length, as indicated in the key to the figure. The broken lines are least-squares linear regressions, with two degrees of freedom each.

The results of the ANOVA are summarized in Table 3. Sag and mean-deviation accounted well for the variance in the data and all the other curved-line attributes failed highly significantly. Sag again provided the best fit. There was some evidence of a quadratic trend in the dependence of the estimate of σ on c [Fig. 4(e)], but the effect did not quite reach accepted levels of significance [$F(1,81) = 3.59$, $P = 0.06$]. A more detailed examination of performance near $c = 0$ arcmin was made in the next experiment.

With regard to the relationship between increment threshold and the standard deviation of increment threshold, the ratio of the estimate of $\sigma/\Delta c$ did not vary significantly over stimulus conditions for either sag or mean-deviation [$F(27,81) = 1.34$, $P > 0.1$] (cf. Crozier, 1936). This constancy suggests that the principal cause of the variance in increment threshold arises *after* whatever internal scaling operations determine Weber's law. This question is considered further in the General Discussion section.

In this and the preceding experiment chord-lengths of the curved-line stimuli were identical in each display, but arc-length necessarily covaried with sag (and of course with other curved-line attributes). On the basis of the corresponding increment-threshold functions (Figs 2 and 3), and standard-deviation functions (Fig. 4), arc-length was rejected as the cue for contour-curvature discrimination (Tables 1, 2, and 3). The possibility that

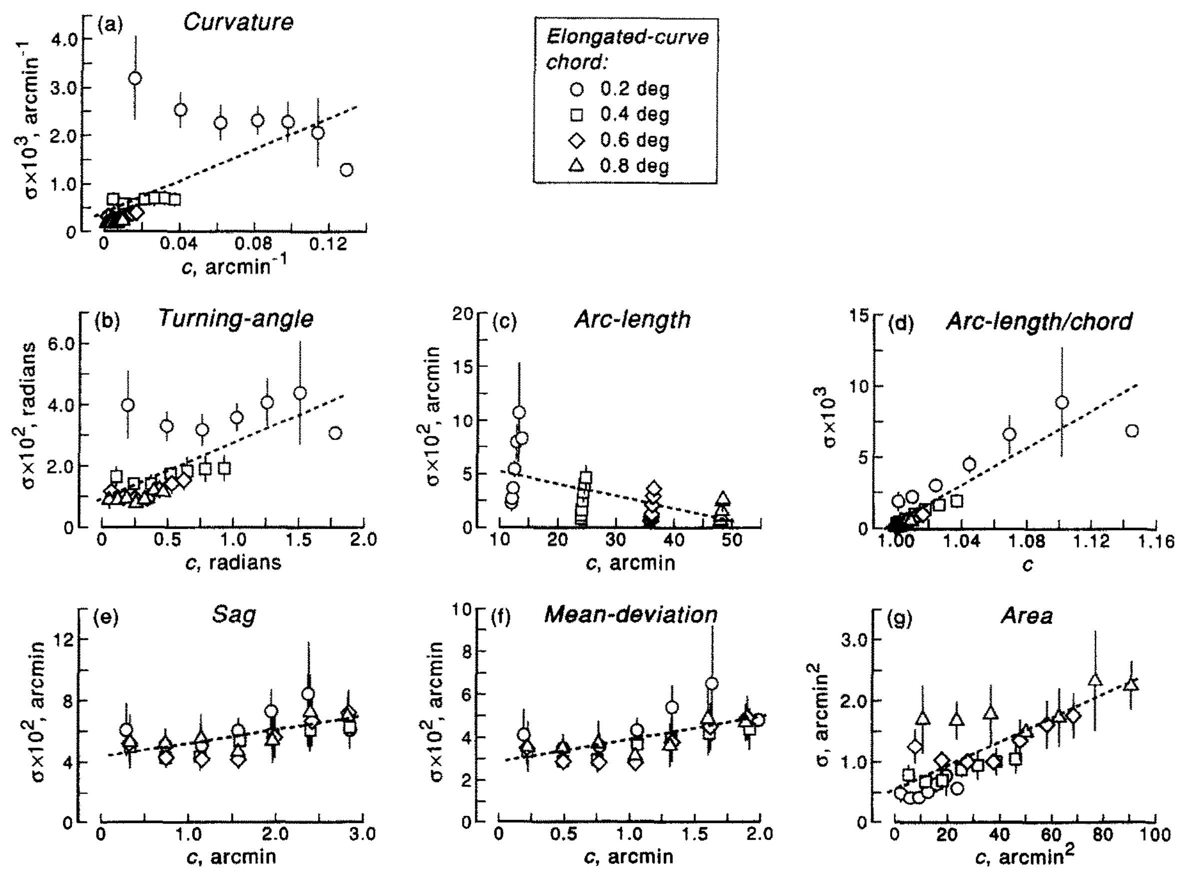

FIGURE 4. Estimated standard-deviation functions under curved-line elongations. For each of the specified curved-line attributes, bootstrap estimates of the standard deviation σ of the increment threshold are plotted against reference value c for four chord-lengths, indicated by different symbols. Each data point is the weighted mean over four subjects, and the vertical bars show ± 1 SEM. Broken lines are least-squares regressions.

differences in arc-length might contribute to the discrimination of curved lines with constant chord-length was explicitly tested in the next experiment.

It has been assumed that, for the purpose of specifying curvature, curved lines subjected to a compression at right angles to their chords could, within the present range of variation, be treated as circular arcs; as noted in the Methods section of Expt 1, differences between elliptical arcs and circular arcs were small, not more than 3% of the sag. To verify this assumption, the next experiment also tested whether increment-threshold functions obtained with elliptical and circular arcs were the same.

EXPERIMENT 3: THRESHOLD FUNCTIONS AND CURVE PARAMETERIZATION

This experiment determined increment-threshold functions for three types of curved-line stimuli: circular and elliptical arcs with constant chord-length, and circular arcs with constant arc-length. Since it was the similarity of the functions that was of interest, performance was specified in terms of one attribute, namely sag. The range of reference values was extended to include zero.

Methods

Stimuli and apparatus. The three types of curved-line stimuli were constructed as follows. (1) For each value of sag s, a unique circular arc was generated with chord-length l fixed at 12 arcmin and radius of curvature ρ given by the reciprocal of formula (4) (Definitions). (2) So that a comparison could be made with previous data, elliptical arcs were also included: for each value of sag s, an elliptical arc was generated by an appropriate compression in the y-direction [Fig. 1(a)] of a circular arc with chord-length l fixed at 12 arcmin and turning angle of 120 deg. By the remark at the end of the definition of turning angle (Definitions), the relationship between s and the scale factor a in the compression $(x,y) \rightarrow (x,ay)$ was $a = \sqrt{12}\ s/l$. (3) For each value of sag s, a unique circular arc was generated with arc-length l also fixed at 12 arcmin and radius of curvature derived from the expression following formula (5) (Definitions).

Procedure and experimental design. The procedure was the same as in Expt 1, except that just one chord-length or one arc-length was used. Stimulus level δc was varied according to n-fold PEST, and trials were performed in sequences of 70. The types of curved lines and values of reference value c were balanced within and across sequences of trials. For each subject, each estimate of the increment threshold Δc at each reference value c for each curved-line type was thus based on approx. 80 trials spread uniformly over sessions.

Data analysis. The methods of data analysis were the same as in Expt 1, except that they were restricted to the attribute sag. A standard repeated-measures ANOVA was performed to determine whether there was a significant effect of curved-line type.

Subjects. Five subjects, aged 19–27 yr, participated in the experiment. Three were male and two were female. Each had normal or corrected-to-normal visual acuity (Snellen acuities were each not worse than 6/5). All subjects except one were unaware of the purpose of the experiment.

Results and comment

Figure 5 shows sag increment threshold value Δc as a function of reference value c for each of the three types of curved-line stimuli. Each data point is the weighted mean over the five subjects, and vertical bars show ± 1 SEM. Different symbols correspond to the different types of curved lines. The broken line is a least-squares linear regression, with two degrees of freedom, fitted to all data values except those at zero sag.

The three increment-threshold functions appeared almost indistinguishable; this assessment was confirmed by the ANOVA, which showed that there was no significant effect of curved-line type [$F(2,8) = 0.24$, $P > 0.5$]. The linear regression of Δc on c, with zero-sag values omitted, yielded a Weber fraction of 0.12 ± 0.02, identical with the value obtained in Expt 2 over a similar range of sag values. When the zero-sag values were included in the present analysis, a highly significant quadratic trend [$F(1,80) = 14.6$, $P \ll 0.001$] was obtained.* This elevation at zero sag duplicates what is found with other sensory attributes (Laming, 1986, 1987), and may reflect the action of a separate discriminatory mechanism (see General Discussion).

Thus, as anticipated, the effects obtained in Expts 1 and 2 cannot be attributed to the use of elliptical arcs rather than circular arcs, nor to the introduction of differences in arc-length to preserve chord-length.

GENERAL DISCUSSION

The approach of the present study to the question of the cue for contour-curvature discrimination was essentially operational: which of the curved-line attributes

*The relative elevation at $c = 0$ arcmin may have been the result of difficulty in determining *which* of two discriminable curved lines was the more curved. Such confusion might occur if there were a small negative offset ($-\Delta c/2$) in the perceived values of sag, so that the comparison was effectively of curved lines with sag values $-\Delta c/2$ and $\Delta c/2$. This interpretation was tested in a "yes–no" control experiment in which "different" displays (containing two curved lines with values c and $c + \Delta c$) were discriminated from "same" displays (containing two curved lines with identical values c). Performance in this yes–no task was overall poorer than in the 2AFC task: one subject showed a small (0.085 arcmin), but non-significant, relative elevation in increment threshold at $c = 0$ arcmin; a second subject showed no relative elevation. Another control experiment was performed that did not require the sequential comparisons of the yes–no design: the 2AFC experiment was repeated, but in each sequence of trials the direction of the two curved lines was fixed (either to the right or to the left); subjects had to report which of the two curved lines was more curved to the right (or to the left depending on the sequence). Both subjects showed small (0.037, 0.017 arcmin), non-significant relative elevations at $c = 0$ arcmin.

TABLE 3. Consistency of estimated standard-deviation data (Fig. 4) with equation (2) under four levels of curved-line elongation for each of seven curved-line attributes

Geometric attribute	Lack of fit: Linear $F(26,81)$	Lack of fit: Quartic $F(23,81)$
Sag	1.38	0.31
Mean-deviation	1.56	0.60
Arc-length/chord-length	1.81*	1.05§
Area	3.11†	3.11†‡
Turning-angle	5.46†	3.62†
Arc-length	5.54†	5.13†§
Curvature	10.82†	8.74†§

$*P < 0.05$, $\dagger P \ll 0.001$.
‡Negligible improvement for quadratic and higher.
§Negligible improvement for cubic and higher.
The F-ratios measure the lack of fit of linear and higher-order (quartic) polynomials in reference values c fitted to observed estimates of standard deviation σ for each attribute.

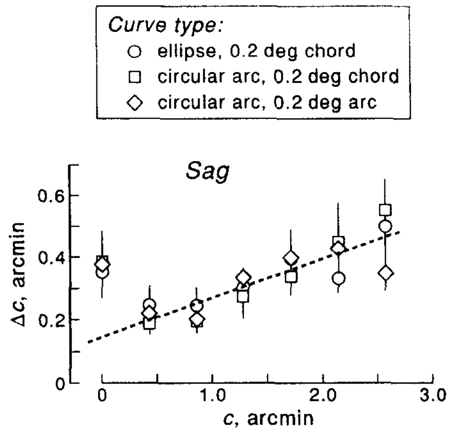

FIGURE 5. Effect of curved-line type. For the curved-line attribute of sag, estimates of increment threshold Δc are plotted against reference value c for each of three types of the curved-line stimuli: circular arcs with constant chord-length (12 arcmin), elliptical arcs with constant chord-length (also 12 arcmin), and circular arcs with constant arc-length (12 arcmin), as indicated by different symbols. Each data point is the weighted mean over five subjects, and vertical bars show ± 1 SEM. The broken line is a least-squares linear regression, with the data values at zero sag omitted from the fit.

gave the most efficient estimators of increment threshold and of standard deviation of increment threshold as a function of cue value? Of the seven curved-line attributes considered, only sag and its close relative mean-deviation accounted well (and arc-length-divided-by-chord-length a little less well) for the variance in the data about the functions defined by equations (1) and (2), as the curved-line stimuli were subjected to one- and two-dimensional scalings. Of the two attributes—mean-deviation and sag—mean-deviation was almost exactly linearly proportional to sag at small sag values [see expression (6), Results and Comment section, Expt 1], and, overall, sag provided the best predictor. As will be shown later, sag also offers some theoretical advantages as a discriminatory cue.

The conclusion from Expts 1, 2, and 3—that sag was the cue for discriminating curved lines, at least over the range of stimuli considered here—does not imply that sag need always be the only cue: clearly, if curved lines have the same sag values and different chord-lengths, or arc-lengths, they may still be discriminated, providing that the differences are sufficiently great. Likewise, curved lines of different curvatures but extending to the limits of the visual field may be discriminated (cf. Wilson & Richards, 1989), even though sag may not be properly defined.* The important result is that for finite curved lines of different curvatures (but with similar chord-lengths), it was sag rather than any other curved-line attribute—including curvature itself—that apparently determined the visual discriminability of the stimuli. Moreover, since sag accounted adequately for the variance in performance over the stimulus range, it was not necessary to consider two or more cues acting over different portions of the range, although such a possibility cannot be excluded (and indeed seems likely) if sufficiently extreme values of sag are included.

The fact that curvature appears not to be the cue for discriminating curved lines need not be taken as evidence of some internal perceptual inconsistency: curvature in the sense of expression (3) (Definitions), or some reformulation based on the rate of change of turning angle, is a formal construct and need not be simply related to what is *perceived* as curvature. Consider, for example, the effect of doubling the size of a circular curved line or halving its length: neither operation need affect perceived "curviness" in the manner predicted by expression (3).

Efficiency relative to an ideal observer

This operational approach based on finding the most efficient estimator—i.e. the one that best predicts the observed data—should be distinguished from a procedure which seeks to determine for each geometric attribute of interest a "relative efficiency" with respect to an ideal observer (Andrews, Butcher & Buckley, 1973). The ideal observer is assumed to know that the stimulus is, in this case, a circular arc, and merely has to determine the value of the selected attribute that gives the best fitting contour to the data points sampled by the retina (Watt & Andrews, 1982). Such a procedure removes the geometrical referent of performance, and a dimensionless quantity—the relative efficiency—defined with respect to this ideal observer may be assessed as the stimulus attribute is manipulated. As a result of such an analysis (Watt, 1984), it was suggested that the mechanism involved in discriminating curved lines operates by calculating curvature. Other curved-line attributes were rejected for the following reasons: turning-angle ("orientation-range"), because relative efficiency varied when orientation range was fixed; sag ("arc-height"), because

*For curved contours of non-constant curvature, points of inflexion might be used to define the effective extent of sections of the contour. In computational studies, sag has itself been used to partition curves: the general method is to select those points on the curve at which the distance of the curve from one or more chords has a local maximum; see e.g. Ramer (1972), and Phillips and Rosenfeld (1987).

of conflicting data (Watt & Andrews, 1982; Watt, 1984); and chord-length, because it had no effect on relative efficiency below about 20 arcmin. There was some evidence that increasing curvature led to increasing relative efficiency, although the effect was weak for small curvatures (as noted by Watt, 1984).

Although there were some differences between the methods of the foregoing studies and those employed here, the different conclusions concerning the identity of the cue for contour-curvature discrimination probably reflect the different criteria used to assess psychophysical performance: the one criterion specifying the attribute that most strongly influenced the use of available stimulus information, the other specifying the attribute that best predicted discrimination performance. It is, in principle and apparently in practice, possible that the cue which best predicts discrimination performance is not the same as the cue which predicts the best use of stimulus information.

Mechanisms for contour-curvature discrimination

Of the seven possible cues, sag is the easiest to estimate by peripheral visual mechanisms: all it entails is a measurement of spatial extent. In fact, a variety of mechanisms for contour-curvature discrimination have been proposed, usually based on specific assumptions about receptive-field structure. Such structures have included the following: a differential combination of Gaussian sensitivity functions forming essentially a 1×3 matrix of inhibitory and excitatory regions (Wilson, 1985); a more complex 2×3 matrix of excitatory and inhibitory regions (Koenderink & Richards, 1988); and a differential combination of different-sized difference-of-Gaussians, specifically in the form of the receptive field of an end-stopped simple cell (Dobbins, Zucker & Cynader, 1989).

An important factor determining visual performance is how information from different receptive fields is sampled as the shape and size of the stimulus changes. In one study (Wilson, 1985), a set of six classes of orientation-selective, spatial-frequency-tuned mechanisms was used to model the dependence of curvature increment threshold on reference curvature for parabolic arcs with straight-line extensions. (Curvature referred there to the maximum curvature of the parabolic arcs.) A subsequent elaboration of the theory (Wilson & Richards, 1989) included a second mechanism to explain changes in discrimination performance at low curvatures.

Direct comparison of those results with the present data, where stimuli had approximately constant curvature, is not straightforward. Nevertheless, parallels have been noted (Wilson & Richards, 1989) with earlier reports suggesting that there are (at least) two mechanisms involved in contour-curvature discrimination. Double peaks have been obtained in measurements of the relative efficiency for curvature discrimination as a function of curvature for short curved lines (Watt & Andrews, 1982); and the elevation here in increment threshold for sag values around zero reference sag (Fig. 5) and the strongly Weberian behaviour for positive reference values beyond about 0.5 arcmin [Fig. 2(e)] also suggest the operation of two distinct mechanisms (but see Laming, 1986, 1987).

A still more marked division in the processing of curved-line stimuli has been demonstrated when the duration of the stimulus display is reduced. Thus in one series of experiments (Foster, 1983) the discriminability of curved lines, differing by a constant increment in sag, was measured at successive values of reference sag, much as in the present experiments, except that percent correct discrimination rather than threshold sag increment was the measure of performance. With long-duration (2-sec) displays, discrimination performance varied linearly with sag reference value (which did not include zero); but with short-duration (100-msec) displays, followed by a random-dot mask, discrimination performance varied rapidly and non-monotonically with reference sag: two peaks, one at 0.87 arcmin, the other at 1.81 arcmin, were obtained. To test the possibility that these peaks represented the boundaries between internal curved-line "categories", other labelling experiments were performed in which the curved-line stimuli in the short-duration displays were assigned labels such as "straight", "just curved" and "more than just curved". These three labels were sufficient to generate theoretical discrimination performances that were closely congruent to the observed ones (Foster, 1983). The data from these and subsequent experiments (Ferraro & Foster, 1986) were explained in terms of the operation of two and possibly three discrete mechanisms for curved-line coding mediating curved-line discrimination with short-duration displays.

These non-monotonicities in discrimination performance away from zero sag were obtained only with short-duration displays. In a separate study (Foster & Cook, 1989), it was shown that as stimulus duration was increased (from 60 msec to 2 sec), discrimination performance became progressively flatter; at stimulus durations of 1–2 sec the linear performance shown here was obtained.

Sag, scale-invariance, and Weber's law

What theoretical advantage might sag offer over other possible cues in determining visual performance? Consider two curved-line stimuli with chords oriented in say the x-direction and with cue values c and $c + \delta c$ for any of the seven cues. Suppose that discrimination performance is characterized by the ratio of the difference in cue values to the reference cue value, that is, by the ratio $\delta c/c$. (This is the Weber fraction.) Consider now what happens if the stimuli are enlarged, by a factor k, with $k > 0$; as before, if the curved lines are embedded in a plane perpendicular to the line of sight, this transformation corresponds to moving the plane closer to or further away from the eye. The ratio $\delta c/c$ remains constant under this transformation for each of the seven possible cues.

Suppose next that the stimuli are elongated or compressed in the y-direction by the factor k (i.e. perpendicular to the chord). This transformation corresponds to rotating the plane containing the embedded curved lines about the x-axis or rotating the plane about the y-axis and moving the plane closer to the eye. It is possible to show analytically or by numerical methods that sag, mean-deviation, and area are the *only* cues of the seven for which $\delta c/c$ remains constant under this transformation. Suppose, finally, that the stimuli are elongated or compressed in the x-direction by the factor k (i.e. along the chord). This transformation corresponds to rotating the plane containing the embedded curved lines about the y-axis or rotating the plane about the x-axis and moving the plane closer to the eye. Again, it is possible to show analytically or by numerical methods that sag, mean-deviation, and area are the only cues for which $\delta c/c$ remains constant under this transformation.

Thus, if determining the relationship between curved lines is regarded as an operation that ought to be robust under the natural changes in an image of an object or a scene arising from relative movement between the observer and the object or scene, then curvature, turning-angle, arc-length and arc-length-divided-by-chord-length all fail to offer an appropriate signal; but with sag as the cue (or mean-deviation or area), the difference it defines between one curved contour and another, expressed as a proportion of one of them, is a constant, independent of relative viewing distance and direction.

Experiments 1 and 2 tested precisely this question of robustness under one- and two-dimensional scalings. In that context, stimuli consisting of elongated or compressed circular arcs (ellipses) may be seen, in retrospect, to be better matched to this general discriminatory task than are enlarged or reduced circular arcs, although the distinction as judged by the effects on discrimination performance for the range of stimuli considered here was unimportant. The findings that sag was the most efficient estimator and that Weber's law was obeyed by this cue over all of its range except near zero are consistent with the hypothesis of the invariance of contour-curvature discriminations.

Finally, it was suggested earlier that the constancy of the relationship between increment threshold and its variance might be a consequence of the variability within the observer arising after the internal scaling operation implied by Weber's law. If so, this variability establishes a uniform level of uncertainty over an image of an object or a scene: all local comparisons of curved contours are then subject to the same level of imprecision, which, like the discrimination of curved contours, is a constant, independent of relative viewing distance and direction.

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