Shape in Picture

Mathematical Description of Shape in Grey-level Images

Edited by

Ying-Lie O

Centre for Mathematics and Computer Science (CWI)
Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

Alexander Toet

Netherlands Organization for Applied Scientific Research
Institute for Human Factors (TNO-IZF)
Kampweg 5, 3769 DE Soesterberg, The Netherlands

David Foster

University of Keele
Department of Communication and Neuroscience
Keele, Staffordshire ST5 5BG, UK

Henk J. A. M. Heijmans

Centre for Mathematics and Computer Science (CWI)
Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

Peter Meer

Rutgers University
Department of Electrical and Computer Engineering
Piscataway, NJ 08855-0909, USA

Springer-Verlag Berlin Heidelberg GmbH

Proceedings of the NATO Advanced Research Workshop "Shape in Picture", held at Driebergen, The Netherlands, September 7–11, 1992

CR Subject Classification (1991): I.3-5

ISBN 978-3-642-08188-0 ISBN 978-3-662-03039-4 (eBook)
DOI 10.1007/978-3-662-03039-4

CIP data applied for.

Originally published by Springer-Verlag Berlin Heidelberg New York in 1994
Softcover reprint of the hardcover 1st edition 1994

Typesetting: Camera ready by authors
45/3140 - 5 4 3 2 1 0 - Printed on acid-free paper

Classical and Fuzzy Differential Methods in Shape Analysis*

David H. Foster

Department of Communication and Neuroscience, Keele University,
Staffordshire ST5 5BG, UK

Abstract. This study considers four means of defining differential operators for extracting local aspects of shape in ill-specified environments: fuzzy differentiation as kernel smoothing; differentiation in the sense of weak or generalized derivatives; differentiation for fuzzy functions between normed spaces; and fuzzy differentiation for mappings between fuzzy manifolds. More consideration is given to the last, norm-free approach, which involves the notions of an abstract fuzzy topological vector space, fuzzy differentiation between fuzzy topological vector spaces, fuzzy atlases, and tangent vectors of fuzzy manifolds.

Keywords: shape description, differential geometry, fuzzy set, fuzzy derivative, fuzzy topological vector space, fuzzy manifold, tangent vector, tangent space.

1 Introduction

A common technique for characterizing shape in an image is to use some kind of differential operator to extract the critical local variations in the light distribution. For images of two-dimensional objects, and their boundaries in particular, one might determine the positions of curvature extrema [1, 38, 9]; and, for images of three-dimensional objects, the positions of extrema in, for example, principal curvatures [17].

Yet in real vision systems, whether machine or human, imprecisions are inherent in the spatial and intensity characterization of the image. At the lowest, most immediate levels of image representation, there are effects of noise in sensory transduction and of limits on sampling frequency, both spatial and temporal. At higher, more removed levels of image representation [13], there are more general imprecisions to do with the specification of image qualities [40]. For the human observer it is unclear what geometrical framework is used to form the representation, and indeed whether a metric structure or the structure of

* I am grateful to P. Fletcher, R. Kopperman, S.R. Pratt, and M.G.A. Thomson for critically reading the manuscript and to J.J. Koenderink for comments on Sect. 2. This work was supported by ESPRIT Basic Research Action No. 6448 (VIVA).

a normed space is part of it [11, 14]. How then should differential operators be defined for these ill-specified environments?

The approaches to this problem have differed in the restrictions they have placed on the class of admissible image characterizations and on the analytic machinery assumed to be available at each processing stage. Four of the main approaches may be summarized as follows.

1. Assume a Euclidean framework and smooth the low-level image representation. The classical differential methods of real analysis may then be applied straightforwardly.
2. Assume that the low-level representation is important only in the way that it "interacts" with certain other functions. For a sufficiently large set of such functions, this interaction defines an operator which is differentiable, in the sense of generalized derivatives, and which can be used in place of the representation.
3. Assume that the image representation is "fuzzy" but constrained in such a way that it may be isometrically embedded in a normed space, which then allows classical differential methods to be applied.
4. Assume that the image representation is fuzzy and introduce a natural fuzzy topological vector space structure—or more generally the structure of a fuzzy differentiable manifold—so that the notion of fuzzy differentiation follows naturally without the imposition of a norm.

This article reviews briefly methods (1)–(3), and then more fully method (4), which involves some relatively unfamiliar topological-geometrical notions. The treatment is not complete: topological [19, 23, 24] and graph-based [22] digital-topological approaches are not considered, nor are synthetic methods [20].

It is assumed, with little loss in generality, that the images of interest are monochromatic, viewed monocularly.

2 Fuzzy Differentiation as Kernel Smoothing

Suppose that the image is represented by some luminance distribution $I(\mathbf{x})$, where $\mathbf{x}$ ranges over the real plane $\mathbb{R}^2$, and suppose that I is non-smooth in some way, that is, I or its first or second derivative is discontinuous in the standard differential structure on $\mathbb{R}^2$. There are various ways of smoothing the data defined by I. A *kernel smoother* uses an explicit set of local weights, defined by the *kernel* K, to produce the smoothed estimate $\tilde{I}$ of I at each $\mathbf{x}$ [42, 16]; thus

$$\tilde{I}(\mathbf{x}) = \int K(\mathbf{x}-\mathbf{x}')\, I(\mathbf{x}')\, \mathrm{d}\mathbf{x}' .$$

If I is obtained by discrete sampling, that is, determined only on a finite subset $\{\mathbf{x}_i\}_{1\le i\le n}$ of points in $\mathbb{R}^2$, the integral is replaced by a summation over i [16].

In general the kernel takes the form

$$K(\mathbf{x}) = (c_0/\sigma)\, d(\|\mathbf{x}\|/\sigma) ,$$

where d is a decreasing function; $\|\cdot\|$ is a norm; σ is the window-width or bandwidth; and c_0 is a normalizing constant. There are several criteria for the choice of kernel [31]; in the present context a natural candidate for d is the standard Gaussian function [42].

For functions of $\mathbb{R}^2$, and for luminance distributions in particular, a definition of a fuzzy derivative has been been proposed [21] that may be viewed as a kernel smoother, the kernel being the derivative of a Gaussian function; that is:

Definition 1. The n^{th} (partial) *fuzzy derivative* at $x \in \mathbb{R}$ is the kernel

$$\phi_n(x; s) = \frac{\partial^n}{\partial x^n}\left(\frac{e^{-x^2/4s}}{\sqrt{4\pi s}}\right),$$

where $\sigma = \sqrt{4s}$ sets the *scale* parameter.

The functions ϕ_n have a ready physical and physiological interpretation [21], and show a concatenation property such that the higher-order derivatives are obtained at lower spatial "resolutions", the resolution corresponding to the inverse of the scale parameter value σ. A discretized version of this scale-space approach has been described in [27, 28], where a discrete analogue of the Gaussian kernel is used.

There is, however, a fundamental problem of deciding how appropriately the fitted surface represents the original surface [10, 4]. A critical question, for example, is whether Gaussian smoothing leads to robust derivatives. As has been noted elsewhere [43], there are two conflicting requirements: accuracy (correct derivatives should be obtained, at least for low orders), and smoothing (the effects of noise and discretization should be minimized). Gaussian kernels can lead to "over-smoothing" errors, but other kernels can be derived that achieve a better compromise between these two requirements [43]. The technique of *adaptive* kernel estimation has been reviewed in [42].

The approach summarized in Definition 1 and developed in [21] differs from some others in that it does not assume necessarily that an "original" surface exists, other than that which can be observed through the kernels (see Sect. 3). This foundational issue has been circumvented in an approach [4] that uses a statistical covariance technique [26] for surface descriptors. By analogy with classical differential methods, the technique yields, for discretely sampled data, definitions of the first and second fundamental forms for a surface in $\mathbb{R}^3$, and the Weingarten equations, which relate the rate of change of the unit normal vector and the corresponding chosen direction of a curve on the tangent plane [4].

The next section considers more generally the notion of derivatives as operators.

3 Generalized Derivatives

Suppose that the image luminance distribution $I(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, is such that it can be associated formally with an operator on a set of "test" functions on $\mathbb{R}^2$.

(The association may be through convolution, as in the preceding section; the test functions are defined shortly, after a natural topology for them is introduced.) Although derivatives of the representation may not be defined in the ordinary way, derivatives of this operator *may* be defined, providing that certain conditions are satisfied.

The set of test functions is given a topology based on a family of seminorms. A *seminorm* on a vector space E is a mapping $\rho : E \to [0, \infty)$ such that:

1. $\rho(\xi + \eta) \leq \rho(\xi) + \rho(\eta)$, for all $\xi, \eta \in E$.
2. $\rho(\alpha\xi) = |\alpha| \, \rho(\xi)$, for all $\xi \in E, \alpha \in \mathbb{C}$ (or $\mathbb{R}$).

A family $\{\rho_\gamma\}_{\gamma \in \Gamma}$ of seminorms *separates points* if

3. $\rho_\gamma(\xi) = 0$ for all $\gamma \in \Gamma$ implies $\xi = 0$.

The *natural topology* on a vector space with a family $\{\rho_\gamma\}_{\gamma \in \Gamma}$ of seminorms separating points is the weakest topology in which all the ρ_γ are continuous and in which the operation of addition is continuous.

The set of test functions on $\mathbb{R}^2$ (or, more generally, $\mathbb{R}^n$) is the set $\mathcal{S}$ of *functions of rapid decrease*; that is, the set of infinitely differentiable functions ϕ on $\mathbb{R}^2$ for which

$$\sup_{(x_1,x_2) \in \mathbb{R}^2} \left| (x_1^{\alpha_1}, x_2^{\alpha_2}) \frac{\partial^{(\beta_1+\beta_2)}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} \phi(x_1, x_2) \right| < \infty , \tag{1}$$

for all non-negative integers $\alpha_1, \alpha_2, \beta_1, \beta_2$. The functions in $\mathcal{S}$ are thus those that together with their derivatives fall off more rapidly than the inverse of any polynomial. The quantity on the left-hand side of (1) defines a seminorm $\| \cdot \|_{\alpha,\beta}$ on $\mathcal{S}$. These seminorms give $\mathcal{S}$ the natural topology.

The space of operators can now be defined as the (topological) dual of $\mathcal{S}$; that is, the set of all continuous linear functions ("functionals") on $\mathcal{S}$. It is denoted by $\mathcal{S}'$ and called the *space of tempered distributions* [41]. The derivative of a tempered distribution is defined as follows.

Definition 2. Let T be a tempered distribution. The *weak* or *generalized derivative* $D^{(\alpha_1,\alpha_2)}T$ (or the *derivative in the sense of distributions*) is given by

$$(D^{(\alpha_1,\alpha_2)}T)(\phi) = (-1)^{|\alpha_1+\alpha_2|} \, T \left(\frac{\partial^{(\alpha_1+\alpha_2)}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \phi \right), \quad \text{for all } \phi \in \mathcal{S} .$$

There is a natural way to associate a certain class of functions f on $\mathbb{R}^2$ with tempered distributions T_f such that if $T_f = T_g$ then $f = g$ almost everywhere. For an image luminance distribution that falls into this class, its derivative may thus be defined as the derivative of the corresponding tempered distribution. The connection between this approach to the differentiation of image functions and the smoothing approach of Sect. 2 is discussed in [10].

4 Normed Spaces of Fuzzy Sets

Suppose, now, that the sampling of the image is less precisely specified. For example, consider a function that assigns to each point $\mathbf{x}$ in $\mathbb{R}^2$ with luminance $I(\mathbf{x})$ some measure of the "goodness" of this characterization of the image at that point, or, more generally, consider a function that assigns to each element of some set X of image attributes, possibly including an estimate of spatial position, a number that specifies the extent to which that attribute is associated with the image or part of the image. Both of these functions are examples of "fuzzy sets", the formal notion of which was introduced by Zadeh [44]. Thus, given an arbitrary set X, a *fuzzy set* (or *fuzzy subset*) in X is a function $A : X \to [0,1]$ such that the value $A(x)$ of A at the point $x \in X$ gives the "grade of membership" of x in A. (Fuzzy set theory should not be confused with probability theory; for discussion of this and related issues, see [45, 33, 34].) For a classical set the grade of membership would be either 0 or 1 (and A would then coincide with its characteristic function). The grade of membership of a fuzzy set may be taken in a complete lattice [15]—that is, a lattice in which every subset has a supremum and an infimum—rather than in the unit interval $[0,1]$; see [33] for examples. A kind of fuzziness for which there is *no* greatest element has been considered in [32], but this weaker structure limits the definition of a topology (Sect. 6).

The set $\mathcal{F}(X)$ of all fuzzy sets in X is a complete distributive lattice. For any fuzzy set A and any number $\alpha \in [0,1]$, the *α-cut* A_α of A is the set $\{ x \in X \mid A(x) \geq \alpha \}$. If X is a vector space, a *convex* fuzzy set A in $\mathcal{F}(X)$ has the property that

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{A(x_1), A(x_2)\},$$

for every x_1, x_2 in X, and λ in $[0,1]$.

The next section considers the differentiation of a "fuzzy" function from a normed vector space into a set of fuzzy sets in a reflexive Banach space Y with norm $\|\cdot\|$. It is possible to introduce a norm on a subset of $\mathcal{F}(Y)$, the set of all fuzzy sets in Y. Recall that the Hausdorff distance $d_{\mathrm{H}}(P,Q)$ between non-empty bounded (classical) subsets P, Q of Y is given by

$$d_{\mathrm{H}}(P,Q) = \max\left\{ \sup_{p\in P} \inf_{q\in Q} \|p-q\|, \sup_{q\in Q} \inf_{p\in P} \|p-q\| \right\}.$$

This distance can be extended to the subset $\mathcal{F}_0(Y)$ of $\mathcal{F}(Y)$ containing those fuzzy sets A with the following properties [36]:

1. A is upper semicontinuous;
2. A is convex;
3. A_α is compact for every α.

For $A, B \in \mathcal{F}_0(Y)$, define the distance $d(A,B)$ between A and B by

$$d(A,B) = \sup_{\alpha>0}\{d_{\mathrm{H}}(A_\alpha, B_\alpha)\}.$$

Then it can be shown [36] that $(\mathcal{F}_0(Y), d)$ is a complete metric space.

The subset $\mathcal{F}_0(Y)$ can be given a linear structure in the following way [36]. For $A, B \in \mathcal{F}_0(Y)$, the *sum* $C = A + B$ of A, B (sometimes denoted by $A \oplus B$) is the fuzzy set in Y defined by

$$C(y) = \sup_{\alpha \in [0,1]} \{ \alpha \mid y \in (A_\alpha + B_\alpha) \}, \quad \text{for all } y \in Y,$$

where $A_\alpha + B_\alpha$ is the (classical) subset $\{ z \in Y \mid z = a + b, a \in A_\alpha, b \in B_\alpha \}$. For any scalar $\alpha \in \mathbb{R}$, the *scalar product* αA of α and A is the fuzzy set in Y defined by

$$(\alpha A)(y) = \begin{cases} A(y/\alpha), & \text{if } \alpha \neq 0, \\ 0, & \text{if } \alpha = 0 \text{ and } y \neq 0, \\ \sup_{z \in Y} A(z), & \text{if } \alpha = 0 \text{ and } y = 0. \end{cases}$$

Although $\mathcal{F}_0(Y)$ is not a vector space with this sum and product [37, 36], the embedding theorem of Rådström [37] may be used to embed $\mathcal{F}_0(Y)$ isometrically in a normed vector space. Let $\mathcal{Y}$ be this normed space and let $j : \mathcal{F}_0(Y) \to \mathcal{Y}$ denote the embedding.

5 Differentiation of a Fuzzy Function between Normed Spaces

One definition [36] of a *fuzzy function* f from an arbitrary set X to an arbitrary set Y is that it is a set-valued mapping or multifunction [3] that assigns to each point $x \in X$ a fuzzy set $f(x) \in \mathcal{F}(Y)$ (but see e.g. [33] for other interpretations). Suppose that X is a normed vector space; U a (classical) open subset of X; Y a reflexive Banach space, as in Sect. 4; and f a fuzzy function from U into Y such that $f(x) \in \mathcal{F}_0(Y)$; that is, for each $x \in X$, the fuzzy set $f(x)$ has the properties (1)–(3) of Sect. 4. Then the differentiability of f at a point in U may be defined [36] by the differentiability of its composition with the embedding j in the normed vector space $\mathcal{Y}$; thus:

Definition 3. The fuzzy function $f : U \to \mathcal{F}_0(Y)$ is *differentiable at a point* $x_0 \in U$ if the composition $\hat{f} = j \circ f$ is differentiable at x_0; that is, if there exists a linear bounded mapping $\hat{f}'(x_0)$ from X into $\mathcal{Y}$ such that

$$\lim_{x \to x_0} \left\{ \frac{\|\hat{f}(x) - \hat{f}(x_0) - \hat{f}'(x_0)(x - x_0)\|}{\|x - x_0\|} \right\} = 0 .$$

Further details are given in [36, 3], where the Hukuhara differential is also discussed.

By definition [46, 30], a *type 2* fuzzy set A in a set X is a fuzzy set characterized by a fuzzy membership function whose values are each fuzzy sets in the unit interval $[0,1]$; that is, for each $x \in X$, the grade of membership $A(x) : J \to [0,1]$, where $J \subset [0,1]$. Type 2 fuzzy sets are a special case of the fuzzy functions just defined. An application of Definition 3 might thus be to those image characterizations that form type 2 fuzzy sets; that is, as in Sect. 4, where each point $\mathbf{x}$ in $\mathbb{R}^2$ of the image is associated with a fuzzy estimate $f(\mathbf{x})$ of (normalized) luminance.

6 Fuzzy Topology and Fuzzy Topological Vector Spaces

Consider, next, fuzzy sets in a set X where there is *no* norm. As will become clear later, all that is needed for a basic definition of differentiation is that X should be equipped with an appropriately fuzzy version of the structure of a topological vector space.

Note. In fact an even simpler framework is possible. R. Kopperman has considered (1992, personal communication) the equivalent definition: $f'(x)$ is a *derivative* for f at x if $f(y) = f(x) + m(x,y)(y-x)$ and $\lim_{y\to x} m(x,y) = f'(x)$, with $m(x,y)$, the *slope* of f between x and y, defined for $x, y \in \mathrm{Dom}(f)$. This definition extends easily to any category of topological abelian groups such that if X, Y are topological abelian groups, then $\mathrm{Hom}(X,Y)$ is also a topological abelian group and $[(f,x) \to f(x)] : \mathrm{Hom}(X,Y) \times X \to Y$ and $[(f,g) \to f \circ g] : \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Z,X) \to \mathrm{Hom}(Z,Y)$ are jointly continuous. In this situation, functions are continuous at points of differentiability and the chain rule and sum rule hold; further, theorems on partial derivatives and the inverse and implicit function theorems, among others, can be formulated and shown in natural settings (see Sect. 8).

For the sake of completeness, some elementary properties of fuzzy sets are briefly recalled [44, 33, 35, 7]. For each $c \in [0,1]$, let k_c denote the *constant* fuzzy set in X, that is, $k_c(x) = c$ for all $x \in X$; and let x_c denote the *fuzzy point* in X, where

$$x_c(y) = \begin{cases} c, & \text{for } y = x; \\ 0, & \text{otherwise.} \end{cases}$$

For a fuzzy set A in X, one writes $x_c \in A$ when $c \leq A(x)$. The set X is identified with the constant fuzzy set k_1 and the empty set is identified with k_0. The inclusion, intersection, union, and complement of two arbitrary fuzzy sets are defined in an obvious fashion [44, 7]; for example, for fuzzy sets A, B in X, the intersection $A \cap B$ is given by $(A \cap B)(x) = \min\{A(x), B(x)\}$, for all $x \in X$.

Let f be a mapping from a set X to a set Y. Let B be a fuzzy set in Y. Then the *inverse image* $f^{-1}[B]$ of B is the fuzzy set in X defined by $f^{-1}[B](x) = B(f(x))$, for all $x \in X$. Conversely, let A be a fuzzy set in X. Then the *image* $f[A]$ of A is the fuzzy set in Y defined by

$$f[A](y) = \begin{cases} \sup_{z \in f^{-1}(y)} A(z), & \text{if } f^{-1}(y) \text{ is nonempty,} \\ 0, & \text{otherwise.} \end{cases}$$

Notice that although f takes fuzzy sets into fuzzy sets, it is not a set-valued mapping in the sense of Sect. 5, where (classical) points are taken into fuzzy sets.

The following definition of a fuzzy topological space is due to Lowen [29]. A *fuzzy topology* on a set X is a family $\mathcal{T}$ of fuzzy sets in X that satisfies the following conditions:

1. For all $c \in [0,1]$, $k_c \in \mathcal{T}$.
2. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$.
3. If $A_j \in \mathcal{T}$ for all $j \in J$ (J some index set), then $\bigcup_{j \in J} A_j \in \mathcal{T}$.

In the definition of a fuzzy topology due to Chang [5], the condition (1) is

1′. $k_0, k_1 \in \mathcal{T}$.

The inclusion in $\mathcal{T}$ of all fuzzy sets that are constant functions on X is required for the fuzzy continuity of the constant functions from X to any other set Y equipped with a fuzzy topology (fuzzy continuity is defined shortly). A fuzzy topology that satisfies condition (1) is called a *proper fuzzy topology*. The pair $(X, \mathcal{T})$ is called a *fuzzy topological space*. An *open* fuzzy set A in X is one which is in $\mathcal{T}$, and a *closed* fuzzy set is one whose complement $\bar{A} = 1 - A$ is in $\mathcal{T}$. A fuzzy set B is a *neighbourhood* of a fuzzy point x_c in X if there is a fuzzy set A in $\mathcal{T}$ such that $x_c \in A \subset B$. A fuzzy topological space is called a *fuzzy* T_1 space if every fuzzy point is a closed fuzzy set.

Let $(X, \mathcal{T}), (Y, \mathcal{V})$ be two fuzzy topological spaces. A mapping f of $(X, \mathcal{T})$ into $(Y, \mathcal{V})$ is *fuzzy continuous* if for each open fuzzy set V in $\mathcal{V}$ the inverse image $f^{-1}[V]$ is in $\mathcal{T}$. Conversely, f is *fuzzy open* if for each open fuzzy set U in $\mathcal{T}$, the image $f[U]$ is in $\mathcal{V}$. For related properties, including the notions of an induced fuzzy topology on a fuzzy set, and relatively fuzzy continous and relatively fuzzy open mappings, see [12].

Suppose that E is a vector space over $\mathbb{K}$ (the real field $\mathbb{R}$ or complex field $\mathbb{C}$). Let A, B be fuzzy sets in E. The definitions of the sum and scalar product (Sect. 4) may be reformulated thus. The *sum* $C = A + B$ of A, B is the fuzzy set in E defined by

$$C(x) = \sup_{a+b=x} \min\{A(a), B(b)\}, \quad \text{for all } x \in E;$$

and, for any scalar $\alpha \in \mathbb{K}$, the *scalar product* αA of α and A is the fuzzy set in E defined by

$$(\alpha A)(x) = \begin{cases} A(x/\alpha), & \text{for } \alpha \neq 0, \\ 0_c(x), & \text{otherwise,} \end{cases}$$

for all $x \in E$, where 0_c is the fuzzy point at 0 in E with $c = \sup_{y \in E} A(y)$.

Suppose that E is equipped with a fuzzy topology $\mathcal{T}$ and that $\mathbb{K}$ is equipped with the usual topology $\mathcal{K}$. A *fuzzy topological vector space* (*ftvs*) is a vector space E over $\mathbb{K}$ such that [18] the two mappings

1. $(x, y) \mapsto x + y$ of $(E, \mathcal{T}) \times (E, \mathcal{T})$ into $(E, \mathcal{T})$,
2. $(\alpha, x) \mapsto \alpha x$ of $(\mathbb{K}, \mathcal{K}) \times (E, \mathcal{T})$ into $(E, \mathcal{T})$,

are fuzzy continuous. Notice that the fuzzy topological vector space E may be proper or improper, but $\mathbb{K}$ is a special case of an improper fuzzy topological vector space. In the sequel, E denotes a ftvs with scalar field $\mathbb{K}$.

7 Fuzzy Differentiation Between Fuzzy Topological Vector Spaces

The following definition of a fuzzy derivative is a generalization of the classical definition for topological vector spaces [25]. Let E, F be two fuzzy ftvs's and let ϕ be a mapping from E into F. Let $o(t)$ be any function of a real variable t such that $\lim_{t\to 0} o(t)/t = 0$. Then ϕ is *tangent to* 0 if given a neighbourhood W of 0_δ in F, $0 < \delta \leq 1$, there exists a neighbourhood V of 0_λ in E, $0 < \lambda < \delta$, such that

$$\phi[tV] \subset o(t)W\,,$$

for some function $o(t)$. If both V, W are classical sets and E, F are normed, then this amounts [25] to the usual condition

$$\|\phi(x)\| \leq \|x\|\psi(x)\,,$$

where $\lim_{\|x\|\to 0} \psi(x) = 0$.

Let E, F be two ftvs's, each endowed with a fuzzy T_1 topology. Let $f : E \to F$ be fuzzy continuous. The fuzzy differentiability of f at a point in E may be defined [6] thus:

Definition 4. The mapping $f : E \to F$ is *fuzzy differentiable at a point* $x \in E$ if there exists a linear fuzzy continuous mapping $f'(x)$ of E into F such that

$$f(x+y) = f(x) + f'(x)(y) + \phi(y), \quad \text{for all } y \in E\,,$$

where ϕ is tangent to 0.

The mapping $f'(x)$ is the *fuzzy derivative* of f at x; it is an element of $L(E,F)$, the set of all linear fuzzy continuous mappings of E into F. The mapping f is *fuzzy differentiable* if it is fuzzy differentiable at every point of E. That $f'(x)$ is unique depends [6] on the fuzzy topology being fuzzy T_1.

An application of Definition 4 might be to those image characterizations which associate with each image point $\mathbf{x}$ in $\mathbb{R}^2$, say, a fuzzy estimate of location (Sect. 4), and with each point $f(\mathbf{x})$ in $\mathbb{R}$, say, a fuzzy estimate of an attribute value such as contour curvature.

The next section considers a generalization of this notion of differentiation to spaces which are only locally like fuzzy topological vector spaces.

8 Fuzzy Differentiation Between Fuzzy Manifolds

Let E, F, G be ftvs's. It may be shown [6] that the composition $g \circ f$ of two fuzzy differentiable mappings $f : E \to F$, $g : F \to G$ is fuzzy differentiable, and that the fuzzy derivative of $g \circ f$ at $x \in E$ is $g'(f(x)) \circ f'(x)$. It may also be shown [6] that if f, g are two fuzzy continuous mappings of E into F that are each fuzzy differentiable at $x \in E$, then $f + g$ is fuzzy differentiable and so is αf for all $\alpha \in \mathbb{K}$. A bijection f of E onto F is a *fuzzy diffeomorphism of class* C^1 if f and its inverse f^{-1} are fuzzy differentiable, and f' and $(f^{-1})'$ are fuzzy continuous.

Classically, one can glue together the open subsets of a topological vector space (more commonly a Banach space) to form a manifold. Fuzzy differentiable manifolds can be defined in the same way; the glue is a family of (local) fuzzy diffeomorphisms between fuzzy topological vector spaces.

Let X be a set. A *fuzzy atlas $\mathcal{A}$ of class C^1* on X is a collection of pairs (A_j, ϕ_j) (here and subsequently j ranges in some index set) that satisfies the following conditions:

1. Each A_j is a fuzzy set in X and $\sup_j A_j(x) = 1$, for all $x \in X$.
2. Each ϕ_j is a bijection, defined on the support of A_j, which maps A_j onto an open fuzzy set $\phi_j[A_j]$ in some ftvs E_j, and, for each l in the index set, $\phi_j[A_j \cap A_l]$ is an open fuzzy set in E_j.
3. For each l in the index set, the mapping $\phi_l \circ \phi_j^{-1}$, which maps $\phi_j[A_j \cap A_l]$ onto $\phi_l[A_j \cap A_l]$, is a C^1 fuzzy diffeomorphism.

Each pair (A_j, ϕ_j) is a *fuzzy chart* of the fuzzy atlas. If a point $x \in X$ lies in the support of A_j then (A_j, ϕ_j) is a *fuzzy chart at x*.

It is then possible to show [8] that given a C^1 fuzzy atlas $\mathcal{A}$ on a set X, the set X may be endowed with a fuzzy topology such that each A_j in $\mathcal{A}$ is an open fuzzy set and each ϕ_j is fuzzy continuous. In fact, the family $\{A_j\}$ of fuzzy sets forms a base for a proper fuzzy topology on X and in this topology the ϕ_j are fuzzy continuous.

Let $(X, \mathcal{T})$ be a fuzzy topological space. Suppose that A is an open fuzzy set in X and that ϕ is a fuzzy continuous bijective mapping which is defined on the support of A and which maps A onto an open fuzzy set V in some ftvs E. The pair (A, ϕ) is *compatible* with the C^1 atlas $\{(A_j, \phi_j)\}$ if each mapping $\phi_j \circ \phi^{-1}$ of $\phi[A \cap A_j]$ onto $\phi_j[A \cap A_j]$ is a fuzzy diffeomorphism of class C^1. Two C^1 fuzzy atlases are compatible if each fuzzy chart of one atlas is compatible with each fuzzy chart of the other atlas. Compatibility between C^1 fuzzy atlases is obviously an equivalence relation. An equivalence class of C^1 fuzzy atlases on X defines a C^1 *fuzzy manifold* on X. In the following, reference is made simply to fuzzy manifolds.

Suppose that X, Y are fuzzy manifolds and that f is a mapping of X into Y. The fuzzy differentiability of f at a point x in X may be defined [8] by its fuzzy differentiability in fuzzy charts at x and $f(x)$; that is:

Definition 5. The mapping $f : X \to Y$ is *fuzzy differentiable at a point $x \in X$* if there is a fuzzy chart (U, ϕ) at $x \in X$ and a fuzzy chart (V, ψ) at $f(x) \in Y$ such that the mapping $\psi \circ f \circ \phi^{-1}$, which maps $\phi[U \cap f^{-1}[V]]$ into $\psi[V]$, is fuzzy differentiable at $\phi(x)$.

It is obvious that this definition does not depend on the choice of fuzzy chart at x and $f(x)$. The mapping f is *fuzzy differentiable* if it is fuzzy differentiable at every point of X; it is a C^1 *fuzzy diffeomorphism* if it is a bijection and both it and its inverse f^{-1} are fuzzy differentiable.

Let X, Y, Z be fuzzy manifolds. The composition $g \circ f$ of two fuzzy differentiable mappings $f : X \to Y$, $g : Y \to Z$ is fuzzy differentiable, and, as a corollary, if f, g are C^1 fuzzy diffeomorphisms, then the composition $g \circ f$ is a C^1 fuzzy diffeomorphism [8].

9 Tangent Vectors in a Fuzzy Manifold

The notion of a directional derivative in Euclidean (or affine) space leads to the classical notion of a tangent vector of a differentiable manifold. A tangent vector of a fuzzy manifold may be defined as follows. Let X be a fuzzy manifold and let x be a (classical) point in X. Consider triples (U, ϕ, v_λ), where (U, ϕ) is a fuzzy chart at x and v_λ is a fuzzy point of the ftvs in which $\phi[U]$ lies. Two such triples (U, ϕ, v_λ), (V, ψ, w_λ) are *related*, written $(U, \phi, v_\lambda) \sim (V, \psi, w_\lambda)$, if the fuzzy derivative of $\psi \circ \phi^{-1}$ at $\phi(x)$ maps v_λ into w_λ; that is,

$$(\psi \circ \phi^{-1})'(\phi(x))v_\lambda = w_\lambda \, .$$

It is straightforward to show that the relation $(U, \phi, v_\lambda) \sim (V, \psi, w_\lambda)$ is an equivalence relation. The equivalence class of triples (U, ϕ, v_λ) constitutes a *tangent vector* of the fuzzy manifold X at x. The *tangent space* $T_x(X)$ at x is the set of all tangent vectors at x.

The set $T_x(X)$ can be given the structure of a vector space. Define the *sum* of two tangent vectors at $x \in X$ as

$$(U_1, \phi_1, v_{1\lambda}) + (U_2, \phi_2, v_{2\lambda}) = (U_2, \phi_2, (\phi_2 \circ \phi_1^{-1})'(\phi_1(x))v_{1\lambda} + v_{2\lambda}) \, ;$$

and the *product* of a tangent vector with a scalar α as

$$\alpha \cdot (U, \phi, v_\lambda) = (U, \phi, \alpha v_\lambda) \, .$$

These two operations do not depend on the choice of fuzzy chart [8]; thus if $(U_1, \phi_1, v_{1\lambda}) \sim (V_1, \psi_1, w_{1\lambda})$ and $(U_2, \phi_2, v_{2\gamma}) \sim (V_2, \psi_2, w_{2\gamma})$, then $(U_1, \phi_1, v_{1\lambda}) + (U_2, \phi_2, v_{2\gamma}) \sim (V_1, \psi_1, w_{1\lambda}) + (V_2, \psi_2, w_{2\gamma})$; and if $(U, \phi, v_\lambda) \sim (V, \psi, w_\lambda)$, then $\alpha \cdot (U, \phi, v_\lambda) \sim \alpha \cdot (V, \psi, w_\lambda)$.

10 Conclusion

Of the possible approaches to defining differential operators in ill-specified environments, the four considered here vary, necessarily, in the directness of their application to image representations. The definitions of differentiation based on convolving image luminance distributions have an immediate applicability, but they may be less suited to the analysis of higher-level image representations. The definitions of differentiation based on fuzzy sets make weaker assumptions about the nature of image representations and the extent of the analytic machinery available; but, for practical applications, they require the construction of an explicit relationship between the physically measurable properties of images and the fuzzy sets that, at some processing level, represent them.

The last issue may be addressed with the aid of a *fuzzy location*; that is, the kind of fuzzy set that, as introduced in Sect. 4, associates with each point $\mathbf{x}$ in $\mathbb{R}^2$

with luminance $I(\mathbf{x})$ a measure of the adequacy of that characterization of the image at that point. At least one experimental procedure has been described [2] for estimating the reliability of visual positional sense, and this procedure could be used to determine a fuzzy location. Based on the notion of fuzzy location and fuzzy orientation, the elements of a fuzzy geometry for visual space have been set out in [7] (see also [39]), where the notions of fuzzy locations for lines and curves have been introduced, and some of the fuzzy relations among them, including fuzzy collinearity, straightness, and tangency.

References

1. Attneave, F. (1954). Some informational aspects of visual perception, Psychological Review 61, pp. 183–193.
2. Attneave, F. (1955). Perception of place in a circular field, American Journal of Psychology 68, pp. 69–82.
3. Banks, H.T., Jacobs, M.Q. (1970). A differential calculus for multifunctions, Journal of Mathematical Analysis and Applications 29, pp. 246–272.
4. Berkmann, J., Caelli, T. (1993) On the relationship between surface covariance and differential geometry, this volume, pp. 343–352.
5. Chang, C.L. (1968). Fuzzy topological spaces, Journal of Mathematical Analysis and Applications 24, pp. 182–190.
6. Ferraro, M., Foster, D.H. (1987). Differentiation of fuzzy continuous mappings on fuzzy topological vector spaces, Journal of Mathematical Analysis and Applications 121, pp. 589–601.
7. Ferraro, M., Foster, D.H. (1993). Elements of a fuzzy geometry for visual space, this volume, pp. 333–342.
8. Ferraro, M., Foster, D.H. (1993). C^1 fuzzy manifolds, Fuzzy Sets and Systems 54, pp. 99–106.
9. Fischler, M.A., Bolles, R.C. (1986). Perceptual organization and curve partitioning, IEEE Transactions on Pattern Analysis and Machine Intelligence 8, pp. 100–105.
10. Florack, L.M.J., ter Haar Romeny, B.M., Viergever, M.A., Koenderink, J.J. (1993). Images: regular tempered distributions, this volume, pp. 651–659.
11. Foster, D.H. (1975). An approach to the analysis of the underlying structure of visual space using a generalized notion of visual pattern recognition, Biological Cybernetics 17, pp. 77–79.
12. Foster, D.H. (1979). Fuzzy topological groups, Journal of Mathematical Analysis and Applications 67, pp. 549–564.
13. Foster, D.H. (1980). A description of discrete internal representation schemes for visual pattern discrimination, Biological Cybernetics 38, pp. 151–157.
14. Foster, D.H. (1991). Operating on spatial relations. In: Watt, R.J. (ed.), Pattern Recognition by Man and Machine, Macmillan, Basingstoke, Hampshire, pp. 50–68.
15. Goguen, J.A. (1967). L-fuzzy sets, Journal of Mathematical Analysis and Applications 18, pp. 145–174.
16. Hastie, T.J., Tibshirani, R.J. (1990). Generalized Additive Models, Chapman & Hall, London.
17. Hoffman, D.D., Richards, W.A. (1984). Parts of recognition, Cognition 18, pp. 65–96.

18. Katsaras, A.K., Liu, D.B. (1977). Fuzzy vector spaces and fuzzy topological vector spaces, Journal of Mathematical Analysis and Applications 58, pp. 135–146.
19. Khalimsky, E.D. (1986). Pattern analysis of N-dimensional digital images, Proceedings 1986 IEEE International Conference on Systems, Man, and Cybernetics, Atlanta, GA, pp. 1559–1562.
20. Kock, A. (1981). Synthetic Differential Geometry (London Mathematical Society Lecture Note Series 51), Cambridge University Press, Cambridge.
21. Koenderink, J.J., van Doorn, A.J. (1987). Representation of local geometry in the visual system, Biological Cybernetics 55, pp. 367–375.
22. Kong, T.Y. Rosenfeld, A. (1991). Digital topology: a comparison of the graph-based and topological approaches. In: Reed, G.M., Roscoe, A.W., Wachter, R.F. (eds.), Topology and Category Theory in Computer Science, Clarendon Press, Oxford, pp. 273–289.
23. Kopperman, R. (1993). The Khalimsky line as a foundation for digital topology, this volume, pp. 3–20.
24. Kovalevsky, V.A. (1993). Topological foundations of shape analysis, this volume, pp. 21–36.
25. Lang, S. (1967). Introduction to Differentiable Manifolds, Interscience (John Wiley and Sons), New York.
26. Liang, P., Todhunter, J.S. (1990). Representation and recognition of surface shapes in range images: a differential geometry approach, Computer Vision, Graphics, and Image Processing 52, pp. 78–109.
27. Lindeberg, T.P. (1990). Scale-space for discrete signals, IEEE Transactions on Pattern Analysis and Machine Intelligence 12, pp. 234–254.
28. Lindeberg, T. (1992). Discrete derivative approximations with scale-space properties: a basis for low-level feature extraction, Technical Report TRITA-NA-P9212 Computational Vision and Active Perception Laboratory, Department of Numerical Analysis and Computing Science, Royal Institute of Technology, Stockholm, Sweden, pp. 1–53.
29. Lowen, R. (1976). Fuzzy topological spaces and fuzzy compactness, Journal of Mathematical Analysis and Applications 56, pp. 621–633.
30. Mizumoto, M., Tanaka, K. (1976). Some properties of fuzzy sets of type 2, Information and Control 31, 312–340.
31. Müller, H.-G. (1984). Smooth optimum kernel estimators of densities, regression curves and modes, Annals of Statistics 12, pp. 766–774.
32. Noest, A.J. (1993). Neural processing of overlapping shapes, this volume, pp. 383–392.
33. Novák, V. (1989). Fuzzy Sets and their Applications, Adam Hilger, Bristol.
34. Pedrycz, W. (1989). Fuzzy Control and Fuzzy Systems, Research Studies Press, Taunton, Somerset.
35. Pu, P.-M., Liu, Y.-M. (1980). Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence, Journal of Mathematical Analysis and Applications 76, pp. 571–599.
36. Puri, M.L., Ralescu, D.A. (1983). Differentials of fuzzy functions, Journal of Mathematical Analysis and Applications 91, pp. 552–558.
37. Rådström, H. (1952). An embedding theorem for spaces of convex sets, Proceedings of the American Mathematical Society 3, pp. 165–169.
38. Richards, W., Dawson, B., Whittington, D. (1986). Encoding contour shape by curvature extrema, Journal of the Optical Society of America A 3, pp. 1483–1491.

39. Rosenfeld, A. (1992). Fuzzy geometry: an overview, IEEE International Conference on Fuzzy Systems, San Diego, CA, pp. 113–117.
40. Russell, B. (1923). Vagueness, Australasian Journal of Psychology and Philosophy 1, p. 84–92.
41. Schwartz, L. (1957, 1959). Théorie des Distributions, Vols I–II, Hermann, Paris.
42. Silverman, B.W. (1986). Density Estimation for Statistics and Data Analysis, Chapman and Hall, London.
43. Weiss, I. (1992). Noise resistant invariants of curves. In: Mundy, J.L., Zisserman, A. (eds.), Geometric Invariance in Computer Vision, MIT Press, Cambridge, MA, pp. 135–156.
44. Zadeh, L.A. (1965). Fuzzy sets, Information and Control 8, pp. 338–353.
45. Zadeh, L.A. (1968). Probability measures of fuzzy events, Journal of Mathematical Analysis and Applications 23, pp 421–427.
46. Zadeh, L.A. (1974). Fuzzy logic and its application to approximate reasoning, Information Processing 74, pp. 591–594.