Shape in Picture

Mathematical Description of Shape in Grey-level Images

Edited by

Ying-Lie O

Centre for Mathematics and Computer Science (CWI)
Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

Alexander Toet

Netherlands Organization for Applied Scientific Research
Institute for Human Factors (TNO-IZF)
Kampweg 5, 3769 DE Soesterberg, The Netherlands

David Foster

University of Keele
Department of Communication and Neuroscience
Keele, Staffordshire ST5 5BG, UK

Henk J. A. M. Heijmans

Centre for Mathematics and Computer Science (CWI)
Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

Peter Meer

Rutgers University
Department of Electrical and Computer Engineering
Piscataway, NJ 08855-0909, USA

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Elements of a Fuzzy Geometry for Visual Space*

Mario Ferraro[1] *and David H. Foster*[2]

[1] Dipartimento di Fisica Sperimentale, Università di Torino, via Giuria 1, 10125 Torino, Italy
[2] Department of Communication and Neuroscience, Keele University, Staffordshire ST5 5BG, UK

Abstract. This study introduces the notions of *fuzzy location* and *fuzzy proximity* to capture the imprecision associated with judgements of absolute and relative visual position. These notions are used to establish the elements of a fuzzy geometry for visual space, including the *fuzzy betweenness* of points, the *fuzzy orientation* of a pair of points, and the *fuzzy collinearity* of three or more points. Fuzzy orientation and fuzzy collinearity are, in turn, used to define the *fuzzy straightness* of a curve and the *fuzzy tangency* of two curves.

Keywords: shape description, differential geometry, fuzzy topology, fuzzy location, proximity, orientation, collinearity, tangency.

1 Introduction

Any description of perceived visual shape is based on certain assumptions concerning the topology and geometry of visual space and the parts of the shape under consideration. For example, the representation of an image by a scalar field $f : \mathbb{R}^2 \to \mathbb{R}$, where $f(\mathbf{x})$ is the light intensity at the point $\mathbf{x} \in \mathbb{R}^2$, assumes that visual space is a manifold, usually smooth, and that $(\mathbf{x}, f(\mathbf{x}))$, $\mathbf{x} \in \mathbb{R}^2$, is a surface, namely a Monge patch, the characteristics of which can be analysed by geometrical methods. Other types of visual representations, oriented more towards graph-theoretic methods, assume that shapes can be partitioned into elementary geometrical components, such as points and lines, which are connected by certain geometrical relations [5].

Although classical topology and geometry provide powerful tools for investigating shape, they fail, by definition, to acknowledge that visual space is not an abstract space and that its properties are determined by the processes that lead to perception. Any visual measurement—that is, any operation performed to estimate the attributes of an image—is affected by imprecision that arises

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from various sources. Thus, if an attribute has numerical values, the accuracy with which these values can be estimated on some absolute scale is limited, by noise and by quantization errors; and, whether or not an attribute has numerical values, the labels used by human observers to characterize those values may still be vague [13]; consider, for example, the notion of the "nearness" of two objects.

This study uses the theory of fuzzy sets [14] as a basis for a more appropriate approach to the geometry of visual space in that it addresses directly the imprecision associated with visual measurements. The structure of fuzzy sets is poorer than that of classical sets since the law of the excluded middle does not hold [10] (see comment after Definition 3). The theory of fuzzy sets has sometimes been interpreted as a part or reformulation of probability theory, but the two theories are distinct, philosophically and operationally. Discussion of related issues can be found in [15, 10, 11], and a review of some other fuzzy geometrical concepts in [12].

2 Fuzzy Sets and Fuzzy Topologies

This section reviews, briefly, some of the basic properties of fuzzy sets. Let X be a set. Any subset W of X has associated with it a *characteristic function* $\chi_W : X \to \{0,1\}$, where $\chi_W(x) = 1$ if $x \in W$ and $\chi_W(x) = 0$ if $x \notin W$. This definition may be generalized to form the notion of a "fuzzy set", which associates with each point $x \in X$ a "grade of membership", usually taken in the unit interval $[0,1]$. Thus a *fuzzy set* A in a set X is a mapping $A : X \to [0,1]$ such that $A(x)$ is the grade of membership of x in A. The grade of membership may be taken in a lattice [6] rather than the interval $[0,1]$.

Definition 1. Let A be a fuzzy set in X. The *support* of A is the classical set $\mathrm{Supp}(A) = \{x \mid A(x) > 0\}$; the *$\alpha$-level* of A for a given $\alpha \in [0,1]$ is the classical set $A^\alpha = \{x \mid A(x) = \alpha\}$; and the *$\alpha$-cut* of A for a given $\alpha \in [0,1]$ is the classical set $A_\alpha = \{x \mid A(x) \geq \alpha\}$.

The following proposition presents a different view of fuzzy sets, namely, as a sequence of classical sets A_α for $\alpha \in [0,1]$.

Proposition 2. *Let A be a fuzzy set. Then*

$$A(x) = \sup_{\alpha \in [0,1]} \{\alpha \mid x \in A_\alpha\} .$$

Proof. See [10].

Relations among fuzzy sets such as equality or inclusion, and operations such as union, intersection, or complement can be defined naturally for fuzzy sets by generalization of the classical definitions. The following summarizes the basic definitions, for the sake of completeness.

Definition 3. Let A, B, C be fuzzy sets in a set X. Then

$$A = B \quad \text{if and only if} \quad A(x) = B(x) \text{ for all } x \in X\,;$$

$$A \subset B \quad \text{if and only if} \quad A(x) \leq B(x) \text{ for all } x \in X\,;$$

$$C = A \cup B \quad \text{if and only if} \quad C(x) = \max\{A(x), B(x)\}, \quad \text{for all } x \in X\,;$$

$$C = A \cap B \quad \text{if and only if} \quad C(x) = \min\{A(x), B(x)\}, \quad \text{for all } x \in X\,;$$

and the complement $\bar{A}$ of A is given by

$$B = \bar{A} \quad \text{if and only if} \quad B(x) = 1 - A(x) \text{ for all } x \in X\,.$$

More generally, for an arbitrary family $\{A_j\}_{j \in J}$ of fuzzy sets, the union $C = \bigcup_{j \in J} A_j$ and the intersection $D = \bigcap_{j \in J} A_j$ are defined by $C(x) = \sup_{j \in J} A_j(x)$, for all $x \in X$, and $D(x) = \inf_{j \in J} A_j(x)$, for all $x \in X$.

It is easy to verify that if membership functions are replaced by characteristic functions the classical definitions result.

For each $c \in [0, 1]$, denote by k_c the fuzzy set in X with membership function $k_c(x) = c$, for all $x \in X$. The fuzzy set k_1 corresponds to the set X and k_0 to the empty set $\emptyset$. Notice that in general for a fuzzy set A the intersection $A \cap \bar{A} \neq k_0$ and the union $A \cup \bar{A} \neq k_1$.

Definition 4. Let f be a mapping from a set X to a set Y. Let B be a fuzzy set in Y. Then the *inverse image* $f^{-1}[B]$ of B is the fuzzy set in X given by

$$f^{-1}[B](x) = B(f(x)), \text{ for all } x \in X\,.$$

Conversely, let A be a fuzzy set in X. The *image* $f[A]$ of A is the fuzzy set in Y given by

$$f[A](y) = \begin{cases} \sup_{z \in f^{-1}(y)} A(z), & \text{if } f^{-1}(y) \text{ is nonempty,} \\ 0, & \text{otherwise,} \end{cases}$$

where $f^{-1}(y) = \{\, x \mid f(x) = y \,\}$.

Definition 5. Let E be a linear space. A fuzzy set A in E is *convex* if

$$A(\lambda x + (1 - \lambda)y) \geq \min\{A(x), A(y)\}\,,$$

for all $x, y \in E$ and $0 \leq \lambda \leq 1$.

Proposition 6. *A fuzzy set is convex if and only if all its α-cuts are (classical) convex sets.*

Proof. See [10].

Given a family of fuzzy sets it is possible to define a fuzzy topology that is a natural generalization of the classical definition.

Definition 7. A *fuzzy topology* on a set X is a family $\mathcal{T}$ of fuzzy sets that satisfies the following conditions [2]:

1. k_0, $k_1 \in \mathcal{T}$.
2. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$.
3. If $A_j \in \mathcal{T}$ for all $j \in J$ (J some index set), then $\bigcup_{j\in J} A_j \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space*, or *fts* for short, and the members of $\mathcal{T}$ are called *open fuzzy sets*. Definition 7 is not completely satisfactory—for instance, it fails to make constant functions between fts's fuzzy continuous—and an alternative definition [7] has been proposed in which condition (1) is replaced by

1′. For all $c \in [0,1]$, $k_c \in \mathcal{T}$.

A fuzzy topology that satisfies condition (1′) is referred to as a *proper* fuzzy topology [3].

Definition 8. A subfamily $\mathcal{B}$ of $\mathcal{T}$ is a *basis* for a fuzzy topology $\mathcal{T}$ if each member of $\mathcal{T}$ can be expressed as the union of members of $\mathcal{B}$.

Proposition 9. *A family $\mathcal{B}$ of fuzzy sets in X is a basis for a proper fuzzy topology on X if it satisfies the following conditions:*

1. $\sup_{B\in\mathcal{B}} B(x) = 1$, *for every* $x \in X$.
2. *If* $B_1, B_2 \in \mathcal{B}$, *then* $B_1 \cap B_2 \in \mathcal{B}$.
3. *For every* $B \in \mathcal{B}$ *and* $c \in [0,1]$, $B \cap k_c \in \mathcal{B}$.

Proof. Let $\mathcal{T}(\mathcal{B})$, or simply $\mathcal{T}$, be the family of fuzzy sets that can each be expressed as a union of elements of $\mathcal{B}$. From condition (1), $k_1 \in \mathcal{T}$, and it is obvious that if $A_j \in \mathcal{T}$ for all $j \in J$ (J some index set), then $\bigcup_{j\in J} A_j \in \mathcal{T}$. Let $\{B_j\}$ and $\{B_l\}$ be subfamilies of $\mathcal{B}$ (j and l ranging in index sets J and L respectively) and let $A = \bigcup_{j\in J} B_j$ and $C = \bigcup_{l\in L} B_l$. Then, for each $x \in X$, $\min\{A(x), C(x)\} = \min\{\sup_j B_j(x), \sup_l B_l(x)\} = \sup_{j,l}\{\min\{B_j(x), B_l(x)\}\}$. Thus if $A, C \in \mathcal{T}$, then $A \cap C \in \mathcal{T}$. Finally, it is necessary to show that k_c belongs to $\mathcal{T}$ for every c, $0 \le c < 1$. Condition (3) implies that for each such c, the fuzzy set with membership function $\sup_{B\in\mathcal{B}}\{\min\{B(x), c\}\}$, $x \in X$, belongs to $\mathcal{T}$. By condition (1) there exists, for each $x \in X$ and c, $0 \le c < 1$, a fuzzy set $B \in \mathcal{B}$ with grade of membership $B(x) \ge c$; hence $\sup_{B\in\mathcal{B}}\{\min\{B(x), c\}\} = c$, which shows that $k_c \in \mathcal{T}$. Thus the family generated by unions of $B \in \mathcal{B}$ is a proper fuzzy topology. □

Notice that in a basis for an improper fuzzy topology, condition (3) is unnecessary.

3 Fuzzy Locations and Fuzzy Proximities

Consider a "physical point" p, for instance, a tiny spot of light, that has position $\mathbf{x}$ in the space $\mathbb{R}^2$. If an observer attempts to locate p visually he or she obtains an estimate that is imprecise, for the reasons mentioned in the Introduction; in addition, each such measurement depends on the experimental conditions under which the determination is made. The effects of this imprecision can be modelled by assuming that, for a given experimental condition, the point p is associated with a fuzzy set, thus:

Definition 10. A *fuzzy location* of a physical point p is a fuzzy set $P_p : \mathbb{R}^2 \to [0,1]$. The family of all fuzzy locations of the point p at a given position $\mathbf{x}$ in $\mathbb{R}^2$ is denoted by $\mathcal{P}_{\mathbf{x}}$, and over all possible positions by $\mathcal{P}$; that is, $\mathcal{P} = \bigcup_{\mathbf{x}\in\mathbb{R}^2} \mathcal{P}_{\mathbf{x}}$. It is assumed that for every physical point p there exists $\mathbf{x}_0 \in \mathbb{R}^2$ such that $\sup_{P_p\in\mathcal{P}_{\mathbf{x}_0}} P_p(\mathbf{x}_0) = 1$, and that $\sup_{P_p\in\mathcal{P}} P_p(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{R}^2$; that is, the family of all fuzzy locations *covers* $\mathbb{R}^2$.

Sometimes an additional assumption is made; namely, that fuzzy locations are convex (Definition 5). Some relevant properties of convex fuzzy sets are given in the following propositions (where the subscript identifying the physical point has been omitted). Convex sets will be used in the next section in the development of the elements of a fuzzy geometry. The notation P_p for a fuzzy location and P_{α_0} for an α_0-cut of P should not be confused.

Proposition 11. *Let P be a convex fuzzy set and let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$ be such that $P(\mathbf{x}_1) = P(\mathbf{x}_2) = \alpha_0$, where $0 < \alpha_0 \leq 1$. Then there exists a set $Y = \{\, y \mid y = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2,\ 0 \leq \lambda \leq 1 \,\}$ such that $P(y) \geq \alpha_0$ for all $y \in Y$.*

Proof. Consider the α_0-cut P_{α_0} (Definition 1). It must be convex, by Proposition 6. Hence Y must be a subset of P_{α_0}, and the assertion follows. □

Proposition 12. *Let P be a convex fuzzy set and let $\alpha_0 = \sup_{\mathbf{x}\in\mathbb{R}^2} P(\mathbf{x})$, where $0 < \alpha_0 \leq 1$. Suppose that there exist $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$ such that for every neighbourhood (in the standard topology on $\mathbb{R}^2$) A_1, A_2 of $\mathbf{x}_1, \mathbf{x}_2$, respectively,*

$$\sup_{\mathbf{x}\in A_1} P(\mathbf{x}) = \sup_{\mathbf{x}\in A_2} P(\mathbf{x}) = \alpha_0 \,.$$

Then there exists a set $Y = \{\, y \mid y = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2,\ 0 \leq \lambda \leq 1 \,\}$ such that for every $y \in Y$ there is a neighbourhood A_y of y for which $\alpha_0 = \sup_{\mathbf{x}\in A_y} P(\mathbf{x})$.

Proof. Suppose that the statement of the proposition is false for some point $y \in Y$. Then there exists ε, $0 \leq \varepsilon < \alpha_0$, such that $\sup_{\mathbf{x}\in A_y} P(\mathbf{x}) < \alpha_0 - \varepsilon$ for every neighbourhood A_y of y. By hypothesis, there exist neighbourhoods A_1, A_2 of $\mathbf{x}_1, \mathbf{x}_2$, respectively, such that $P(\mathbf{z}_1) \geq \alpha_0 - \varepsilon$ and $P(\mathbf{z}_2) \geq \alpha_0 - \varepsilon$ for some $\mathbf{z}_1 \in A_1$ and $\mathbf{z}_2 \in A_2$. Choose λ_0, $0 \leq \lambda_0 \leq 1$, and y_0 belonging to a neighbourhood A_y of y such that $y_0 = \lambda_0\mathbf{z}_1 + (1-\lambda_0)\mathbf{z}_2$. Set $\alpha_1 = \min\{P(\mathbf{z}_1), P(\mathbf{z}_2)\}$ and consider the α_1-cut P_{α_1}. This set is not convex, which implies that P is not convex (compare Proposition 6), contrary to the hypothesis. □

Proposition 13. *Suppose that a convex fuzzy set P has a maximum value, α_0 say. Then the α_0-cut P_{α_0} is a point, or an interval of a line, or a convex subset of $\mathbb{R}^2$.*

Proof. Obvious, since P_{α_0} must be convex. □

Next, a fuzzy set is defined that determines the grade of proximity of any two physical points.

Definition 14. Given two physical points p, q, with fuzzy locations P_p, P_q respectively, the *fuzzy proximity* of p, q, denoted by $\delta(p, q)$, is given by the fuzzy set $P_p \cap P_q$. The two points p, q are said to be *fuzzy proximal* if $\delta(p, q) \neq k_0$; that is, there exists $\mathbf{x}_0 \in \mathbb{R}^2$ such that $\delta(p, q)(\mathbf{x}_0) = \min\{P_p(\mathbf{x}_0), P_q(\mathbf{x}_0)\} > 0$.

Notice that if p, q are fuzzy proximal, then $\sup_{\mathbf{x} \in \mathbb{R}^2} \min\{P_p(\mathbf{x}), P_q(\mathbf{x})\} > 0$. The fuzzy set $\delta(p, q)$ can be thought of as quantifying the vague description "near to". This definition extends naturally to pairs of any (not necessarily finite) number of points: if $\{p_j\}_{j \in J}, \{p_l\}_{l \in L}$ are two sets of physical points, then, with an abuse of notation, their fuzzy proximity is given by

$$\delta(\{p_j\}, \{p_l\}) = \left(\bigcup_{j \in J} P_{p_j} \right) \cap \left(\bigcup_{l \in L} P_{p_l} \right).$$

A fuzzy proximity δ for physical points and their fuzzy locations satisfies conditions analogous to those characterizing a proximity for classical sets, except for a separation condition (see [9]); thus:

Proposition 15. *Let p, q, r be physical points. Then:*

1. $\delta(p, q) = \delta(q, p)$.
2. $\delta(p, q) \neq k_0$ *implies* $P_p \neq k_0$ *and* $P_q \neq k_0$, *where* P_p, P_q *are the fuzzy locations of* p, q *respectively.*
3. $\delta(\{p, q\}, r) \neq k_0$ *if and only if* $\delta(p, r) \neq k_0$ *or* $\delta(q, r) \neq k_0$.

Proof. Statements (1) and (2) are obviously true. To prove statement (3), consider the following. Let P_r be the fuzzy location of r. Suppose that there exists a point $\mathbf{x}_0 \in \mathbb{R}^2$ such that $\min\{\max\{P_p(\mathbf{x}_0), P_q(\mathbf{x}_0)\}, P_r(\mathbf{x}_0)\} > 0$ and $\min\{P_p(\mathbf{x}_0), P_r(\mathbf{x}_0)\} = 0$ and $\min\{P_q(\mathbf{x}_0), P_r(\mathbf{x}_0)\} = 0$. Then $\min\{\max\{P_p(\mathbf{x}_0), P_q(\mathbf{x}_0)\}, P_r(\mathbf{x}_0)\} = 0$, contrary to the hypothesis. Conversely, suppose that there exists a point $\mathbf{x}_0 \in \mathbb{R}^2$ such that $\min\{P_p(\mathbf{x}_0), P_r(\mathbf{x}_0)\} > 0$ or $\min\{P_q(\mathbf{x}_0), P_r(\mathbf{x}_0)\} > 0$. Then $\min\{\max\{P_p(\mathbf{x}_0), P_q(\mathbf{x}_0)\}, P_r(\mathbf{x}_0)\} \geq \min\{P_p(\mathbf{x}_0), P_r(\mathbf{x}_0)\} > 0$, or $\min\{\max\{P_p(\mathbf{x}_0), P_q(\mathbf{x}_0)\}, P_r(\mathbf{x}_0)\} \geq \min\{P_q(\mathbf{x}_0), P_r(\mathbf{x}_0)\} > 0$. □

The form of the fuzzy locations P_p, P_q determines the form of $\delta(p, q)$, as follows.

Proposition 16. *If fuzzy locations P_p, P_q are convex fuzzy sets, then $\delta(p, q)$ is a convex fuzzy set.*

Proof. It is enough to recall that the intersection of two convex classical sets is convex and the result follows from Proposition 6. □

Notice that it is possible to define the fuzzy proximity of an arbitrary, finite number of points, $p_1, p_2, \ldots, p_n$:

$$\delta(p_1, p_2, \ldots, p_n) = \bigcap_{i=1}^{n} P_{p_i} ,$$

and if all the P_{p_i} are convex fuzzy sets, then $\delta(p_1, p_2, \ldots, p_n)$ is a convex fuzzy set.

Although not developed here, it is easy to see that fuzzy proximities define a basis for an *improper* fuzzy topology on visual space.

Proposition 17. *The family of fuzzy sets formed by fuzzy proximities of the form $\delta(p_1, p_2, \ldots, p_n)$, for every finite integer n, is a basis for an improper fuzzy topology.*

Proof. Let $\mathcal{B}$ be the family of all fuzzy proximities $\delta(p_1, p_2, \ldots, p_n)$, n finite. For all $P_p \in \mathcal{P}$, the family of all fuzzy locations, $\delta(p,p) = P_p$ and hence $\mathcal{P} \subset \mathcal{B}$. Then condition (1) of Proposition 9 is satisfied, since $\sup_{P_p \in \mathcal{P}} P_p(x) = 1$ for all $\mathbf{x} \in \mathbb{R}^2$ (Definition 10). Next, given two fuzzy proximities $\delta(p_1, p_2, \ldots, p_m)$, $\delta(p_{m+1}, p_{m+2}, \ldots, p_{m+n})$, their intersection

$$\delta(p_1, p_2, \ldots, p_m) \cap \delta(p_{m+1}, p_{m+2}, \ldots, p_{m+n}) = \delta(p_1, p_2, \ldots, p_{m+n})$$

belongs to $\mathcal{B}$ and thus condition (2) of Proposition 9 holds. □

If fuzzy locations are assumed to be convex, then fuzzy locations and fuzzy proximities are open fuzzy sets of a proper fuzzy topology.

Proposition 18. *The family of all convex fuzzy sets defined in $\mathbb{R}^2$ is a basis for a proper fuzzy topology.*

Proof. Conditions (1) and (2) of Proposition 9 are obviously satisfied because the fuzzy set k_1 is convex and the intersection of convex fuzzy sets is a convex fuzzy set (see Proposition 16). To prove condition (3) it is enough to observe that the fuzzy sets k_c, $c \in [0,1]$, are convex fuzzy sets. □

4 Elements of a Fuzzy Geometry

In this section, the notion of fuzzy location is extended to the notion of the fuzzy orientation of a pair of points and the fuzzy collinearity of three or more points. It is shown that these notions make it possible to define the fuzzy straightness of a curve and the fuzzy tangency of two curves. First, the notion of the fuzzy betweenness of points is introduced.

Definition 19. Let p, r be two fuzzy proximal physical points. A physical point q is *fuzzy between* p and r if $\delta(p,q) \supset \delta(p,r)$ and $\delta(r,q) \supset \delta(r,p)$.

Consider the set of all possible orientations θ, $0 \leq \theta < 2\pi$, in the plane.

Definition 20. For any pair of physically distinguishable points (p,q), the *fuzzy orientation* of (p,q) is a fuzzy set $O_{p,q} : [0,2\pi) \to [0,1]$. If the points are not physically distinguishable, then $O_{p,q} : [0,\pi) \to [0,1]$.

It is, in principle, possible to derive a fuzzy orientation from the fuzzy location of two distinct physical points p,q. Let A_θ be the set of all pairs $(\mathbf{x},\mathbf{y})$, $\mathbf{x} \in \mathrm{Supp}(P_p)$, $\mathbf{y} \in \mathrm{Supp}(P_q)$, $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$, for which the orientation of the line joining $\mathbf{x}$ and $\mathbf{y}$ is θ. The derived fuzzy orientation $O'_{p,q}$ of (p,q) is then defined as the fuzzy set

$$O'_{p,q}(\theta) = \sup_{(\mathbf{x},\mathbf{y})\in A_\theta} \{\min\{P_p(\mathbf{x}), P_q(\mathbf{y})\}\}, \quad \text{for all } \theta \in [0,2\pi).$$

Observed orientation estimates need not, however, follow such a rule.

The notion of fuzzy proximity may be extended to fuzzy orientations.

Definition 21. Let $(p,q),(r,s)$ be two pairs of physical points with fuzzy orientations $O_{p,q}, O_{r,s}$ respectively. Their *fuzzy proximity with respect to orientation*, denoted by $\gamma(p,q;r,s)$, is given by the fuzzy set $O_{p,q} \cap O_{r,s}$. The two pairs of points are said to be *fuzzy proximal with respect to orientation* if $\gamma(p,q;r,s) \neq k_0$.

Notice that $\gamma(p,q;r,s) = \gamma(r,s;p,q)$. Fuzzy proximity with respect to orientation makes it possible to define for three physical points their fuzzy collinearity.

Definition 22. Let p,q,r be physical points with fuzzy orientations $O_{p,q}$, $O_{q,r}$, $O_{p,r}$ taken a pair at a time. Then the *fuzzy collinearity* of p,q,r, denoted by $\eta(p,q,r)$, is given by the fuzzy set $O_{p,q} \cap O_{q,r} \cap O_{p,r}$. The points p,q,r are said to be *fuzzy collinear* if $\eta(p,q,r) \neq k_0$.

The definition may be extended to four or more points.

In the following, a physical *curve* in $\mathbb{R}^2$ is considered as the image of a mapping of an interval in $\mathbb{R}$ into $\mathbb{R}^2$ rather than as the mapping itself.

Definition 23. Let c be a physical curve in $\mathbb{R}^2$. The *fuzzy location* P_c of c is the union $\bigcup_{p\in c} P_p$ of the fuzzy locations P_p for all $p \in c$; that is, $P_c(\mathbf{x}) = \sup_{p\in c} P_p(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^2$.

The fuzzy proximity $\delta(p,c)$ of a physical point p and a physical curve c is defined by the extension of Definition 14 as the intersection $P_p \cap P_c$ of their fuzzy locations P_p and P_c. If p and c are fuzzy proximal, then $P_p \cap P_c \neq k_0$; that is, there exists $\mathbf{x}_0 \in \mathbb{R}^2$ such that $\delta(p,c)(\mathbf{x}_0) = \min\{P_p(\mathbf{x}_0), \sup_{q\in c} P_q(\mathbf{x}_0)\} > 0$. It is easy to show that

$$\delta(p,c)(\mathbf{x}) = \min\left\{P_p(\mathbf{x}), \sup_{q\in c} P_q(\mathbf{x})\right\} = \sup_{q\in c}\{\min\{P_p(\mathbf{x}), P_q(\mathbf{x})\}\},$$

for all $\mathbf{x} \in \mathbb{R}^2$. That is, the fuzzy proximity $\delta(p,c)$ of a point and a curve is the union of the fuzzy proximities $\delta(p,q)$ of p and the points q belonging to c.

Proposition 24. *Let p be a physical point and c a physical curve. Then p and c are fuzzy proximal if and only if there exists a point q in c such that p and q are fuzzy proximal.*

Proof. Suppose that there exists no $q \in c$ such that $\delta(p,q) \neq k_0$; then $\delta(p,c)(\mathbf{x}) = \sup_{q\in c}\{\min\{P_p(\mathbf{x}), P_q(\mathbf{x})\}\} = \sup_{q\in c}\delta(p,q)(\mathbf{x}) = 0$, for all $\mathbf{x} \in \mathbb{R}^2$. Conversely, let $\mathbf{x}_0 \in \mathbb{R}^2$ be such that for some $q \in c$, $\delta(p,q)(\mathbf{x}_0) > 0$; then $\delta(p,c)(\mathbf{x}_0) = \sup_{q\in c}\delta(p,q)(\mathbf{x}_0) > 0$. □

Given a physical curve c and two fuzzy proximal physical points p, r, the curve is said to be *fuzzy between* the two points if there exists $q \in c$ such that $\delta(p,q) \supset \delta(p,r)$ and $\delta(r,q) \supset \delta(r,p)$. The fuzzy proximity $\delta(c,h)$ of two physical curves c and h is also defined by the extension of Definition 14; that is, as the intersection $P_c \cap P_h$ of their fuzzy locations P_c and P_h.

Proposition 25. *Two physical curves c, h are fuzzy proximal if and only if there exist points p in c and q in h such that p and q are fuzzy proximal.*

Proof. Omitted, since it is analogous to the proof of Proposition 24.

The definition of the fuzzy collinearity $\eta(p,q,r)$ of three points p,q,r (Definition 22) can be extended to define the fuzzy straightness of a curve.

Definition 26. Let l be a physical curve. The *fuzzy straightness* of l, denoted by $\eta(l)$, is given by the fuzzy set $\bigcap_{p,q,r\in l}\eta(p,q,r)$. The curve l is said to be *fuzzy straight* if $\eta(l) \neq k_0$.

Notice that the fuzzy straightness of a curve is simply the intersection of all fuzzy orientations $O_{p,q}$ for all p, q in the curve.

The definition of the fuzzy proximity with respect to orientation $\gamma(p,q;r,s)$ of two pairs of points $(p,q),(r,s)$ (Definition 21) leads to a definition of fuzzy tangency (a different approach to fuzzy tangency is discussed in [4, 3]).

Definition 27. Let c, h be two physical curves and suppose that there exists a point p that is fuzzy proximal to both c and h. Then c, h are *fuzzy tangent* at p if, for any two points $q \in c$, $r \in h$ that are each fuzzy proximal to but distinct from p, the pairs $(q,p),(r,p)$ are fuzzy proximal with respect to orientation; that is, $\gamma(q,p;r,p) \neq k_0$.

The notion of fuzzy betweenness can be extended to fuzzy orientations. Let $(p,q),(r,s),(u,v)$ be three pairs of distinct points. Then (r,s) is said to be fuzzy between (p,q) and (u,v) with respect to orientation if the fuzzy proximities $\gamma(p,q;r,s) \supset \gamma(p,q;u,v)$ and $\gamma(u,v;r,s) \supset \gamma(u,v;p,q)$.

Proposition 28. *Suppose that two physical curves c, h are fuzzy tangent at a point p, and a third curve g is fuzzy proximal to p. Suppose further that, for any three points q in c, r in g, s in h that are each fuzzy proximal to but distinct from p, (r,p) is fuzzy between (q,p) and (s,p) with respect to orientation. Then g is fuzzy tangent to both c and h at p.*

Proof. The fuzzy proximity with respect to orientation $\gamma(q,p;s,p) \neq k_0$, and both $\gamma(q,p;r,p) \supset \gamma(q,p;s,p)$ and $\gamma(s,p;r,p) \supset \gamma(s,p;q,p)$. □

5 Conclusion

The approach of this study to the geometry of visual space has been formal in that the construction of geometrical properties and relations was not founded on a particular set of empirical data. It is, however, possible to determine by experimental measurement—for a given observer and experimental paradigm—typical instances of fuzzy locations and fuzzy orientations, and typical instances of (in principle) dependent relations and properties such as fuzzy betweenness, fuzzy collinearity, fuzzy straightness, and fuzzy tangency. A possible experimental procedure for making these measurements has been described by Attneave [1]. This procedure could be used to generate examples of fuzzy locations, and extended to the generation of other properties and relations. Whether the results of such measurements can be related to each other according to the present analysis may offer a test of its physical appropriateness.

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