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Almost equivalence of combinatorial and distance processes for discrimination in multielement images

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Abstract. Under certain experimental conditions, visual discrimination performance in multielement images is closely related to visual identification performance: elements of the image are distinguished only insofar as they appear to have distinct, discrete, internal characterizations. This report is concerned with the detailed relationship between such internal characterizations and observable discrimination performance. Two types of general processes that might underline discrimination are considered. The first is based on computing all possible internal image characterizations that could allow a correct decision, each characterization weighted by the probability of its occurrence and of a correct decision being made. The second process is based on computing the difference between the probabilities associated with the internal characterizations of the individual image elements, the difference quantified naturally with an $l^{(p)}$ norm. The relationship between the two processes was investigated analytically and by Monte Carlo simulations over a plausible range of numbers n of the internal characterizations of each of the m elements in the image. The predictions of the two processes were found to be closely similar. The relationship was precisely one-to-one, however, only for $n = 2$, $m = 3, 4, 6$, and for $n > 2$, $m = 3, 4$, $p = 2$. For all other cases tested, a one-to-one relationship was shown to be impossible.

1 Introduction

In some visual tasks, principally involving short-duration displays or spatially complex images, the number of internal descriptions or characterizations used by the visual system to represent the parts of an image appears to be very small. For example, the discrimination of line segments in short-duration images presented in the peripheral visual field is dominated by mechanisms sensitive only to the general orientation of the line segments, that is the vertical, horizontal, and oblique (Beck and Ambler 1972; Beck 1983); in the discrimination of brief images containing curvilinear elements, performance is determined mainly by mechanisms sensitive to just 2–3 values of perceived curvature or sag (Foster 1983; Ferraro and Foster 1986); and in the discrimination of brief images containing pairs of longitudinally and laterally displaced line segments, performance is determined mainly by mechanisms sensitive to just two values of relative position in each of the two directions of displacement (Foster and Ferraro 1989). Possible rationales for this behaviour have been offered by Ferraro and Foster (1986) and by Watt (1987).

This report is concerned with the detailed relationship between internal characterizations of the kind just described and observable discrimination performance. Two types of processes that might underlie discrimination are considered, each general in its formulation and representative of a class of discrimination model not necessarily restricted to vision. Despite the fact that the processes depend on markedly different algorithms, they agreed closely in their predictions of discrimination performance. Except for certain parameter values, however, their relationship was not one-to-one.

2 Discrimination and internal image characterizations

Suppose that each image is composed of m spatially distinct elements or subpatterns, and that the shape (or other attribute) of each element is parameterized by a 1-dimensional variable s (a condition that may be relaxed). Suppose that all the elements in the image but one have identical shape parameter values; let the "odd" element have parameter value s_1 and all the other elements have parameter value s_2, $s_2 \neq s_1$. The probability of achieving a correct discrimination of the odd element from the others is assumed to depend on the probabilities governing the choice of possible internal characterizations. Let $p_i(s)$ be the probability that internal characterization r_i, $1 \leqslant i \leqslant n$, is given to each element with parameter value s. We make no assumption about the form of the dependence of the $p_i(s)$ on s, or about the similarity of the p_i (contrast with Quick 1974; Nachmias 1981). As to the set of labels r_i used to mark different internal characterizations we assume only that it is finite, without metric structure.

3 A combinatorial process

Let E_i be the subset of m_i elements in the image given internal characterization r_i. The subsets E_i are disjoint and form a partition of the image, so that $\sum_{i=1}^{n} m_i = m$. The number M of possible partitions is given by the formula:

$$M = \frac{(m+n-1)!}{m!(n-1)!}\,.$$

Assume that the probability $P^{\mathrm{C}}(s_1, s_2)$ of a correct discrimination of the odd element with parameter value s_1 from the other identical elements with parameter value s_2 is equal to the probability that the smallest (not necessarily unique) subset E_{i_k} contains element s_1 multiplied by the probability that this element is chosen from among the elements with the same characterization in each of the smallest subsets, of which there are l in all. Then $P^{\mathrm{C}}(s_1, s_2)$ is given by the following:

$$P^{\mathrm{C}}(s_1, s_2) = \sum_{j=1}^{n} \frac{c_j}{lk} p_j(s_1) p_j(s_2)^{m_j - 1} \prod_{i=1,\, i \neq j}^{n} p_i(s_2)^{m_i}, \quad (1)$$

where

$$c_j = \frac{(m-1)!}{(m_j - 1)! \prod_{i \neq j} m_i!}\,,$$

$$k = \min\{m_i : 1 \leqslant i \leqslant n\}\,,$$

and m_j is subject to the condition $m_j = k$. A formula analogous to (1) holds for $P^{\mathrm{C}}(s_2, s_1)$, the probability of a correct discrimination of the odd element with parameter value s_2 from the other identical elements with parameter value s_1. In a balanced experimental design, expected discrimination performance may be then summarized by the mean of the two probabilities $P^{\mathrm{C}}(s_1, s_2)$, $P^{\mathrm{C}}(s_2, s_1)$ as a function of $s = (s_1 + s_2)/2$.

4 A generalized-distance process

Formally, each image element with parameter value s may be represented by an n-tuple $(p_1(s), \ldots, p_n(s))$. Assume that the probability $P_q^{\mathrm{D}}(s_1, s_2)$ of a correct discrimination of the odd element with parameter value s_1 from the other identical elements with parameter value s_2 is determined by the magnitude of the difference $e_q(s_1, s_2)$ between the two n-tuples $(p_1(s_1), \ldots, p_n(s_1))$, $(p_1(s_2), \ldots, p_n(s_2))$, the magnitude defined naturally by the $l^{(p)}$ norm, as follows, where for clarity the index p has been replaced by q.

$$e_q(s_1, s_2) = \left(\sum_{i=1}^{n} |p_i(s_1) - p_i(s_2)|^q \right)^{1/q}, \quad (2)$$

where q is fixed and $1 \leqslant q < \infty$. The case $q = 1$ corresponds to the city-block metric and $q = 2$ to the Euclidean metric; by definition, $q = \infty$ corresponds to the sup metric. The maximum and minimum values of $e_q(s_1, s_2)$ are $2^{1/q}$ and 0, respectively (Ferraro and Foster 1984). The probability $P_q^{\mathrm{D}}(s_1, s_2)$ of a correct discrimination may then be obtained from $e_q(s_1, s_2)$ by an affine transformation, which serves to normalize the range:

$$P_q^{\mathrm{D}}(s_1, s_2) = \frac{1}{m} + \frac{m-1}{m} \cdot \frac{e_q(s_1, s_2)}{2^{1/q}}\,. \quad (3)$$

The minimum value of $P_q^{\mathrm{D}}(s_1, s_2)$ is then $1/m$, corresponding to chance-level performance, and the maximum value is obviously 1. Before differences (2) are taken and the affine transformation (3) applied the $p_i(s)$ may be linearized, for example by the inverse Φ^{-1} of the standardized normal integral,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-u^2}{2}\right) \mathrm{d}u\,.$$

Other linearizing transformations include the logistic and the angular, but in practice the particular choice of transformation may not be important (Cox 1970). As with the combinatorial process, expected discrimination performance is summarized by the mean of the two probabilities $P_q^{\mathrm{D}}(s_1, s_2)$, $P_q^{\mathrm{D}}(s_2, s_1)$ as a function of $s = (s_1 + s_2)/2$, although here $P_q^{\mathrm{D}}(s_1, s_2) = P_q^{\mathrm{D}}(s_2, s_1)$.

5 Relationship of combinatorial and generalized-distance processes

Given (1), (2) and (3) it is in principle possible to write P^{C} as a function of P_q^{D}. In practice, however, the number of terms in (1) rapidly becomes unmanageably large, as the number of possible internal characterizations of each image element and the number of image elements increases. It is therefore necessary to resort to numerical simulation, although an analysis can be performed when the number of characterizations $n = 2$. In this case, since $\sum_{i=1}^{n} p_i(s_j) = 1, j = 1, 2$, (2) becomes

$$e_q(s_1, s_2) = 2^{1/q} |p_1(s_1) - p_1(s_2)|\,.$$

Consequently $P_q^{\mathrm{D}}(s_1, s_2)$ is independent of q:

$$P_q^{\mathrm{D}}(s_1, s_2) = \frac{1}{m} + \frac{m-1}{m} |p_1(s_1) - p_1(s_2)|\,. \quad (4)$$

We therefore omit the subscript q and write simply P^{D}, the dependence on the parameter s being understood; likewise for P^{C}. Suppose that the number of elements $m = 3$ or 4. It is straightforward to show from equation (1) that the probability P^{C} is given by

$$P^{\mathrm{C}} = \frac{1}{m} + \frac{m-1}{m} (p_1(s_1) - p_1(s_2))^2\,, \quad (5)$$

from which it follows that

$$P^{\mathrm{C}} = \frac{1}{m} + \frac{m}{m-1} \left(P^{\mathrm{D}} - \frac{1}{m} \right)^2. \quad (6)$$

Equation (6) shows that in this case P^C can be derived from P^D by means of a nonlinear monotonic transformation and vice-versa, and, in particular, that P^C is a quadratic function of P^D. The two processes for expected discrimination performance are thus equivalent in that the respective probabilities of correct discrimination can be derived from each other by a monotonic invertible transformation. In general, however, this relation does not hold. If the number of elements $m > 4$, P^C can be written as follows:

$$P^C = \frac{1}{m} + \frac{1}{4m}(m-1)(m-2)(p_1(s_1) - p_1(s_2))^2 + a(p_1(s_1)^3 + p_1(s_2)^3) + b(p_1(s_1)^2 p_1(s_2) + p_1(s_1)p_1(s_2)^2) + O(p_1^4), \quad (7)$$

where the constants a and b can be calculated from (1), and $O(p_1^4)$ consists of terms of order $\geqslant 4$ in $p_1(s_1)$ and $p_1(s_2)$. It is apparent from (7) that unless the sum of the terms of order $\geqslant 3$ is identically zero, the relationship between P^C and P^D is not one-to-one: P^C depends on the values of $p_1(s_1)$ and $p_1(s_2)$, whereas P^D depends solely on their differences. It is easy to ascertain that the sum of the terms of order $\geqslant 3$ cannot be zero, the only possible exception arising when $m = 6$, as will be shown in the numerical simulations. Note that if the number of characterizations $n \geqslant 3$ one cannot expect, in general, a one-to-one relationship between P^C and $P^D = P_q^D$, for each q, since P_q^D and $P_{q'}^D$, $q \neq q'$, are not in one-to-one correspondence.

6 Monte Carlo simulations

As indicated earlier, a general understanding of the relationship between P^C and P_q^D is feasible only with the aid of numerical simulation. In the present Monte Carlo analysis, the probabilities $p_i(s_j)$ were generated from a uniform random distribution over the interval [0, 1], subject to the constraint that $\sum_{i=1}^n p_i(s_j) = 1$. The discrimination probabilities P_q^D for $q = 1, 2, \infty$ and P^C were computed from (1), (2), and (3).

It may be noted that data derived from measurements of visual pattern discrimination and categorical identification (Foster 1983; Foster and Ferraro 1989), as well as theoretical considerations (Foster 1980; Ferraro and Foster 1984), provide some indication of the form of the dependence of the $p_i(s)$ on s; for example, if $S = [a, b]$ is the domain of the map $s \mapsto p_i(s)$, it may be assumed that there exists, for each i, a single interval $[c_i, d_i] \subset S$, $c_i \leqslant d_i$, on which $p_i(\cdot)$ achieves its maximum value and such that on $[a, c_i]$ $p_i(\cdot)$ is monotonically increasing and on $[d_i, b]$ it is monotonically decreasing. Since the aim of the study was to analyze the general characteristics of the relationship between the two modes of computing correct discriminations, the present simulations entailed no assumption of this kind.

Plots of P^C vs P_q^D are shown in Figs. 1–3. If $n = 3$ the sup and city-block metrics coincide. The results of the simulations can be summarized as follows.

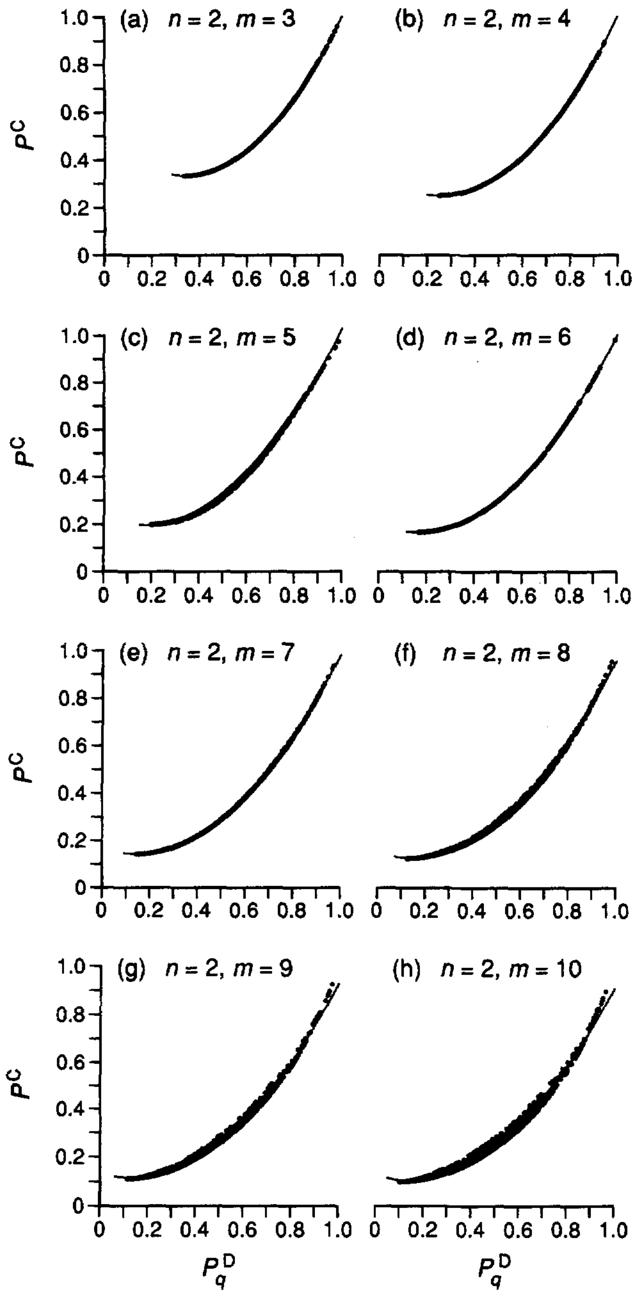

Fig. 1a–h. Comparison of expected discrimination performances predicted from the combinatorial (P^C) and generalized-distance (P_q^D) processes. The number n of possible characterizations of each image element and the number m of elements in each image are indicated. The value of the index q in the distance process is immaterial. The continuous line shows a quadratic function fitted to the data points by least squares. In **c** and **e** the spread of data points about the quadratic function is small but positive (RMSE 0.0043 and 0.0016 respectively)

(1) Number of characterizations $n = 2$

When the number of elements $m = 3,4$ the dependence of P^C on P^D is given, as expected, by (6) and the same is true for $m = 6$ (Fig. 1a, b, d). In all other cases the correspondence ceases to be one-to-one, although there is an evident quadratic trend in P^C vs P^D (confirmed by

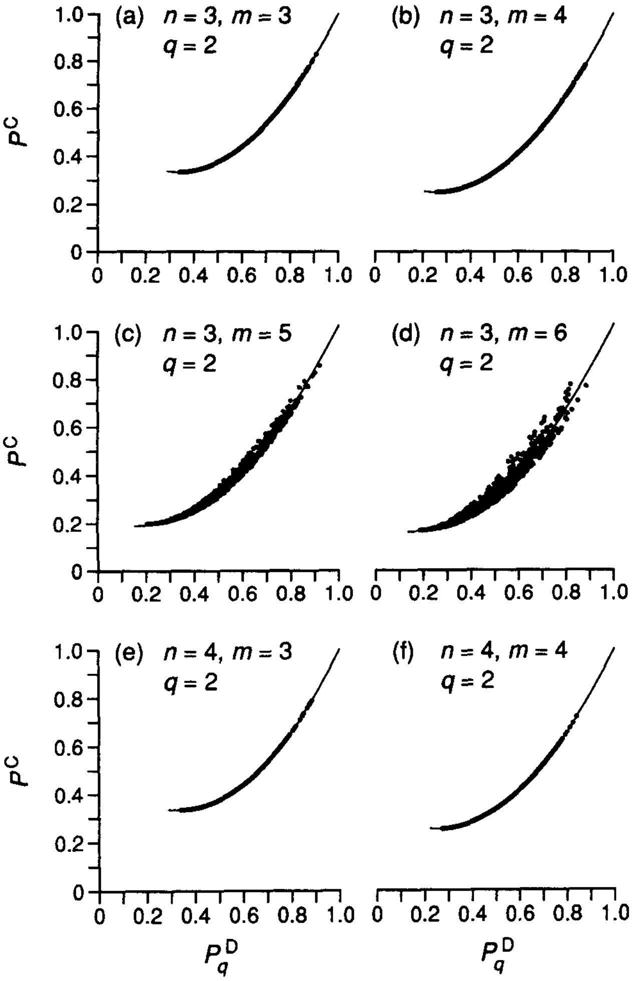

Fig. 2a–f. Comparison of expected discrimination performances predicted from the combinatorial (P^C) and generalized-distance (P^D_q) processes. The number n of possible characterizations of each image element, the number m of elements in each image, and the value of the index q in the distance process are indicated. The continuous line shows a quadratic function fitted to the data points by least squares

formal analyses). In general the spread of points about this trend increases as the number of elements increases (in Fig. 1e–h, RMSEs about the best-fitting quadratic are 0.0016, 0.0057, 0.0082, and 0.0121, respectively).

(2) Number of characterizations $n \geqslant 3$

When the number of elements $m = 3, 4$, and $q = 2$, it is seen (Fig. 2a, b, e, f) that the relationship of P^C to P^D_2 is still determined by (6), whereas for other values of m (Fig. 2c, d) the results appear to be similar to those for $n = 2$ categorizations (with the obvious exception of $m = 6$). If $q \neq 2$ then even for $m = 3, 4$ a one-to-one correspondence no longer holds for, as noted earlier, the relationship between P^D_q and $P^D_{q'}$, $q \neq q'$ is not one-to-one (Fig. 3a–h). Again, the spread of points about a quadratic trend increases as the number of elements increases (in Fig. 3a–d, RMSEs about the best-fitting quadratic are 0.0143, 0.0149, 0.0169, and 0.0214, respectively).

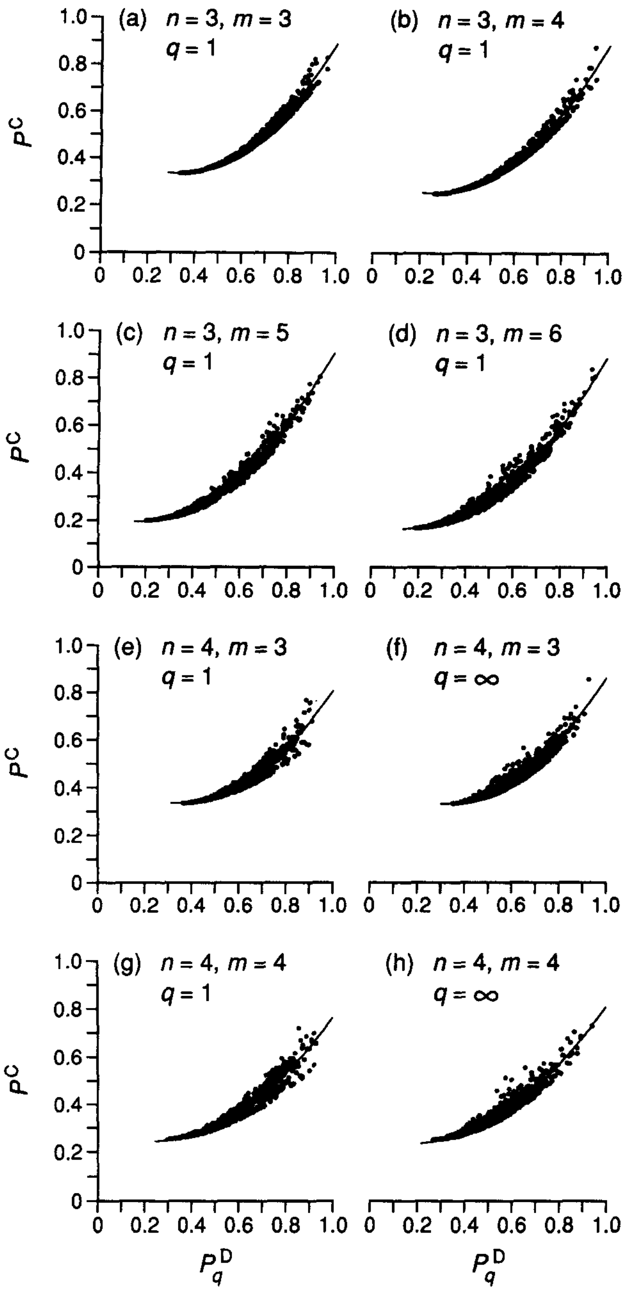

Fig. 3a–h. Comparison of expected discrimination performances predicted from the combinatorial (P^C) and generalized-distance (P^D_q) processes. The number n of possible characterizations of each image element, the number m of elements in each image, and the value of the index q in the distance process are indicated. The continuous line shows a quadratic function fitted to the data points by least squares

7 Discussion

There is a parallel between the results of the present study and those obtained by Quick (1974) who compared a probability-summation model and a vector-magnitude model of contrast detection. The construction of those two models, however, was fundamentally different from the processes considered in the present study. Quick was concerned with the probability of one or more similar mechanisms being excited in order for detection – rather than discrimination – of the stimuli to occur. The two models predicted similar results. The mechanisms were required to satisfy a homogeneity condition, and some technical difficulties associated with that assumption have been noted by Nachmias (1981).

The predictions of the combinatorial and generalized-distance processes considered here were found to be closely similar to each other, the underlying trend of the relationship being quadratic. This relationship was precisely one-to-one, however, only when the number of internal characterizations $n = 2$ and the number of elements in the image $m = 3, 4, 6$, or when $n > 2$, $m = 3, 4$, and the index of the norm $q = 2$. For all other cases tested, a one-to-one relationship was shown to be impossible. The reason for this failure was shown to be central to the structure of the processes: for the combinatorial process, the probability P^C of a correct discrimination depends on the values of the individual probabilities $p_i(s_j)$, whereas for the generalized-distance process, the probability P_q^D of a correct discrimination depends only on their differences.

In its application, the generalized-distance process is attractive in that it measures directly the difference between the probabilities governing different internal representations of the images. It is, moreover, computationally undemanding. Although the generalized distance $e_q(s_1, s_2)$ does not depend on the number of image elements a dependence of the expected value of P_q^D on m was introduced into the formula (3) to scale P_q^D to the correct range. In contrast, the combinatorial model is less immediate and as the number of elements in the image increases it becomes more computationally demanding. On the other hand, it takes into account the number of image elements and the number of categorizations in an intrinsic fashion.

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References

Beck J (1983) Textural segmentation, second-order statistics, and textural elements. Biol Cybern 48:125–130

Beck J, Ambler B (1972) Discriminability of differences in line slope and in line arrangement as a function of mask delay. Percept Psychophys 12:33–38

Cox DR (1970) The analysis of binary data. Methuen, London

Ferraro M, Foster DH (1984) Characterization of discrete and continuous modes of visual pattern discrimination. Biol Cybern 50:9–13

Ferraro M, Foster DH (1986) Discrete and continuous modes of curved-line discrimination controlled by effective stimulus duration. Spatial Vision 1:219–230

Foster DH (1980) A spatial perturbation technique for the investigation of discrete internal representations of visual patterns. Biol Cybern 38:159–169

Foster DH (1983) Visual discrimination, categorical identification, and categorical rating in brief displays of curved lines: implications for discrete encoding processes. J Exp Psychol: Hum Percept Perform 9:785–806

Foster DH, Ferraro M (1989) Visual gap and offset discrimination and its relation to categorical identification in brief line-element displays. J Exp Psychol: Hum Percept Perform 15:771–784

Nachmias J (1981) On the psychometric function for contrast detection. Vision Res 21:215–223

Quick RFJr (1974) A vector-magnitude model of contrast detection. Kybernetik 16:65–67

Watt RJ (1987) Scanning from coarse to fine spatial scales in the human visual system after the onset of a stimulus. J Opt Soc Am (Ser A) 4:2006–2021

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