Introduction to Partial Differential Equations for Engineers

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1 What are Partial Differential Equations?

Solving ordinary differential equations involves finding a function (or a set of functions) of one independent variable but partial differential equations are for *functions of two or more variables*. Examples of physical models using partial differential equations are the heat equation for the evolution of the temperature distribution in a body, the wave equation for the motion of a wavefront, the flow equation for the flow of fluids and Laplace's equation for an electrostatic potential or elastic strain field. In such cases we need to have not only the initial conditions, but also **boundary conditions** for the region in which the model applies; thus we have to solve **boundary value problems**.

As with ODEs, we call a PDE *linear homogeneous* if a linear combination of derivatives is equal to zero—and then a linear combination of solutions is another solution. Here are typical examples of the commonest types of linear homogeneous PDEs, for the simplest case—just two independent variables (x, t or x, y—it is easy to see how they would generalize to more variables x, y, z, t)

$$\text{Flow Equation} \quad c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0, \quad \text{given initial or boundary values for } u \tag{1.1}$$

$$\text{Heat Equation} \quad c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \quad \text{given initial or boundary values for } u \tag{1.2}$$

$$\text{Wave Equation} \quad c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad \text{given initial or boundary values for } u \tag{1.3}$$

$$\text{Laplace's Equation} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{given boundary values for } u. \tag{1.4}$$

An example of a *linear but non homogeneous PDE*—Poisson's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y). \tag{1.5}$$

An example of a *nonlinear PDE*—a nonlinear heat equation:

$$c^2\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial t} = 0. \tag{1.6}$$

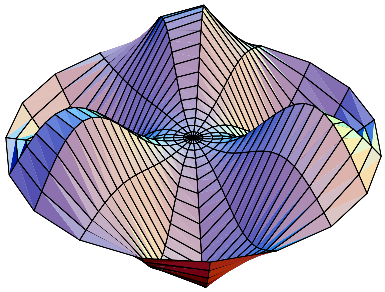

Figure 1: *A snapshot of a particular solution of the wave equation, for a vibrating drum.*

1.1 Notation

Sometimes we denote partial differentiation by subscripts, as in u_x for $\frac{\partial u}{\partial x}$ or u_{tt} for $\frac{\partial^2 u}{\partial t^2}$. So, for example, the heat equation can be written

$$c^2 u_{xx} - u_t = 0.$$

If we have more variables, or need to write programs for solving a PDE, then often we use numbered variables, such as, $x_1, x_2, \ldots, x_n$ and then we denote a partial derivative by the subscript number. For example, u_{22} is the second partial derivative of u with respect to x_2.

1.2 Exercises

Note that the Exercises in these lecture notes are intended to be done, since the results they give are often used in the theory we develop.

1. Sketch the graph of the function $u(x,y) = x^2+y^2$, and find a partial differential equation of which it is a solution. Can you find other solutions?

2. Write out in subscript form for partial derivatives the other PDEs we have mentioned.

3. Look up other examples of PDEs, from engineering books—or final year exams!

4. Think about how Laplace's equation changes its appearance when you change from Cartesian coordinates, (x, y), to polar coordinates, (r, θ). Certainly, polar coordinates would be a good choice if we had to solve an equation in a 2-dimensional circular region. A snapshot in time of one such solution, of the 2-dimensional wave equation

$$c^2(u_{xx} + u_{yy}) - u_{tt} = 0$$

for the displacement of a vibrating drum, is shown in Figure 1. The boundary condition is for zero displacement round the circular rim of the drumskin and the initial condition would have specified the state of the membrane at $t = 0$, a deformation of some sort such as caused by striking it with drumsticks. The graphic shown is at one instant of time; the solution will of course change with time so to show it properly it needs a movie.

2 Classification of PDEs

The general form of a linear second order PDE, in the two variables x_1, x_2, is given by

$$Au_{11} + 2Bu_{12} + Cu_{22} + Du_1 + Eu_2 + Fu = f \tag{2.7}$$

where A, B, C, D, E, F and f all depend only on x_1, x_2. There is a classification scheme depending on the values of A, B, C; we say that PDE (2.7) is:

$$\begin{aligned} \text{Hyperbolic if} \quad B^2 - AC &> 0 & (2.8)\\ \text{Parabolic if} \quad B^2 - AC &= 0 & (2.9)\\ \text{Elliptic if} \quad B^2 - AC &< 0. & (2.10) \end{aligned}$$

If $f = 0$ then the PDE (2.7) is homogeneous.

If PDE (2.7) is to represent a nonlinear PDE, then some of the functions A, B, C, D, E, F depend on u, as well as on x_1, x_2. We do not consider nonlinear PDEs in detail in this course.

3 Solutions to PDEs

Solutions to practical models using these equations are usually very difficult to obtain analytically and computers are used to obtain numerical approximate solutions by standard iterative procedures. You can see examples of some numerical programs in BASIC and C for solving simple PDEs via the webpage:
http://www.ma.umist.ac.uk/kd/comp/comp.html.

Nonlinear PDEs are particularly difficult to solve, but they are important in many practical problems—eg when the resistance to a flow is a function of the flow strength or when the elastic modulus of a membrane is a function of the strain.

In some simple cases, solutions can be found in terms of sums and products of elementary functions. Find all the first and second order partial derivatives of the following functions $u(x,t)$ or $u(x,y)$ and try to match the functions to the above linear homogeneous partial differential equations with suitable initial and boundary conditions:

$$3e^{-at}\sin\sqrt{a/b}x, \quad e^{-ax}\cos ay, \quad \log(x^2+y^2), \quad \sin(ax)\sinh(ay),$$
$$e^{-x}\cos(t-x), \quad e^{-a^2t}\sin(ax), \quad 2\cos(ax)\cos(act)$$

3.1 Exercises

1. If u is a solution of any of the above-mentioned linear homogeneous PDEs, then so also is ku a solution for any constant k.

2. If u is a solution of a linear non homogeneous PDE and v is a solution of the homogeneous version of the PDE, then, for all constants k, another solution of the non homogeneous PDE is given by $u + k\,v$.

3. Show that $u(x,t) = \sin(x - ct)$ is a solution of the wave equation; for what initial and boundary conditions?. Can you find any more solutions; for what initial and boundary conditions?

4. Consider the PDE

$$u_{xy} = 0. \tag{3.11}$$

Show that this must mean that u_x is a function of x only. Let this function be $f(x)$. Then show that it follows that every solution of (3.11) must be of the form

$$u(x,y) = F(x) + g(y), \quad \text{where } F(x) = \int f(x)\,dx$$

4 Analytical Methods of Solution

As we see, given a function of two or more variables, it is quite easy to find many PDEs of which it is a solution and to invent suitable boundary conditions. The reverse process, finding solutions to a given PDE with prescribed boundary conditions is much harder and usually impossible analytically. Nevertheless, there are three things that help engineers in this regard:

- Many problems in engineering and physics involve one of a relatively small number of types of PDE involving derivatives up to two only.

- There are a number of standard analytic methods that yield solutions to the important linear PDEs arising in models of real processes.

- Computer software, such as Mathematica, Maple and Matlab can perform analytic manipulations which would be prohibitively tedious by hand, so the range of analytic methods is extended enormously for engineers. Often, part of the work in an engineering problem is a numerical procedure, for which there are many standard packages. For more information, see:
http://www.ma.umist.ac.uk/kd/comp/comp.html.

In this course we shall consider some of the simplest analytic methods.

5 D'Alembert's solution of $c^2u_{xx} = u_{tt}$

The wave equation (1.3) has the D'Alembert solution $\phi(x-ct)+\psi(x+ct)$, for some choices of the functions ϕ and ψ to suit the given conditions. Such solutions are waves travelling at constant speed c, in both directions along the x-axis. To see how this is arrived at we need to do some applications of the chain rule for partial derivatives.

5.1 Exercises

1. Let $v = x + ct$ and $w = x - ct$. Use the chain rule to show that

$$\begin{aligned} u_x &= u_v + u_w && (5.12)\\ u_t &= c(u_v - u_w) && (5.13)\\ u_{xx} &= u_{vv} + 2u_{vw} + u_{ww} && (5.14)\\ u_{tt} &= c^2(u_{vv} - 2u_{vw} + u_{ww}) && (5.15) \end{aligned}$$

2. Substitute these expressions into the wave equation, $c^2u_{xx} = u_{tt}$.

Now we find that $c^2u_{xx} = u_{tt}$ is equivalent to

$$c^2(u_{vv} + 2u_{vw} + u_{ww}) = c^2(u_{vv} - 2u_{vw} + u_{ww})$$

and this simplifies to

$$4c^2u_{vw} = 0, \quad \text{or,} \quad u_{vw} = 0,$$

since $c > 0$. But, from a previous exercise, it follows that u must be expressible as a sum of a function of v and a function of w, such as

$$u(v, w) = \phi(w) + \psi(v).$$

Hence we have our result when we substitute back for the variables (x, t),

$$u(x, t) = \phi(x - ct) + \psi(x + ct) \quad \text{D'Alembert's solution.} \tag{5.16}$$

We need some initial and boundary conditions to find the form of the functions in (5.16). Suppose we have been given

$$u(x, 0) = f(x) \quad \text{and} \quad u_t(x, 0) = 0. \tag{5.17}$$

We substitute (5.17) into (5.16) and deduce as follows:

$$\begin{aligned} u(x,0) &= \phi(x) + \psi(x) = f(x) \quad \text{since } t = 0 && (5.18)\\ u_t(x,0) &= -c\phi'(x) + c\psi'(x) = 0 && (5.19)\\ \phi'(x) = \psi'(x) \quad &\text{so} \quad \phi(x) = \psi(x) + k \;\; \text{for some constant } k && (5.20)\\ f(x) &= \phi(x) + (\phi(x) + k) \;\; \text{by } (5.18) && (5.21)\\ \phi(x) = \frac{1}{2}(f(x) - k) \quad &\text{and} \quad \psi(x) = \frac{1}{2}(f(x) + k) && (5.22)\\ u(x,t) &= \frac{1}{2}\left((f(x - ct) + f(x + ct)\right). && (5.23) \end{aligned}$$

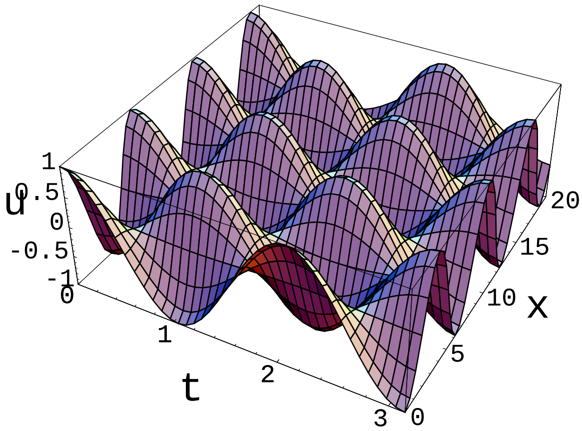

Figure 2: *A particular D'Alembert solution of the 1-D wave equation* $c^2 u_{xx} = u_{tt}$; *this solution is given by* $u(x,t) = \frac{1}{2}\left((\cos(x-3t) + cos(x+3t)\right)$. *A slice parallel to the x-axis at* $t = t_0$ *gives the shape of the wave along the x-direction, at that chosen instant of time* t_0. *A slice parallel to the t-axis at* $x = x_0$ *gives the variation in time of the shape at that chosen point* x_0.

Such solutions are waves travelling at constant speed c, in both directions along the x-axis. We can think of $f(x)$ as being the shape of the wave, or the 'wave profile' that moves along the x-axis.

For example, if we take

$$u(x,t) = \frac{1}{2}\left((\cos(x-3t) + cos(x+3t)\right)$$

so f is a cosine function and the speed is $c = 3$, then the appearance of the solution is shown in Figure (2). Here, the wavelength is $\lambda = 2\pi$ and the period is $T = \lambda/c = 2\pi/3$.

5.2 Exercises

1. Is $u(x,t) = A\cos m(x-kt)$ a solution of the 1-D wave equation for constant A, k? What about $u(x,t) = A\cos m(x+kt)$?

2. Investigate D'Alembert solutions with cosine wave profiles of higher frequencies.

3. The method generalises. For a parabolic equation of Euler type

$$au_{xx} + 2hu_{xy} + bu_{yy} = 0, \tag{5.24}$$

there is only one solution of the associated quadratic

$$a + 2h\lambda + b\lambda^2 = 0,$$

so the general solution of (5.24) is of form

$$u(x,y) = f(x+\lambda y) + (rx+sy)g(x+\lambda y).$$

4. For a hyperbolic equation of Euler type

$$au_{xx} + 2hu_{xt} + bu_{tt} = 0, , \tag{5.25}$$

there are two solutions λ_1, λ_2 of the associated quadratic

$$a + 2h\lambda + b\lambda^2 = 0,$$

so the general solution of (5.25) is of form

$$u(x,t) = f(x + \lambda_1 t) + g(x + \lambda_2 t).$$

5. Find the general solution of

$$u_{xy} - u_{yy} = 0$$

by using the change of variables: $v = x,\ w = x + y$.

6. Find the general solution of

$$x\, u_{xy} - y\, u_{yy} = u_y$$

by using the change of variables: $v = x,\ w = xy$.

7. Transform the equation

$$u_{vw} = 0$$

to other PDEs with variables x, y by choosing some different simple expressions for x and y as functions of v, w, eg $x = v,\ y = w - v$.

6 Separation of variables

A general method for attempting to solve PDEs is to suppose that the solution function u is a product of functions, each one depending on one only of the independent variables. This converts the PDE into two (or more) ODEs which may be soluble. We shall find the corresponding ODEs for solution of the wave equation (1.3) by this method of separation of variables.

In $c^2 u_{xx} = u_{tt}$ we substitute $u(x,t) = X(x)T(t)$. To simplify the notation we shall denote differentiation with respect to x by $'$ and differentiation with respect to t by a dot; thus

$$\dot{T} = \frac{dT}{dt} \quad \text{and} \quad X' = \frac{dX}{dx}.$$

We obtain

$$u_{tt} = X\ddot{T} \quad \text{and} \quad u_{xx} = X''T, \quad \text{so} \quad X\ddot{T} = c^2 X''T.$$

If $XT \neq 0$, we obtain

$$\frac{\ddot{T}}{c^s T} = \frac{X''}{X} = k, \quad \text{a constant.} \tag{6.26}$$

If X or T *is* zero, then we cannot divide here but we can still use $X\ddot{T} = c^2X''T$. If $X = 0$ then either also $T = 0$ or $X'' = 0$ and $X = ax + b$ for some constants a, b. Similarly, if $T = 0$ then either also $X = 0$ or $T = mt + n$ for some constants m, n. In either case, we obtain that $u(x, t)$ is linear in x and t so its second partial derivatives both vanish identically.

We can proceed once we choose some boundary conditions. Let the boundary conditions be given at $x = 0$ and $x = l$ for all t by the following:

$$u(0, t) = X(0)T(t) = 0 \quad \text{and} \quad u(l, t) = X(l)T(t) = 0 \quad \text{for all } t$$

If T is the zero function then so is u. If T is not the zero function then the boundary conditions tell us that

$$X(0) = 0 \quad \text{and} \quad X(l) = 0.$$

We need to consider the three possibilities for the constant k, zero, positive or negative.

If $k = 0$, then $X'' = 0$ and $X = ax+b$ but the boundary conditions tell us that then $a = b = 0$, so X is the zero function.

Next, suppose that $k > 0$, let $k = \mu^2$, say. Now our ODE to solve is

$$X'' - \mu^2 X = 0$$

which has the general solution

$$X(x) = Ae^{\mu x} + Be^{-\mu x}.$$

Substituting the boundary conditions, $X(0) = 0$ and $X(l) = 0$, we obtain that $A + B = 0$ from putting $x = 0$ and so when $x = l$ we find

$$X(l) = A(e^{\mu l} - e^{-\mu l}) = 0.$$

But $e^{\mu l} - e^{-\mu l} = 2\sinh(\mu l) = 0$ implies $\mu l = 0$ so $\mu = 0$ and then $k = 0$, which is a contradiction.

Hence, $k < 0$, so suppose $k = -p^2$. Now our ODE is

$$X'' + p^2 X = 0$$

which has general solution

$$X(x) = A\cos px + B\sin px.$$

From $X(0) = 0$, we have $A = 0$, so $X(x) = B\sin px$. But also $X(l) = B\sin pl = 0$ and so $\sin pl = 0$ and hence $pl = n\pi$ for some integer n. That is, $p = n\pi/l$ and so we have infinitely many solutions, one for each integer value of n; we label them by subscripts:

$$X_n = B_n \sin\frac{n\pi x}{l}, \quad \text{for } n = 0, \pm1, \pm2, \pm3, \ldots. \tag{6.27}$$

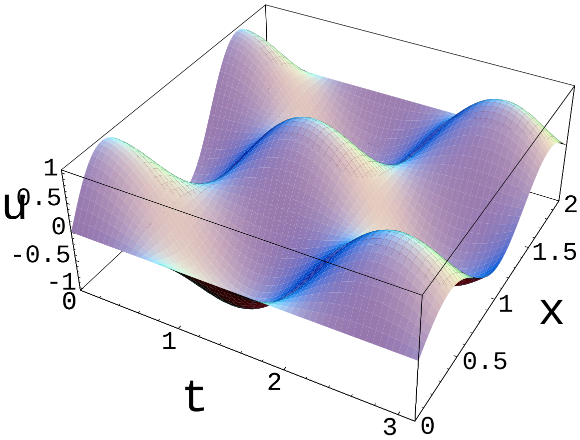

Figure 3: *One of the solutions (6.29) of the 1-D wave equation* $c^2 u_{xx} = u_{tt}$ *with* $c = \frac{1}{2},\ l = 2,\ n = 3$ *given by* $u(x,t) = \sin\frac{3\pi x}{2}\cos\frac{3\pi t}{4}$. *The boundary conditions were:* $u = 0$ *at* $x = 0$ *and* $x = 2$, *for all* t, *as you can see in the Figure. A slice parallel to the x-axis at* $t = t_0$ *gives the shape of the wave along the x-direction, at that chosen instant of time* t_0. *A slice parallel to the t-axis at* $x = x_0$ *gives the variation in time of the shape at that chosen point* x_0.

6.1 Exercises

1. Sketch the solution for the case $n = 1$, $B_1 = 1$ and $l = 2$, in (6.27).

2. Repeat the steps with the ODE for T starting from

$$\ddot{T} - c^2 kT = \ddot{T} + c^2 p^2 T = \ddot{T} + c^2(\frac{n\pi}{l})^2 T = 0.$$

3. Deduce that there are infinitely solutions given by

$$T_n = D_n \cos\frac{cn\pi t}{l} + E_n \sin\frac{cn\pi t}{l}, \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots. \tag{6.28}$$

Finally, putting the above results together, we have infinitely many solutions for u, labelled by the integer n, as follows:

$$u_n(x,t) = \left(D_n \cos\frac{cn\pi t}{l} + E_n \sin\frac{cn\pi t}{l}\right) B_n \sin\frac{n\pi x}{l}, \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots. \tag{6.29}$$

One particular example is shown in Figure 3 with $c = \frac{1}{2},\ l = 2,\ n = 3,\ B_n = D_n = 1,\ E_n = 0$.

Remark Note that the ODEs arising from the wave equation reject all except the periodic solutions. The wavelengths of the wave profiles along the x-axis are controlled by the original boundary conditions. The period of the oscillations in time are controlled by the length l and the speed c.

6.2 Exercises

1. Consider the heat equation:

$$
\begin{aligned}
c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} &= 0. && (6.30)\\
\text{Substitute } u(x,t) &= F(x)G(t) && (6.31)\\
\text{to obtain } \frac{1}{c^2G}\frac{dG}{dt} &= \frac{1}{F}\frac{d^2F}{dx^2} \quad \text{if } (F(x)G(t) \neq 0). && (6.32)
\end{aligned}
$$

2. In the ODE (6.32), each side depends on a different single variable only, hence both sides must be equal to the same constant. That yields two ODEs to solve subject to the given conditions for the problem. What are the two ODEs to solve?

3. What happens if F or G is the zero function?

7 Method of characteristics

We shall consider a flow problem, in the form

$$a(x,t)u_x + b(x,t)u_t = 0. \qquad (7.33)$$

Using the *method of characteristics*, we shall see that u is constant along a certain curve in (x,t)-space given by $\frac{dx}{dt} = \frac{a(x,t)}{b(x,t)}$ and then from initial conditions we can obtain u.

Suppose that we have a solution u of (7.33) in a region R of (x,t)-space. Now we consider a general curve in R given parametrically by

$$x = x(s),\ t = t(s), \quad \text{for parameter } s \text{ with } s_0 \leq s \leq s_1. \qquad (7.34)$$

Note that often it is greatly simplifying if we choose $s = t$ as parameter. On this curve (7.34), the values of u are given by

$$u(s) = u(x(s), t(s)), \quad \text{for } s_0 \leq s \leq s_1. \qquad (7.35)$$

Differentiating through this with respect to s, using the chain rule, gives

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds}, \quad \text{for } s_0 \leq s \leq s_1. \qquad (7.36)$$

Next, we choose the curve (7.34) in such a way that for some function λ, depending on (x,t), we have along the curve the following dependence on the functions a and b in the PDE (7.33):

$$\frac{dx}{ds} = \lambda a \quad \text{and} \quad \frac{dt}{ds} = \lambda b, \quad \text{for } s_0 \leq s \leq s_1. \qquad (7.37)$$

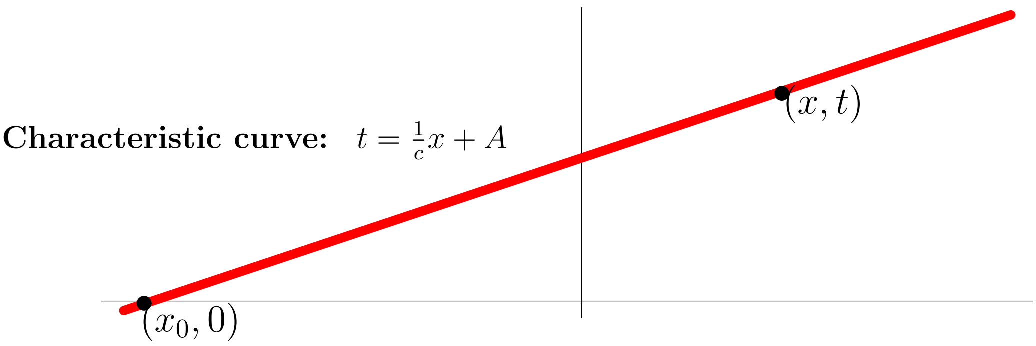

Figure 4: *Characteristic curve for PDEs of form* $c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} =$ *function of* t.

Now, substituting in (7.36) we obtain

$$\frac{du}{ds} = \lambda(a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial t}) = 0, \quad \text{for } s_0 \leq s \leq s_1. \tag{7.38}$$

So, u is constant along such a curve and to determine it we need to find the function λ. We do this by rearranging (7.37) to yield a differential relationship between x and t along the curve

$$\frac{dx}{a(x,t)} = \frac{dt}{b(x,t)} = \lambda(x,t)\, ds. \tag{7.39}$$

These are called *characteristic curves* of the PDE (7.33).

7.1 Exercises

1. Let $b = 1$, and $a(x,t) = c$, a positive constant, so we have the homogeneous linear PDE

$$c\, u_x + u_t = 0 \tag{7.40}$$

   Show that the characteristics can be obtained from (7.39) by $\frac{dt}{dx} = \frac{1}{c}$ so here these curves are just lines given by $t = \frac{1}{c}x + A$, for some constant A.

2. Sketch the characteristics as lines in (x,t) space for several values of A, cf. Figure 4.

To obtain an expression for $u(x,t)$, find the characteristic that passes through (x,t). Follow this characteristic back to the initial point $(x_0, 0)$. But, $t = \frac{1}{c}x + A$, so at $t = 0$ we have $0 = \frac{1}{c}x_0 + A$, and therefore $\frac{1}{c}x_0 = -A$. On the other hand, we know that on this characteristic $A = t - \frac{1}{c}x$, and so $\frac{1}{c}x_0 = \frac{1}{c}x - t$, which rearranges to $x_0 = x - ct$.

Suppose that we are given the initial condition

$$u(x,0) = f(x) \tag{7.41}$$

for some known function f. Now we know from the initial condition (7.41) that

$$u(x_0, 0) = f(x_0). \tag{7.42}$$

Finally, by arrangement, u is constant on the characteristic joining (x, t) to $(x_0, 0)$ so we have that

$$u(x, t) = f(x_0) = f(x - ct) \tag{7.43}$$

which we recognise as a wave travelling at constant speed c in the positive x-direction, with wave profile $f(x)$.

Inhomogeneous flow equation
Consider next the case that the flow equation is not homogeneous. We illustrate in this Exercise with a linear inhomogeneous example.

7.2 Exercises

1. Consider the inhomogeneous flow equation

$$c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = kt. \tag{7.44}$$

   Show that along a characteristic we have

$$\frac{dx}{c} = \frac{dt}{1} = \frac{du}{kt}.$$

2. Deduce that along the characteristic

$$t = \frac{1}{c}x + A, \text{ for some constant } A, \text{ and so } u(x, t) = \frac{1}{2}kt^2 + B, \text{ by integration of } du = kt\,dt.$$

3. Follow the characteristic back from (x, t) to $(x_0, 0)$ so $x_0 = -cA$, and hence we deduce $x_0 = x - ct$, cf. Figure 4.

4. Use the initial condition $u(x, 0) = f(x)$ to show that $u(x_0, 0) = B$ so $B = f(x_0) = f(x - ct)$ and finally obtain the solution to (7.44) as

$$u(x, t) = \frac{1}{2}kt^2 + f(x - ct).$$

So, the characteristics are given as before, cf. Figure 4, but, unlike for the homogeneous case (7.40), now u is not constant along the characteristic, it satisfies an ODE because of the nonzero term on the right of (7.44). There is a particular solution $u_{PI}(x, t) = \frac{1}{2}kt^2$, and so a general solution of (7.44) is of the form $u(x, t) = \frac{1}{2}kt^2 + f(x - ct)$. As before, $f(x - ct)$ represents a wave travelling along the x-axis at constant speed c with no change in its shape.

Inhomogeneous nonlinear flow equation
Here we have the most general case, a nonzero function on the right hand side, and all terms may depend on u as well as x, t. This is the equation:

$$a(x, t, u)u_x + b(x, t, u)u_t = c(x, t, u). \tag{7.45}$$

The algebraic manipulation is similar, but now we obtain the following differential equation along the characteristics:

$$\frac{dx}{a(x,t,u)} = \frac{dt}{b(x,t,u)} = \frac{du}{c(x,t,u)}. \tag{7.46}$$

The presence of u means that we can no longer solve for the characteristic in terms of x, t and then solve for u along it, as we did in the linear case. The two differential equations contained in (7.46) have to be solved simultaneously for the characteristic and for u along it; this can often be done only by numerical methods.

Sometimes, we can find an implicit solution, as in the following Exercise.

7.3 Exercises

Consider this inhomogeneous nonlinear flow equation

$$(1+t)uu_x + u_t = u, \quad u(x,0) = f(x). \tag{7.47}$$

1. Along a characteristic,
$$\frac{dx}{(1+t)u} = \frac{dt}{1} = \frac{du}{u},$$
show that $u = Ae^t$ and use this to obtain
$$\frac{dx}{dt} = (1+t)u = A(1+t)e^t.$$
2. Deduce that $x = Ate^t + B = ut + B$.
3. Use the initial condition to deduce that $u(x_0, 0) = f(x_0) = A$ and $x_0 = B = x - ut$.
4. Hence obtain the implicit form of a solution to (7.47)
$$u(x,t) = e^t f(x - ut).$$

We can see that in general it would be difficult to rearrange this implicit solution into an explicit solution for u in terms only of x, t. It might be possible if f is a simple function.

8 Appendix

8.1 Scientific Wordprocessing with LaTeX

This pdf document with its hyperlinks was created using LaTeX which is the standard (free) mathematical wordprocessing package; more information can be found via the webpage [1].

References

[1] On-line mathematical materials:

Mathematicians:
http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/

Alfred Gray's Mathematica NoteBooks on differential geometry:
http://library.wolfram.com/infocenter/Books/3759

Elementary Notes on:
Curves http://www.maths.manchester.ac.uk/ kd/curves/curves.pdf
Surfaces http://www.maths.manchester.ac.uk/ kd/curves/surfaces.pdf
Knots http://www.maths.manchester.ac.uk/ kd/curves/knots.pdf

LaTeX Tutorial:
http://www.maths.manchester.ac.uk/ kd/latextut/pdfbyex.htm

[2] S. Wolfram. **The Mathematica Book** Cambridge University Press, Cambridge 1996.