# Introducing Knots

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These notes supplement the lectures and provide practise exercises. We begin with some material you will have met before, perhaps in other forms, to set some terminology and notation. Further details on unfamiliar topics may be found in, for example Cohn [3] for algebra, Dodson and Poston [5] for linear algebra, topology and differential geometry, Gray [6] for curves, surfaces and calculations using the computer algebra package *Mathematica*, and Wolfram [11] for *Mathematica* itself. Several on-line hypertext documents are available to support this course [1].

## Introduction

This document briefly summarizes definitions and hints at proofs of principal results for a first course on knots, beginning with an informal introduction to homotopy and the fundamental group. It is intended as an *aide memoire*—a companion to lectures, tutorials and computer lab classes, with exercises and proofs to be completed by the student. Exercises include the statements to be verified—mathematics needs to be done, not just read!

The prereqisites are: elementary knowledge of Euclidean geometry and the definition of $\mathbb{R}^n$, familiarity with vector and scalar product, norms and basic linear algebra
(remember $dim\ dom\ =\ dim\ ker\ +\ dim\ im$?)
some basic topological concepts—compactness, covering space—and elementary group theory—free groups and quotients, presentation by relations, commutator subgroup.

Where possible, we encourage use of computer algebra software to experiment with the mathematics, to perform tedious analytic calculations and to plot graphs of functions that arise in the studies. For this purpose, we shall make use of Gray's book [6]—which contains all of the theory we need for curves and surfaces—and we use the computational packages he provides free in the form of Mathematica NoteBooks via the webpage: http://library.wolfram.com/infocenter/Books/3759

For general information about the Mathematica software, see Wolfram's book [11] and the website Mathematica. For further study of more general differential geometry and its applications to relativity and spacetime geometry, see Dodson and Poston [5]. For an introduction to algebraic topology see Armstrong [2] and for more advanced topics and their applications in analysis, geometry and physics, see Dodson and Parker [4]. The abovementioned books contain substantial bibliography lists for further reference.

The document you are reading was created with LaTeX and a LaTeX tutorial is available at [1].



## 1 Sets and maps

We recall some basic definitions. **Definitions**
Let $X$ and $Y$ be non-empty sets. A **relation** from $X$ to $Y$ is a subset $\rho \subseteq X \times Y$, which means that it can be represented equivalently by its **graph** in the $X - Y$ space. We write $x\rho\, y$ if $(x, y) \in \rho$ and define for $\rho$ its **domain** dom $\rho = \{x \in X \mid \exists\, y \in Y \text{ with } (x, y) \in \rho\}$ and its **image** im $\rho = \{y \in Y \mid \exists\, x \in \text{dom}\, \rho \text{ with } (x, y) \in \rho\}$. When dom $\rho = X$ we say that $\rho$ is an

**entire** relation; we shall use *only* entire relations so we shall not need this qualification. A relation $\rho \subseteq X \times X$ may have any or none of the following properties:

| | |
|---|---|
| **symmetry** | $x\rho\, y$ if and only if $y\rho\, x$ |
| **reflexivity** | for all $x \in X$, $x\rho\, x$ |
| **transitivity** | for all $x, y, z \in X$, $x\rho\, y$ and $y\rho\, z$ implies $x\rho\, z$ |
| **equivalence** | symmetry, reflexivity, and transitivity |
| **antisymmetry** | $x\rho\, y$ and $y\rho\, x$ implies $x = y$ |
| **partial order** | antisymmetry, reflexivity, and transitivity. |
| **total order** | for all $x, y \in X$, either $x\rho\, y$ or $y\rho\, x$ |

A **function** or **map** from a set $X$ to a set $Y$ is a set of ordered pairs from $X$ and $Y$ (pairs like $(x, y)$ are the coordinates in the graph of the function) satisfying the **uniqueness of image** property:

for all $x \in X$, there exists a **unique** $y \in Y$ that is related to the given $x$

Then we usually write $y = f(x)$ or $y = fx$, and make the sets involved clear by

$$f : X \to Y : x \mapsto f(x).$$

Note that a map is equivalent to its graph, as a set of ordered pairs of coordinates in $X - Y$ space; the graph of a map must not pass twice through any point in its domain—unlike a general relation.
A map $f : X \to Y$ may have any or none of the following properties:

| | |
|---|---|
| **injectivity (1 to 1)** | $f(x) = f(y)$ implies $x = y$ |
| **surjectivity (onto)** | im $f = Y$; denoted $f : X \twoheadrightarrow Y$ |
| **bijectivity (both)** | injectivity and surjectivity |

The **inclusion map** of a subset $A \subseteq X$ is the map

$$i : A \hookrightarrow X : a \mapsto a.$$

The **restriction** of a map $f : X \to Y$ to a subset of its domain $A \subseteq X$ is the composite map $f|_A = fi$.
The **Axiom of Choice** is required for a number of constructions in topology and a convenient form is this:

**Every surjection has a right inverse.**

That is, if $f : X \to Y$ is surjective, then we can always find a map $s : Y \to X$ such that $f \circ s = 1_Y$. Then $s$ is called a **section** of $f$. Equivalently, given *any* collection (not necessarily countable) of sets, it is possible to choose one element from each.
Given a map

$$f : X \to Y : x \mapsto f(x)$$

we get free two maps on subsets, one going each way:

$$f_{\to} : \mathrm{Sub}X \to \mathrm{Sub}Y : A \mapsto \{f(x)|\ x \in A\}$$

$$f^{\leftarrow} : \mathrm{Sub}Y \to \mathrm{Sub}X : B \mapsto \{x \in X|\ f(x) \in B\}$$

Normally, we just use $f$ to denote what is really $f_{\to}$, but it is important to be very careful in writing $f^{-1}$ instead of $f^{\leftarrow}$, because whereas $f^{\leftarrow}$ always exists, $f^{-1}$ may not.

# 2 Topology

A topological space is a set with the least structure necessary to define the concepts of nearness and continuity; you have met examples in real and complex analysis and perhaps also as a metric space (a set with the least structure necessary to support the concept of distance).

General topology is concerned with the study of topological spaces and maps among them while algebraic topology is concerned with the casting of topological problems into easier algebraic form using functors.

**Definitions**

A **metric space** $(X, d)$ is a nonempty set $X$ and a map $d : X \times X \to \mathbb{R}$, called a **metric** or **Hausdorff distance function**, satisfying the natural requirements of a distance function:

1. $d(x, y) = d(y, x) \quad \forall\, x, y \in X$ (Symmetry)
2. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$ (Positive definiteness)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)

The standard metric on a normed vector space is simply the norm of the difference between two points. In geometry, **Euclidean n-space**, $\mathbb{E}^n$, is the metric space of points in $\mathbb{R}^n$ with distance function

$$d(p, q) = ||q - p||.$$

A **topological space** $(X, \mathcal{T})$ is a set $X$ together with a collection $\mathcal{T}$ of subsets, so $\mathcal{T} \subseteq P(X)$, satisfying:

1. $\emptyset, X \in \mathcal{T}$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

We call $\mathcal{T}$ **the topology** of the space $(X, \mathcal{T})$ or **a topology** on the set $X$. Elements of $\mathcal{T}$ are called **open sets** in the topological space $(X, \mathcal{T})$ or they are called **$\mathcal{T}$-open sets** of $X$. A **base** for a topology $\mathcal{T}$ on $X$ is a collection $\mathcal{B} \subseteq \mathcal{T}$ of open sets of $X$ such that every member of $\mathcal{T}$ is expressible as a union of members of $\mathcal{B}$.

Every metric space $(X, d)$ has a topology $\mathcal{T}_d$ determined by $d$. A subset $A$ of $X$ is $d$-open in $(X, d)$ if it contains an open ball around each of its points, and we define $\mathcal{T}_d$ to be the set of $d$-open subsets. So a base for a metric topology is the collection of all open balls.

Let $(X, \mathcal{T})$ be a topological space. A set $A$ is **closed** in $(X, \mathcal{T})$ if $X \setminus A$ is open in $(X, \mathcal{T})$ (that is, closed if it is the complement of an open set). Sometimes $X \setminus A$ is denoted $X - A$. A point $x \in X$ is a **limit point** of $A \subseteq X$ in $(X, \mathcal{T})$ if every neighborhood of $x$ meets $A \setminus \{x\}$ non-emptily. A limit point of $A$ need not be in $A$, but it turns out that $A$ is closed in $(X, \mathcal{T})$ if and only if $A$ contains all of its limit points.

The **closure** $\bar{A}$ of a set $A$ in a topological space is the union of $A$ with all of its limit points; that is, the smallest closed set containing $A$. The **interior**, int $A$ (also denoted $(A)^\circ$, when convenient) of $A$ is the largest open set contained in $A$. $A$ is **dense in** $(X, \mathcal{T})$ if $\bar{A} = X$. The **boundary** or **frontier** of a set $A$ is $\partial A = \bar{A} \cap \overline{X \setminus A}$.

A map between topological spaces $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is called **continuous** if

$$\forall\, B \in \mathcal{T}',\ f^{\leftarrow} B \in \mathcal{T}.$$

A continuous map $f$ is called
**open** if $U \in \mathcal{T} \ \Rightarrow\ fU \in \mathcal{T}'$
and is called
**closed** if $U \in \mathcal{T} \ \Rightarrow\ f(X \setminus U)$ is closed in $Y$.
Open, closed, and continuous are independent properties.
A map $f : X \to Y$ is called a **homeomorphism** if

$$f \text{ is continuous}, \quad f^{-1} \text{ exists, and} \quad f^{-1} \text{ is continuous;}$$

that is, if $f$ is **bijective and bicontinuous**; then we say that $X$ and $Y$ are **homeomorphic** and write $X \cong Y$.

For proofs in topology we usually juggle the properties of open and closed sets. We can get open sets in relatively few ways:

1. directly from $\mathcal{T}$;
2. as complements of closed sets;
3. as inverse images of open sets by continuous maps.

# 3 Not the Knot

The concept of a knot as a continuous thread in space is intuitively clear from common experience with string and shoelaces, but it is less clear how to decide when two knots should be viewed as equivalent or different. The novel idea from algebraic topology is to study the space where the knot is not, that is, the complement of a knot, which is a subset of Euclidean 3-space, $\mathbb{E}^3$, from which an embedded circle has been removed. The interest arises from the variety of ways in which a circle can be continuously embedded (ie as a homeomorphic image) into $\mathbb{E}^3$, even when we disallow infinite sequences of loops in the image (the so-called wild knots). In order to make a satisfactory attempt at classifying knots, we need to introduce some homotopy theory; this allows us to probe the complement of a knot with loop curves. The loops give rise to the **knot group**; then knots yielding different groups are different but the converse is not true so the classification is incomplete.

# 4 Homotopy

Topological spaces are enormously varied and homeomorphisms in general give much too fine a classification to be useful. Algebraic topology involves the classification of topological spaces in terms of algebraic objects (groups, rings) that are invariant under usefully large classes of homeomorphisms. The fundamental concept here is that of **homotopy equivalence**, for maps and spaces. Homotopy is studied in detail in Dodson and Parker [4], which contains many applications. We follow that approach here.

A pair of continuous maps $f, g : X \to Y$ which agree on $A \subseteq X$ is said to admit a **homotopy** $H$ from $f$ to $g$ **relative to** $A$ if there is a map

$$X \times [0,1] \xrightarrow{H} Y : (x,t) \longmapsto H_t(x)$$

with $H_t(a) = H(a,t) = f(a) = g(a)$ for all $a \in A$, $H_0 = H(\ ,0) = f$, and $H_1 = H(\ ,1) = g$. Then we write $f \overset{H}{\sim} g\ (relA)$. If $A = \emptyset$ or $A$ is clear from the context (such as $A = *$ for pointed spaces when $A$ is a point), then we write $f \overset{H}{\sim} g$, or sometimes just $f \sim g$ and say that $f$ and $g$ are **homotopic**.

We can also think of $H$ as either of:

- a 1-parameter family of maps

$$\{H_t : X \longrightarrow Y \ \mid\ t \in [0,1]\ \} \ \text{ with } \ H_0 = f \ \text{ and } \ H_1 = g\,;$$

- a curve $c_H$ from $f$ to $g$ in the function space $Y^X$ of maps from $X$ to $Y$

$$c_H : [0,1] \longrightarrow Y^X : t \longmapsto H_t\,.$$

We call $f$ **nullhomotopic** or **inessential** if it is homotopic to a constant map. Intuitively, we picture $H$ as a continuous deformation of the *graph* of $f$ into that of $g$. **Exercises**

1. Use the standard homeomorphism

$$h : [a,b] \longrightarrow [0,1] : s \longmapsto \frac{s-a}{b-a}$$

to show that

$$f : [0,1] \longrightarrow [0,1] : s \longmapsto \begin{cases} 2s & s \in [0, \frac{1}{4}] \\ s + \frac{1}{4} & s \in [\frac{1}{4}, \frac{1}{2}] \\ (s+1)/2 & s \in [\frac{1}{2}, 1] \end{cases}$$

is homotopic to the identity on $[0,1]$. Deduce that being homotopic is a transitive relation on paths and on loops in any space. Observe that a loop in $X$ is a path

$$c : [0,1] \longrightarrow X \quad \text{with } c(0) = c(1)\,,$$

so for loops we are interested in homotopy $rel\{0,1\}$.

2. Supply the proof that $\sim$ determines an equivalence relation.
3. Consider the identity map, $1_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$, as a closed curve on the torus $\mathbb{S}^1 \times \mathbb{S}^1$ and find two other closed curves on the torus such that all three belong to different homotopy classes.
4. If two continuous maps $f, g : X \to \mathbb{S}^n$ have $f(x) \neq -g(x)$ for all $x \in X$ then $f$ and $g$ are homotopic. For, otherwise consider

$$\frac{tf + (1-t)g}{\|tf + (1-t)g\|}\,.$$

5. Any two continuous maps into a contractible space are homotopic.

Two topological spaces $X, Y$ are said to be **of the same homotopy type** or **homotopy equivalent** or, loosely, just **homotopic** if there exist (continuous) maps

$$f : X \longrightarrow Y, \quad g : Y \longrightarrow X$$

$$\text{with } \; gf \sim 1_X \; \text{ and } \; fg \sim 1_Y\,.$$

Then we write $X \simeq Y$ and say that $f$ and $g$ are **mutual homotopy inverses** or **inverse up to homotopy**.

Similarly to the case for maps, $\simeq$ is an equivalence relation on any collection of topological spaces and one sometimes speaks (loosely) of spaces in the same class as being **homotopic**.

The spaces in the homotopy equivalence class determined by a singleton space are called **contractible**; we often use $*$ to denote a singleton space.

It turns out that equivalences up to homotopy are sufficient for easy proof of a wide range of important results in topology and analysis, (like fixed point theorems, extension and lifting theorems, fundamental theorem of algebra, hairy ball theorem . . . ) as may be seen in [4].

**Exercises**

The following $X, Y$ are homotopically equivalent spaces which are not homeomorphic in the usual topologies (*cf.* Figure 1). Here, the **wedge product** $\mathbb{S}^1 \vee \mathbb{S}^1$ is the quotient of the disjoint union of two circles, obtained by identifying one point in each circle to each other.

1. $X = \mathbb{S}^n$, $Y = \mathbb{S}^n \times \mathbb{R}^m$;
2. $X = \mathbb{R}^n$, $Y = \{0\}$;
3. $X = \mathbb{S}^{n-1}$, $Y = \mathbb{R}^n \setminus 0$;
4. $X = \mathbb{S}^1 \vee \mathbb{S}^1$, $Y =$ punctured Klein bottle;
5. $X = \mathbb{S}^1 \vee \mathbb{S}^1$, $Y =$ punctured torus.

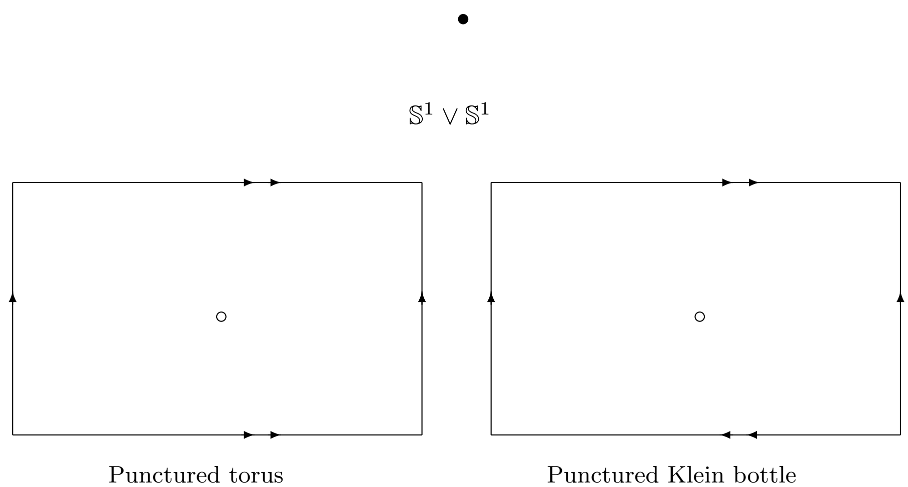


Figure 1: Homotopy equivalent spaces that are not homeomorphic

# 5 Groups

A **group** is the least structure in which we can define an internal operation which generalizes the multiplication and division on nonzero real numbers. A **field** is the nicest way in which two distinct groups can be fitted together so as to preserve the two identity elements as in the familiar example $(\mathbb{R}, +, \times)$ which is given by $\big((\mathbb{R}, +), (\mathbb{R} \setminus \{0\}, \times)\big)$. A **ring** is slightly weaker, lacking division, like $(\mathbb{Z}, +, \times)$.

A **vector space**, or **linear space**, is the nicest way in which a group (with a commutative operation $+$) can be combined with a field so as to preserve all three identity elements; a **module** is similar, but uses a ring instead of a field for its scalars.

For each of these, the appropriate maps which preserve the operations (hence all identities and inverses), between two structures of the same type, are called **homomorphisms**. Invertible homomorphisms are called **isomorphisms** and, for a given structure the set of self-isomorphisms or **automorphisms** forms a group. The fundamental concepts in group theory are enshrined in what we now call the category *Grp* of groups and group homomorphisms.

**Definitions**

A **group** is a set $G$ together with a map

$$* : G \times G \ \to \ G : (g, h) \ \mapsto \ g * h,$$

called a **binary operation**, satisfying:

1. $*$ is associative: $(a * b) * c \ = \ a * (b * c)$ $(\forall\, a, b, c \in G)$;

2. $*$ has an identity element $e \in G$: $a * e = e * a = a$ $(\forall\, a \in G)$;

3. $*$ admits inverses: $(\forall\, a \in G)(\exists\, a^{-1} \in G) : \ a * a^{-1} = a^{-1} * a = e$.

When there is no risk of confusion, we may omit the product symbol $*$ and write $ab$ for $a * b$; however, it is quite common to be dealing with more than one group structure on the same set so care is needed. A group $(G, *)$ is called **abelian** or **commutative** if $a * b = b * a$ for all $a, b \in G$.

A map $\phi : G \to H$ between groups $(G, *), (H, \star)$ is a **group homomorphism** if it preserves the group operations:

$$\phi(a * b) \; = \; \phi(a) \star \phi(b) \qquad (\forall\, a, b \in G)\,.$$

If the homomorphism is from a group to itself then we call it an **endomorphism**.

A subset $G'$ of a group $G$ is a **subgroup** of $G$ if the inclusion map $G' \overset{i}{\hookrightarrow} G$ is a group homomorphism; then $G'$ is itself a group with the restriction of the operation of $G$. The **kernel** of a homomorphism $\phi : G \to H$ is the subgroup $\ker \phi = \phi^{\leftarrow} 1_H$, where $1_H$ denotes the identity element of $H$.

If a group homomorphism $\phi : G \to H$ is bijective, then its inverse is also a group homomorphism, $\phi$ is called an **isomorphism** and the groups $G$ and $H$ are called **isomorphic**, written $G \cong H$. If an isomorphism is from a group to itself, then we call it an **automorphism**.

If $H$ is a subgroup of $G$, then we define for each $g \in G$:

1. $gH = \{g * h \mid h \in H\}$, and $\{gH \mid g \in G\}$ the set of **right cosets** of $H$ in $G$.
2. $Hg = \{h * g \mid h \in H\}$, and $\{Hg \mid g \in G\}$ the set of **left cosets** of $H$ in $G$.

There is always a bijection between the sets of right and left cosets, but it may not be natural. When it is, we call the subgroup **normal** if $gH = Hg$ for all $g$. In this case the set of cosets itself forms a group, the **quotient group** denoted by $G/H$.

The number of elements in $G$ is called the **order** of $G$, denoted $|G|$; if the order of a group is finite then we call it a finite group. If the smallest number of elements in a generating set is finite, then we call the group **finitely generated**. If $|G|$ is finite and $G$ has a subgroup $H$, then $H$ has a finite number of right cosets in $G$, called the **index** of $H$ in $G$ and denoted by $(G : H)$. It follows that, if $|G|$ is finite,

$$|G| = (G : H)\ |H|\,. \qquad \text{(Remember: } |G| \text{ finite!)}$$

This gives the famous theorem of Lagrange: if $G$ is a *finite* group then the order of any subgroup divides the order of $G$. Hence groups of prime order have no nontrivial subgroups.

We can construct a group from a given set of elements by simple juxtaposition of the elements; the group consists of the set of all finite **words** made up from the given elements and their inverses, with composition of words by juxtaposition. This group is called the **free group** on the given elements. The free group on one generator is isomorphic to $(\mathbb{Z}, +)$; a free group on more than one generator cannot be abelian. Many groups arise in practice as a set of generating elements together with some rules of combination. The **free product** $G * F$ of two groups consists of words made from both, with all internal products simplified in each.

The **direct product** $(G \times H, * \times \circ)$ of two groups $(G, *), (H, \circ)$ is the group defined on the product set $G \times H$ by

$$(g, h) * \times \circ\, (g', h') = (g * g', h \circ h')\,.$$

If two normal subgroups $J, K$ of a group $G$ can be found such that every $g \in G$ can be written uniquely in the form $g = jk$ for some $j \in J,\ k \in K$ and $J \cap K = \{e\}$, the trivial subgroup, then we say that $G$ decomposes into the direct product of $J$ and $K$.

The **commutator** of two elements $a, b$ in a group $G$ is the element $a^{-1}b^{-1}ab$. The subgroup $[G, G]$ of $G$ generated by all of its commutators is called the **commutator subgroup**; the quotient group $G/[G, G]$ is always abelian. We call the process of taking this quotient **abelianizing** $G$.

**Exercises**

1. Show that the identity element in a group is unique, as are inverses.
2. The set $\{z \in \mathbb{C} \mid \ |z| = 1\}$, of unimodular complex numbers, forms an infinite group under multiplication. This is actually a topological group, homeomorphic to the unit circle.

3. The set of $n^{th}$ roots of unity forms a group under multiplication.

4. Find all possible groups of orders 2, 3 and 4 by writing out possible entries in the matrix of products (the **group table**).

5. Given a finite group $G$ and any set $a_1, a_2, \ldots, a_n$ of distinct elements from $G$, prove that these elements and their products among themselves and their inverses generate a subgroup of $G$.

6. The set of $n \times n$ nonsingular real matrices forms a group $GL(n, \mathbb{R})$, often just written $GL(n)$, the **general linear group**, under matrix multiplication. So does $O(n)$, the subset consisting of **orthogonal** matrices, and its subset $SO(n)$ consisting of those with determinant $+1$.

7. Prove that $GL(2)$ has a subgroup consisting of

$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \ \theta \in \mathbb{R} \right\}.$$

   This is actually $SO(2)$, the **special orthogonal group** of $2 \times 2$ real matrices.

8. Find an isomorphism

$$f : SO(2) \to \{z \in \mathbb{C} |\ |z| = 1\}$$

   and give its inverse.

9. Prove that, for all elements $a$ in group $G$, the map

$$c_a : G \to G : x \mapsto a^{-1}xa$$

   is an automorphism; find the inverse of $c_a$.

10. Prove that if we have a homomorphism $f : G \to H$ and $H$ is abelian, then $\ker f$ contains all of the commutators in $G$.

11. If $\phi : G \longrightarrow H$ is a group homomorphism and $e_H$ denotes the identity element in $H$, then:

    (a) $\ker\phi = \{g \in G \mid \phi(g) = e_H\}$ is a subgroup of $G$.

    (b) $\text{im }\phi = \{\phi(g) \in H \mid g \in G\}$ is a subgroup of $H$.

12. The set of $n^{th}$ roots of unity forms a subgroup of the abelian group of unimodular complex numbers.

13. The map

$$\phi : \mathbb{Z} \longrightarrow \mathbb{S}^1 : k \longmapsto e^{ik2\pi}$$

    is a group homomorphism from the additive group of integers $(\mathbb{Z}, +)$ to the multiplicative group of unimodular complex numbers.

14. $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{R}, +)$.

15. $GL(n; \mathbb{R})$ is not abelian if $n > 1$.

16. The **symmetric group** $S_n$ of permutations of $n$ objects is not abelian for $n > 2$.

17. Given groups $G_1, G_2$, show that the natural projections

$$p_i : G_1 \times G_2 \ \longrightarrow \ G_i : \ (g_1, g_2) \ \mapsto \ g_i \qquad (i = 1, 2)$$

    are group homomorphisms from the direct product group.

18. If $H, K$ are subgroups of $G$ then $H \cap K$ is also a subgroup, but $H \cup K$ is a subgroup of $G$ if and only if $H \subseteq K$ or $K \subseteq H$. If $H, K$ are normal subgroups then so is $HK$.

19. If $G$ has no nontrivial subgroups (that is, only $\{e\}$ and $G$ are subgroups of $G$) then $G$ is generated by one element (so $G$ is called a **cyclic** group) and has prime order.

# 6 Fundamental Group

The first nontrivial homotopy class of topological spaces is represented by $\mathbb{S}^0$, since it has two (disjoint connected) components; indeed, it is elementary to show that connectedness may be characterised by the nonexistence of continuous maps from a space onto $\mathbb{S}^0$, and actually, 'up to homotopy' is good enough for such characterization.

Next comes $\mathbb{S}^1$ which has only one component but it admits also loops which cannot be 'homotopied' to a point. We denote by $[\mathbb{S}^1, \mathbb{S}^1]$ the collection of homotopy equivalence classes of continuous maps from $\mathbb{S}^1$ to itself (ie loops in $\mathbb{S}^1$) which preserve a given basepoint $*$. Intuitively, the set of classes is parametrized by $\mathbb{Z}$, since the loops either go forwards a net integer number of times round the circle, or backwards, or are the trivial loop, up to homotopy. The fact that $[\mathbb{S}^1, \mathbb{S}^1] \equiv \mathbb{Z}$ as a set (and, later, as a group) and is not a singleton tells us that $\mathbb{S}^1$ bounds a 2-dimensional 'hole', whereas $\mathbb{R}^1$ and $\mathbb{S}^0$ do not.

For any *pointed* $X$, that is a topological space with a chosen basepoint $*$, its **fundamental group** or **first homotopy group** is the set of homotopy classes of loops based at $*$ :

$$\pi_1(X) \ = \ [\mathbb{S}^1, X] \tag{1}$$

which has a natural group structure by composition of curves. Also, we denote by $\pi_0(X) = [\mathbb{S}^0, X]$ the (pointed) set of path components of $X$; so $X$ is **connected** if $\pi_0(X)$ is a singleton.

> The higher homotopy groups, $[\mathbb{S}^n, X]$, are extremely powerful tools in analysis, geometry and topology [4]; we shall not need that theory here but to whet your appetite for it we note the beautiful and surprising result of Hopf which led to homotopy theory (cf [4] p 100):
>
> $$[\mathbb{S}^3, \mathbb{S}^2] \cong \mathbb{Z}, \quad \text{so } \pi_3(\mathbb{S}^2) = \mathbb{Z}$$
>
> in contrast to
>
> $$\pi_m(\mathbb{S}^1) = 0 \ \text{ for } m > 1$$
>
> Find out more about Hopf via the bigraphies of mathematicians at
> http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/

If $\pi_1(X)$ is the trivial group we write $\pi_1(X) = 0$, and if $\pi_1(X) = 0$ and $\pi_0(X)$ is a singleton, then we say that $X$ is **simply-connected**—which is independent of the choice of basepoint. When the basepoint must be denoted, because $X$ has more than one connected component, we write $\pi_1(X, x_0)$ for example.

A common way to make use of this construction is to show that two spaces have different fundamental groups; then it follows that they must have different homotopy types and hence cannot be homeomorphic. Continuous maps between spaces induce group homomorphisms between their fundamental groups, a powerful way to study families of spaces.

**Exercises**

1. The result $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ can be approached as follows.
   - (a) $p : \mathbb{R} \to \mathbb{S}^1 : x \mapsto e^{2\pi i x}$ is a continuous surjection.
   - (b) $\phi : \mathbb{Z} \to \pi_1(\mathbb{S}^1) : n \mapsto [p\sigma_n]$ where $\sigma_n : I \to \mathbb{R} : t \mapsto nt$ is a group homomorphism.
   - (c) Paths in $\mathbb{S}^1$ admit unique lifts to $\mathbb{R}$.
   - (d) $\mathbb{R} \xrightarrow{p} \mathbb{S}^1$ has the homotopy lifting property.
   - (e) $\phi$ is an isomorphism.
2. If $X = U \cup V$ for some open 1-connected subsets $U, V$, and $U \cap V$ is 0-connected, then $X$ is 1-connected since loops in $X$ are homotopic to a product of loops in $U$ or in $V$. Hence $\pi_1(\mathbb{S}^n) = 0$ for $n \geq 2$.

3. Consider the two paths $c$ and $a$ going half counterclockwise and half clockwise respectively round $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$ :

$$c : [0,1] \longrightarrow \mathbb{S}^1 : s \longmapsto e^{1\pi s}\,,$$

$$a : [0,1] \longrightarrow \mathbb{S}^1 : s \longmapsto e^{-1\pi s}\,.$$

   Show they induce the same isomorphisms between $\pi_1(\mathbb{S}^1, 1)$ and $\pi_1(\mathbb{S}^1, -1)$.

4. The fundamental group of $\mathbb{S}^1 \vee \mathbb{S}^1$ is $\mathbb{Z} * \mathbb{Z}$ and hence is nonabelian. The paths in $\mathbb{S}^1 \vee \mathbb{S}^1$ corresponding to $a, c$ in the previous example do not induce the same isomorphisms.

5. No continuous map from the unit disk to its boundary can restrict to the identity on the boundary—simply consider the fundamental groups and the homomorphism induced by the inclusion map of the boundary.

6. Show that

$$\begin{aligned} \pi_1(\mathbb{E}^3) &= 0, \\ \pi_1(\mathbb{S}^1 \times \mathbb{S}^1) &= \mathbb{Z} \times \mathbb{Z}. \end{aligned}$$

# 7 Simplicial Complex

For computation of fundamental groups of compact subsets of $\mathbb{E}^n$ it is convenient to represent the subsets as simplicial complexes, made up from suitably joined copies of the standard n-simplices [2, 4].

The **standard** $n$**-simplex** is

$$\Delta^n \;=\; \{(x_i) \in \mathbb{R}^{n+1} \mid \sum x_i = 1,\; x_i \in [0,1]\; \forall i\}$$

and its **vertices** are the $n+1$ points

$$v_0 = (1, 0, \ldots, 0),\;\; v_1 = (0, 1, 0, \ldots, 0), \ldots, v_n = (0, \ldots, 0, 1)\,.$$

Conversely, given $n+1$ points $v_0, v_1, \ldots, v_n$ in $\mathbb{R}^{n+1}$ such that the $n$ vectors $\{v_i - v_0 \mid i = 1, \ldots, n\}$ are linearly independent, then they define an $n$-simplex. It is the subset:

$$(v_0, v_1, \ldots, v_n) = \{x \in \mathbb{E}^{n+1} \mid x = \sum t_i v_i,\;\; t_i \in [0,1],\;\; \sum t_i = 1\}$$

which we say **spans** the vertices $v_0, v_1, \ldots, v_n$ with **barycentric coordinates** $(t_i)$.

A **geometric (finite) simplicial complex** $K$ is a (finite) collection $\{\sigma_i \in \mathbb{E}^m \mid i = 1, 2, \ldots, p\}$ of simplices, all in $\mathbb{E}^m$ for some finite $m$, satisfying:

(i) $\sigma_i \cap \sigma_j$ is a face of $\sigma_i$ and of $\sigma_j$

(ii) every face of a simplex in $K$ is itself a simplex in $K$. A simplicial complex $K$ inherits the *subspace topology* and we denote this topological space by $|K|$. It is called a **realization** of $K$. Then a space $X$, homeomorphic to $|K|$ is called a **polyhedron** and we say that $K$ is a **triangulation** of $X$.

**Van Kampen's Theorem** allows us to compute fundamental groups of spaces in terms of those of constituent simpler subspaces [2, 4]:

> Let $M$ be a polyhedron with a triangulation as the union of two simplicial complexes $M \equiv J \cup K$ where $J, K, J \cap K$ are all path-connected with inclusions of the underlying spaces
>
> $$|J| \overset{j}{\hookleftarrow} |J \cap K| \overset{i}{\hookrightarrow} |K|\,.$$

Then, for all vertices $v$ in $J \cap K$,

$$\pi_1(|M|, v) \cong \pi_1(|J|, v) * \pi_1(|K|, v)/\sim$$

where $\sim$ denotes the set of relations:

$$j_*(z) = k_*(z) \qquad \text{for all } z \in \pi_1(|J \cap K|, v)\,.$$

The simplicial complex structures actually generate a family of groups, called **simplicial homology groups** [2, 4], one for each dimension present in the complex. These groups are easier to compute than the homotopy groups but they do in a sense approximate the latter.

# 8 Knot Group

A **knot** $k$ is an embedding (ie a continuous 1-1 map) of $\mathbb{S}^1$ into $\mathbb{E}^3$, and we shall restrict our attention to **tame knots** which have only a finite number of loops in $\mathbb{E}^3$. In some situations it is convenient for the proofs to consider the knots to be embedded in $\mathbb{S}^3$ by adding a point at infinity to $\mathbb{E}^3$.

The **knot group** of $k$ is $\pi_1(\mathbb{E}^3 \setminus k)$, so the **trivial knot,** $k = \mathbb{S}^1$, has knot group $\mathbb{Z}$. Armstrong [2] Chapter 10 gives a procedure for using Van Kampen's theorem to obtain knot groups in terms of generators (one per overpass) and relations (one per crossing) in suitable projections of the knot onto a plane. For example, the **trefoil knot** has knot group $\{a, b|aba = bab\}$. Abelianizing a knot group $G$, that is by taking its quotient by the commutator subgroup

$$C = \{x^{-1}y^{-1}xy|x, y \in G\}$$

always yields the free group on one generator, $G/C \cong \mathbb{Z}$. The **first homology group** $H_1(X)$ of a connected compact subset $X \subset \mathbb{E}^n$ is generated by classes of loops round the edges of simplices and actually coincides with the quotient of the fundamental group by its commutator subgroup.

The knot group can be represented by the so-called **Alexander Polynomial** through a combinatorial algorithm applied to a suitably 'nice' projection (eg. no crossings project onto one another) of the knot onto a plane [2]. The Alexander polynomial is unaltered by a mirror reflection of the knot. For the knots with 6 or less overcrossings in a nice projection, the inequivalent Alexander polynomials up to a factor $t^k$ are given for you to check in the table below. It uses the notation of Lickorish[8], where knot $m_n$ is the $n^{th}$ knot type having $m$ overcrossings.

| Knot $m_n$ | Alexander Polynomial |
|---|---|
| $3_1$ (Trefoil) | $+1 - t + t^2$ |
| $4_1$ (Figure eight) | $-1 + 3t - t^2$ |
| $5_1$ | $+1 - t + t^2 - t^3 + t^4$ |
| $5_2$ | $+2 - 3t + 2t^2$ |
| $6_1$ (Stevedore's) | $-2 + 5t - 2t^2$ |
| $6_2$ | $-1 + 3t - 3t^2 + 3t^3 - t^4$ |
| $6_3$ | $+1 - 3t + 5t^2 - 3t^3 + t^4$ |

Find out more about Alexander via the bigraphies of mathematicians at
http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/

# References


[1] On-line mathematical materials:

Mathematicians: http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/

Alfred Gray's Mathematica NoteBooks: http://library.wolfram.com/infocenter/Books/3759

Elementary Notes on:
Curves http://www.maths.manchester.ac.uk/ kd/curves/curves.pdf
Surfaces http://www.maths.manchester.ac.uk/ kd/curves/surfaces.pdf
Knots http://www.maths.manchester.ac.uk/ kd/curves/knots.pdf

LaTeX Tutorial:
http://www.maths.manchester.ac.uk/ kd/latextut/pdfbyex.htm

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