

# Quantifying galactic clustering and departures from randomness of the inter-galactic void probability function using information geometry \*

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## Abstract

A number of recent studies have estimated the inter-galactic void probability function and investigated its departure from various random models. We study a family of parametric statistical models based on gamma distributions, which do give realistic descriptions for other stochastic porous media. Gamma distributions contain as a special case the exponential distributions, which correspond to the ‘random’ void size probability arising from Poisson processes. The space of parameters is a surface with a natural Riemannian metric structure. This surface contains the Poisson processes as an isometric embedding and a recent theorem [1] shows that it contains neighbourhoods of all departures from randomness. The method provides thereby a geometric setting for quantifying departures from randomness and on which may be formulated cosmological evolutionary dynamics for galactic clustering and for the concomitant development of the void size distribution.

The 2dFGRS data [5, 6] offer the possibility of more detailed investigation of this approach than was possible when it was originally suggested [7, 8, 9] and some parameter estimations are given.

**Key words:** galaxies: clustering, statistics; cosmology: voids, statistics, parametric models, information geometry.

## 1 Introduction

Several years ago, the author presented an information geometric approach to modelling a space of perturbations of the random state for galactic clustering and cosmological void statistics [7]. The present note is intended to update somewhat and draw attention to this approach as a possible contribution to the interpretation of the data from the 2-degree field Galaxy Redshift Survey (2dFGRS), cf Croton et al. [5, 6].

The classical random model is that arising from a Poisson process of mean density  $\bar{N}$  galaxies per unit volume in a large box. Then, in a region of volume  $V$ , the probability of finding exactly  $m$  galaxies is

$$P_m = \frac{(\bar{N}V)^m}{m!} e^{-\bar{N}V} \quad (1)$$

So the probability that the given region is devoid of galaxies is  $P_0 = e^{-\bar{N}V}$ . It follows that the probability density function for the continuous random variable  $V$  in the Poisson case is

$$p_{\text{random}}(V) = \bar{N} e^{-\bar{N}V} \quad (2)$$

In practice of course, measurements will depend on algorithms that specify threshold values for density ranges of galaxies in cells and the lowest range will represent the ‘underdense’ regions which include the voids; Benson et al. [3] discuss this.

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A hierarchy of  $N$ -point correlation functions needed to represent clustering of galaxies in a complete sense was devised by White [21] and he provided explicit formulae, including their continuous limit. In particular, he made a detailed study of the probability that a sphere of radius  $R$  is empty and showed that formally it is symmetrically dependent on the whole hierarchy of correlation functions. However, White concentrated his applications on the case when the underlying galaxy distribution was a Poisson process, the starting point for the present approach which is concerned with geometrizing the parameter space of departures from a Poisson process. Croton et al. [6] found that the negative binomial model for galaxy clustering gave a very good approximation to the 2dFGRS, pointing out that this model is a discrete version of the gamma distribution.

## 2 Modelling statistics of galaxy void sizes

For a general account of large-scale structures in the universe, see Fairall [12]. Kauffmann and Fairall [15] developed a catalogue search algorithm for larger nearly spherical regions devoid of bright galaxies and obtained a spectrum for radii of significant voids. This indicated a peak radius near  $4 h^{-1}Mpc$ , a long tail stretching at least to  $32 h^{-1}Mpc$ , and is compatible with the recent extrapolation models of Baccigalupi et al [2] which yield an upper bound on void radii of about  $50 h^{-1}Mpc$ . This data has of course omitted the expected very large numerical contribution of smaller voids. More recent work, notably of Croton et al. [6] provide much larger samples with improved estimates of void size statistics and Benson et al. [3] gave a theoretical analysis in anticipation of the 2dFGRS survey data, including the evaluation of the void and underdense probability functions. Hoyle and Vogeley [13] provided detailed results for the statistics of voids larger than  $10 h^{-1}Mpc$  in the 2dFGRS survey data; they concluded that such voids constitute some 40% of the universe and have a mean radius of about  $15 h^{-1}Mpc$ .

The count density  $N(V)$  of galaxies observed in zones using a range of sampling schemes each with a fixed zone volume  $V$  results in a decreasing variance  $Var(N(V))$  of count density with increasing zone size, roughly of the form

$$Var(N(V)) \approx V(0) e^{-V/V_k} \text{ as } V \rightarrow 0 \quad (3)$$

where  $V_k$  is some characteristic scaling parameter. This monotonic decay of variance with zone size is a natural consequence of the monotonic decay of the covariance function, roughly isotropically and of the form

$$Cov(r) \approx e^{-r/r_k} \text{ as } r \rightarrow 0 \quad (4)$$

where  $r_k$  is some characteristic scaling parameter of the order of magnitude of the diameter of filament structures; this was discussed in [8]. Then

$$Var(N(V)) \approx \int_0^\infty Cov(r) b(r) dr \quad (5)$$

where  $b(r)$  is the probability density of finding two points separated by distance  $r$  independently and at random in a zone of volume  $V$ . The power spectrum using, say, cubical cells of side lengths  $R$  is given by the family of integrals

$$Pow(N(R)) \approx \int_R^\infty Cov(r) b(r) dr. \quad (6)$$

Fairall [12] (page 124) reported a value  $\sigma^2 = 0.25$  for the ratio of variance  $Var(N(1))$  to mean squared  $\bar{N}^2$  for counts of galaxies in cubical cells of unit side length. In other words, the coefficient of variation for sampling with cells of unit volume is

$$cv(N(1)) = \frac{\sqrt{Var(N(1))}}{\bar{N}} = 0.5 \quad (7)$$

and this is dimensionless.

We choose a family of parametric statistical models for void volumes that includes the random model (2) as a special case. There are of course many such families, but we take one that a recent theorem [1] has shown contains neighbourhoods of all departures from randomness and it has been successful in modelling void size distributions in terrestrial stochastic porous media with similar departures from randomness [11]. Also, the

complementary logarithmic version has been used in the representation of clustering of galaxies [8, 9]. The family of gamma distributions has event space  $\Omega = \mathbb{R}^+$ , parameters  $\mu, \beta \in \mathbb{R}^+$  and probability density functions given by

$$f(V; \mu, \beta) = \left(\frac{\beta}{\mu}\right)^\beta \frac{V^{\beta-1}}{\Gamma(\beta)} e^{-V\beta/\mu} \quad (8)$$

Then  $\bar{V} = \mu$  and  $Var(V) = \mu^2/\beta$  and we see that  $\mu$  controls the mean of the distribution while the spread and shape is controlled by,  $1/\beta$ , the square of the coefficient of variation.

The special case  $\beta = 1$  corresponds to the situation when  $V$  represents the random or Poisson process in (2) with  $\mu = 1/n$ . Thus, the family of gamma distributions can model a range of stochastic processes corresponding to non-independent ‘clumped’ events, for  $\beta < 1$ , and dispersed events, for  $\beta > 1$ , as well as the random case (cf. [1, 10, 11]). Thus, if we think of this range of processes as corresponding to the possible distributions of centroids of extended objects such as galaxies that are initially distributed according to a Poisson process with  $\beta = 1$ , then the three possibilities are:

**Chaotic or random structure** with no interactions among constituents,  $\beta = 1$ ;

**Clustered structure** arising from mutually attractive type interactions,  $\beta < 1$ ;

**Dispersed structure** arising from mutually repulsive type interactions,  $\beta > 1$ .

For our gamma-based void model we consider the radius  $R$  of a spherical void with volume  $V = \frac{4}{3}\pi R^3$  having distribution (8). Then the probability density function for  $R$  is given by

$$p(R; \mu, \beta) = \frac{4\pi R^2}{\Gamma(\beta)} \left(\frac{\beta}{\mu}\right)^\beta \left(\frac{4\pi R^3}{3}\right)^{\beta-1} e^{-\frac{4\pi R^3\beta}{3\mu}} \quad (9)$$

The mean  $\bar{R}$ , variance  $Var(R)$  and coefficient of variation  $cv(R)$  of  $R$  are given, respectively, by

$$\bar{R} = \left(\frac{3\mu}{4\pi\beta}\right)^{\frac{1}{3}} \frac{\Gamma(\beta + \frac{1}{3})}{\Gamma(\beta)} \quad (10)$$

$$Var(R) = \left(\frac{3\mu}{4\pi\beta}\right)^{\frac{2}{3}} \frac{\Gamma(\beta) \Gamma(\beta + \frac{2}{3}) - \Gamma(\beta + \frac{1}{3})^2}{\Gamma(\beta)^2} \quad (11)$$

$$cv(R) = \frac{\sqrt{Var(R)}}{\bar{R}} = \sqrt{\frac{\Gamma(\beta) \Gamma(\beta + \frac{2}{3})}{\Gamma(\beta + \frac{1}{3})^2} - 1} \quad (12)$$

The fact that the coefficient of variation (12) depends only on  $\beta$  gives a rapid parameter fitting of data to the probability density function for void radii (9). Numerical fitting to (12) gives  $\beta$ ; this substituted in (10) yields an estimate of  $\mu$  to fit a given observational mean.

However, there is a complication: necessarily in order to have a physically meaningful definition for voids, observational measurements introduce a minimum threshold size for voids. For example, Hoyle and Vogeley [13] used algorithms to obtain statistics on 2dFGRS voids with radius  $R > R_{min} = 10 h^{-1} Mpc$ ; for voids above this threshold they found their mean size is about  $15 h^{-1} Mpc$  with a variance of about 8.1. This of course is not directly comparable with the above distribution for  $R$  in equation (9) since the latter has domain  $R > 0$ . Now, from (9), the probability that a void has radius  $R > A$  is

$$P_A = \frac{\Gamma(\beta, \frac{4A^3\pi\beta}{3\mu})}{\Gamma(\beta)} \quad (13)$$

and hence the mean, variance and coefficient of variation for the void distribution with  $R > A$  become:

$$\overline{R_{>A}} = \left(\frac{3\mu}{4\pi\beta}\right)^{\frac{1}{3}} \frac{\Gamma(\beta + \frac{1}{3}, \frac{4A^3\pi\beta}{3\mu})}{\Gamma(\beta, \frac{4A^3\pi\beta}{3\mu})} \quad (14)$$

$$Var(R_{>A}) = \left(\frac{3\mu}{4\pi\beta}\right)^{\frac{2}{3}} \frac{\Gamma(\beta) \Gamma(\beta + \frac{2}{3}, \frac{4A^3\pi\beta}{3\mu}) - \Gamma(\beta + \frac{1}{3}, \frac{4A^3\pi\beta}{3\mu})^2}{\Gamma(\beta, \frac{4A^3\pi\beta}{3\mu}) \Gamma(\beta)} \quad (15)$$

$$cv(R_{>A}) = \frac{\sqrt{Var(R_{>A})}}{\overline{R_{>A}}} = \sqrt{\frac{\Gamma(\beta) \Gamma(\beta + \frac{2}{3}, \frac{4A^3\pi\beta}{3\mu})}{\Gamma(\beta + \frac{1}{3}, \frac{4A^3\pi\beta}{3\mu})^2} - 1} \quad (16)$$

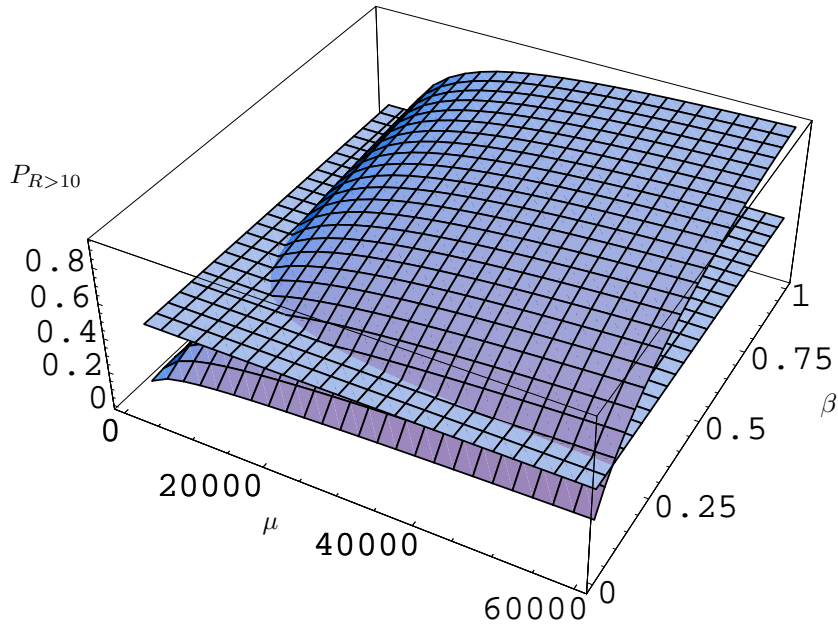


Figure 1: Probability that a void will have radius  $R > 10 h^{-1} \text{Mpc}$  as a function of parameters  $\mu$ ,  $\beta$  from equation (13). The range  $\beta < 1$  corresponds to clustering regimes. The plane at level  $P_{R>10} = 0.4$  corresponds to the fraction 40% of the universe filled by voids, as reported by Hoyle and Vogeley [13].

where

$$\Gamma(\beta, A) = \int_A^\infty t^{\beta-1} e^{-t} dt \text{ is the incomplete gamma function with } \Gamma(\beta) = \Gamma(\beta, 0).$$

Summarizing from Hoyle and Vogeley [13]:

$$A = 10 h^{-1} \text{Mpc}, P_A \approx 0.4, \overline{R_{>A}} \approx 15 h^{-1} \text{Mpc}, \text{Var}(R_{>A}) \approx 8.1 (h^{-1} \text{Mpc})^2 \text{ so } cv(R_{>A}) \approx 0.19.$$

### 3 Model coupling of clustering densities and voids

Next we follow the methodology introduced in [8, 9] to provide a model that links the number counts in cells and the void probability function and which contains perturbations of the random case. This exploits the central role of the gamma distribution in providing neighbourhoods of randomness [1] that contain all maximum likelihood nearby perturbations of the random case and it allows a direct use of the linked information geometries for the coupled stochastic processes of voids and galaxies. Clearly, in regions where the local void volume  $V$  tends to be small the local matter density  $N$  will tend to be large. Since the matter density must be bounded, then a simple phenomenological model that couples the two random variables in the stochastic process is  $N(V) = e^{-V}$ , where the upper bound on  $N$  has been set to unity. This model was explored in [8, 9] and it is easy to show that the probability density function for  $N$  is given by the log gamma distribution

$$g(N; \mu, \beta) = \left(\frac{\beta}{\mu}\right)^\beta \frac{N^{\beta/\mu-1}}{\Gamma(\beta)} |\log N|^{1-\beta} \quad (17)$$

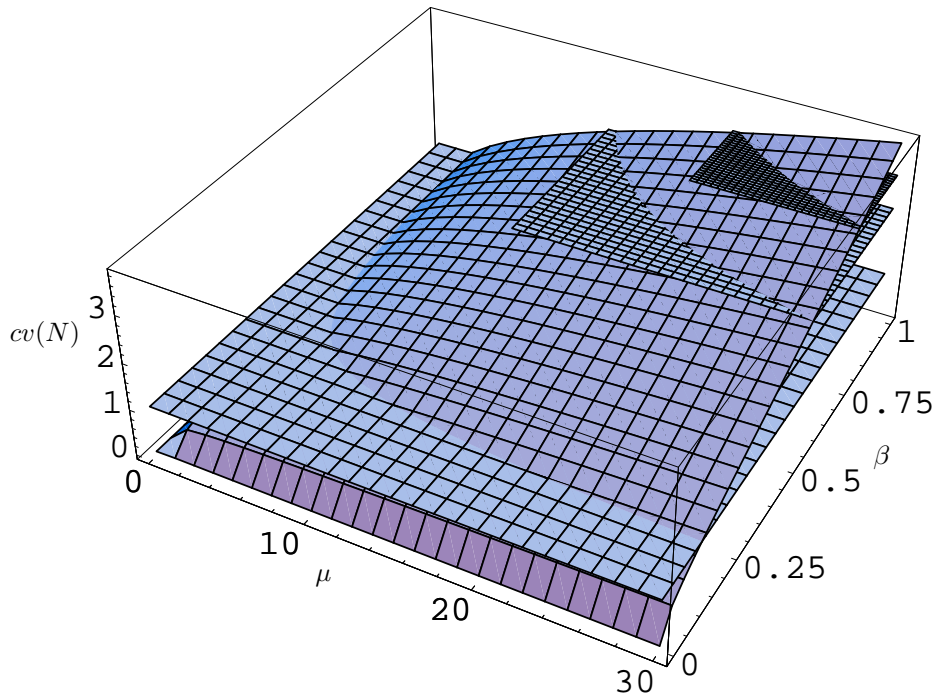


Figure 2: Coefficient of variation of counts in cells  $N$  for log gamma distribution equation (17). The range  $\beta < 1$  corresponds to clustering regimes. The three planes show the levels  $cv(N) = 1, \sqrt{6}, \sqrt{10}$  as reported by Fairall [12] and Croton et al. [5, 6].

This distribution for local galactic number density has mean  $\bar{N}$ , variance  $Var(N)$  and coefficient of variation  $cv(N) = \sqrt{Var(N)}/\bar{N}$  given by

$$\bar{N} = \left( \frac{\beta}{\beta + \mu} \right)^\beta \quad (18)$$

$$Var(N) = \left( \frac{\beta}{\beta + 2\mu} \right)^\beta - \left( \frac{\beta}{\beta + \mu} \right)^{2\beta} \quad (19)$$

$$cv(N) = \frac{\sqrt{Var(N)}}{\bar{N}} = \sqrt{\left( \frac{\beta}{\beta + \mu} \right)^{-2\beta} \left( \frac{\beta}{\beta + 2\mu} \right)^\beta - 1}. \quad (20)$$

Using the reported value  $cv(N(1)) = 0.5$  from Fairall [12] (page 124) for cubical volumes with side length  $R = 1 h^{-1} Mpc$ , the curve so defined in the parameter space for the log gamma distributions (17), has maximum clustering for  $\beta \approx 0.6, \mu \approx 0.72$ .

From the 2-degree field Galaxy Redshift Survey (2dFGRS), Croton et al. [5] in Figure 2 reported the decay of normalised variance,  $\bar{\xi}_2 = cv(N(R))^2$  with scale radius  $R$  and the associated departure from randomness in the form  $\chi = -\frac{\log_{10} P_0(R)}{\bar{N}}$ , where  $P_0(R)$  is the probability of finding zero galaxies in a spherical region of radius  $R$  when the mean number is  $\bar{N}$ . From that Figure 2 we see that, for the data of the Volume Limited Catalogue with magnitude range  $-20$  to  $-21$  and  $\bar{N}(1) = 1.46$ :  $cv(N(1))^2 \approx 6$  and  $\chi \approx 0.9$  at  $R \approx 1$  also  $cv(N(7))^2 \approx 1$  and  $\chi \approx 0.4$  at  $R \approx 7$ .

Croton et al. [6] in Table 1 reported  $\bar{N}$  values for cubical volumes with side length  $R = 1 h^{-1} Mpc$  in the range  $0.11 \leq \bar{N} \leq 11$ . From Figure 3 in that paper we see that, at the scale  $R = 1 h^{-1} Mpc$ ,  $\log_{10} \bar{\xi}_2 \approx 1$  which gives a coefficient of variation  $cv(N(1)) \approx \sqrt{10}$ .

The above-mentioned observations  $cv(N) = 1, \sqrt{6}, \sqrt{10}$  are shown in Figure 2 on a plot of the coefficient of variation for the number counts in cells from the log gamma family of distributions equation (17). The range  $\beta < 1$  corresponds to clustering regimes.

Theoretical models for the evolution of galactic clustering through an evolving stochastic process subordinate to the log gamma distribution (17) of densities could be represented as curves on the space of parameters with

the metric (25) interpreting the parameter changes with time in the appropriate way. The coupling with the void probability function controlled by the gamma distribution (9) allows the corresponding void evolution to be represented. It is of course very unlikely that this simple model is suitable in all respects but given a different family of distributions the necessary information geometry can be computed for the representation of evolutionary processes.

## 4 Information geometry of gamma models for void volume statistics

For stochastic processes subordinate to a given family of parametric statistical models, Shannon's information theoretic 'entropy' or 'uncertainty' (cf. eg. Jaynes [14]) is given, up to a factor, by the negative of the expectation of the logarithm of the probability density function. For the family of models we propose for void volumes (8) this entropy is,

$$S_f(\mu, \beta) = - \int_0^\infty \log(f(V; \mu, \beta) f(V; \mu, \beta) dV \quad (21)$$

$$= \beta + (1 - \beta) \frac{\Gamma'(\beta)}{\Gamma(\beta)} + \log \frac{\mu \Gamma(\beta)}{\beta} \quad (22)$$

In particular, at unit mean, the maximum entropy (or maximum uncertainty) occurs at  $\beta = 1$ , which is the random case, and then  $S_f(\mu, 1) = 1 + \log \mu$ .

The 'maximum likelihood' estimates  $\hat{\mu}, \hat{\beta}$  of  $\mu, \beta$  can be expressed in terms of the mean and mean logarithm of a set of independent observations  $X = \{X_1, X_2, \dots, X_n\}$ . These estimates are obtained in terms of the properties of  $X$  by maximizing the 'log-likelihood' function

$$l_X(\mu, \beta) = \log \text{lik}_X(\mu, \beta) = \log \left( \prod_{i=1}^n p(X_i; \mu, \beta) \right)$$

with the following result

$$\hat{\mu} = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad (23)$$

$$\log \hat{\beta} - \psi(\hat{\beta}) = \overline{\log X} - \log \bar{X} \quad (24)$$

where  $\overline{\log X} = \frac{1}{n} \sum_{i=1}^n \log X_i$  and  $\psi(\beta) = \frac{\Gamma'(\beta)}{\Gamma(\beta)}$  is the digamma function, the logarithmic derivative of the gamma function

The usual Riemannian information metric on the 2-dimensional parameter space  $\mathcal{S} = \{(\mu, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+\}$  is given by

$$ds_{\mathcal{S}}^2 = \frac{\beta}{\mu^2} d\mu^2 + \left( \psi'(\beta) - \frac{1}{\beta} \right) d\beta^2 \quad \text{for } \mu, \beta \in \mathbb{R}^+. \quad (25)$$

The important point about this non-Euclidean metric on the space of parameters is that it derives from log-likelihood properties and so it is the 'correct' one for this family of distributions. Given a different family then the information geometry can be computed for that; the geometries are known also for bivariate gamma distributions, Gaussian and multivariate Gaussian distributions among others. For more details about the geometry see [10].

The 1-dimensional subspace parametrized by  $\beta = 1$  corresponds to the available 'random' processes. A path through the parameter space  $\mathcal{S}$  of gamma models determines a curve

$$c : [a, b] \rightarrow \mathcal{S} : t \mapsto (c_1(t), c_2(t)) \quad (26)$$

with tangent vector  $\dot{c}(t) = (\dot{c}_1(t), \dot{c}_2(t))$  and norm  $\|\dot{c}\|$  given via (25) by

$$\|\dot{c}(t)\|^2 = \frac{c_2(t)}{c_1(t)^2} \dot{c}_1(t)^2 + \left( \psi'(c_2(t)) - \frac{1}{c_2(t)} \right) \dot{c}_2(t)^2. \quad (27)$$

The information length of the curve is

$$L_c(a, b) = \int_a^b \|\dot{c}(t)\| dt \quad (28)$$

and the curve corresponding to an underlying Poisson process has  $c(t) = (t, 1)$ , so  $t = \mu$  and  $\beta = 1 = \text{constant}$ , and the information length is  $\log \frac{b}{a}$ .

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